Double Phase Representation of Analytic Functions*

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The double phase representation is discussed for the elastic scattering amplitude $A(s,t,u)$ as a function of the covariant Mandelstam variables s, t, and u. This representation is written as $A(s,t,u)=[P_1(s,t,u)/P_2(s,t,u)]Q(s,t,u)$, where $P_1(s,t,u)$ and $P_2(s,t,u)$ are both finite polynomials in s, t, and u, and $Q(s,t,u)$ has no zeros or poles except at infinity and is expressed in terms of the phase of $A(s,t,u)$ along the cuts. Thus, $P_1(s,t,u)$ and $P_2(s,t,u)$ account for all the zeros and poles of $A(s,t,u)$, respectively, except for a zero or a pole at infinity. The conditions for the above double phase representation to exist are, besides the usual Mandelstam assumption, that a finite polynomial $P_1(s,t,u)$ accounts for all the zeros of $A(s,t,u)$ except for the one at infinity and no others, and that $A(s,t,u)$ has even or odd crossing symmetry with respect to the interchange of some pair of s, t, and u. These conditions imply that the phase of $A(s,t,u)$ has no extra branch points in the momentum-transfer plane other than those which belong to $A(s,t,u)$ and remains finite in the physical regions even in the limit of infinite energy. The asymptotic forms of this double phase representation when some of s, t, and u become infinite are derived in the case when the phase approaches the limit at infinity not too slowly. This is the case when the elastic scattering amplitude exhibits asymptotically a power behavior in energy (usually called the Regge behavior). In particular, the case when the forward peak of high-energy elastic scattering does not shrink is examined closely. The case of no shrinkage is found to be the case when the phase in the crossed channel does not diverge logarithmically at infinity in its momentum-transfer plane. If the forward peak shrinks, the above phase diverges logarithmically at infinity. In the case of no shrinkage, the asymptotic shape of the forward peak is determined solely by the phase in the crossed channel. Furthermore, the above shape assumes a pure exponential function of the covariant momentum-transfer squared when momentum transfer is small, and approaches a power-law behavior in the same variable for large momentum transfer. In the case of the $\pi^0+\pi^0\rightarrow\pi^0+\pi^0$ amplitude, high symmetry available in this amplitude enables one to determine almost uniquely the polynomials in the double phase representation. In particular, the only possibility in the case of no shrinkage is $P_1(s,t,u)/P_2(s,t,u)=c_0+c_2(s^2+t^2+u^2)$, where c_0 and c_2 are real constants. No shrinkage also implies that the S-wave scattering length must not be negative for the $\pi^0+\pi^0\rightarrow\pi^0+\pi^0$ amplitude. Some of the specific predictions of the phase representation approach to high-energy elastic scattering are listed at the end of the last section.

1. INTRODUCTION

THE analytic function $A(s)$ has under the conditions given below the phase representation[1]

$$A(s)=\frac{P_1(s)}{P_2(s)}\exp\left\{\frac{s}{\pi}\int_{\text{cuts}}\frac{\delta(s')ds'}{s'(s'-s)}\right\}, \qquad (1)$$

where $P_1(s)$ and $P_2(s)$ are finite polynomials and $\delta(s)$ is the (real) phase of $A(s)$ along the cuts which are assumed to occur on the real axis. Thus, $\delta(s)$ is given by

$$A(s+i\epsilon)=\pm|A(s+i\epsilon)|e^{i\delta(s)}, \qquad (2)$$

where s is real and ϵ is an infinitesimal positive number. The representation (1) is valid independently of the specific normalization of $\delta(s)$. However, it appears most convenient to require that $\delta(s)$ vanishes on the real axis where no cuts occur and the discontinuities in $\delta(s)$ are smaller than π in magnitude. With this normalization the exponential factor in (1) has no zeros or poles except at infinity. Therefore, the polynomials $P_1(s)$ and $P_2(s)$ accommodate all the zeros and poles of $A(s)$, respectively, except for the one at infinity. The conditions under which (1) is valid are that (a) $A(s)$ is analytic everywhere in s except for cuts on the real axis and a finite number of poles; (b) $A(s)$ is real in the sense that $A^*(s)=A(s^*)$; (c) $A(s)$ is bounded by a finite polynomial at infinity, and either (d) $\delta(s)$ has finite limits $\delta(\pm\infty)$ as $s\rightarrow\pm\infty$, or (d′) $A(s)$ has a finite number of zeros. The dispersion relation exists for $A(s)$ under the conditions (a), (b), and (c). Therefore, the condition (d) or (d′) is the extra condition for (1) to exist.

The purpose of the present paper is to generalize the phase representation (1) when the analytic function has two independent variables. We consider in particular the representation of the elastic scattering amplitude $A(s,t,u)$ as a function of covariant variables s, t, and u. In terms of the c.m. momentum q and the c.m. scattering angle θ, these variables are given by

$$\begin{gathered} s=(E_1+E_2)^2\,,\quad t=-2q^2(1-\cos\theta)\,,\\ u=-2q^2(1+\cos\theta)+(E_1-E_2)^2\,,\\ s+t+u=2m_1^2+2m_2^2\equiv a\,, \end{gathered} \qquad (3)$$

where E_1 and E_2 are the c.m. energies of two colliding particles with masses, m_1 and m_2, respectively.

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[1] M. Sugawara and A. Tubis, Phys. Rev. Letters, **9**, 355 (1962). For the details, see M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).

The representation to be discussed in this paper is written as

$$A(s,t,u)=[P_1(s,t,u)/P_2(s,t,u)]Q(s,t,u)\,, \qquad (4)$$

where $P_1(s,t,u)$ and $P_2(s,t,u)$ are finite polynomials in s, t, and u and $Q(s,t,u)$ has no zeros or poles except at infinity and is expressed in terms of the phase of $A(s,t,u)$ along the cuts. The explicit expressions for $Q(s,t,u)$ are given by (14), (15), and (16) of Sec. 3, all of which are equivalent to each other. We call this representation (4) the double phase representation.

The conditions for this double phase representation to be valid are as follows:

(i) $A(s,t,u)$ is analytic with respect to two independent variables everywhere except for three cuts given by $\infty > s \geq s_0$, $\infty > t \geq t_0$, and $\infty > u \geq u_0$, and a finite number of poles at $s=s_1, \cdots$, $t=t_1, \cdots$, and $u=u_1, \cdots$, where all these constants s_0, s_1, $\cdots$, etc., are real and positive.

(ii) $A(s,t,u)$ is real in the sense that $A^*(s,t,u) = A(s^*,t^*,u^*)$.

(iii) $A(s,t,u)$ is bounded by finite polynomials in s, t, and u at infinity.

(iv) The zeros of $A(s,t,u)$ occur in such a way that a finite polynomial $P_1(s,t,u)$ accommodates all of them except for the one at infinity and no others.

(v) $A(s,t,u)$ has crossing symmetry, either even or odd. For example, $A(s,t,u)=\pm A(u,t,s)$.

The conditions (i), (ii), and (iii) are what one calls the Mandelstam assumption. Condition (i) implies that there is a finite, real polynomial $P_2(s,t,u)$ which accommodates all the poles of $A(s,t,u)$ except for the one at infinity and no others. Condition (iv) prescribes similar situation regarding the zeros of $A(s,t,u)$. Condition (v) can always be satisfied by any elastic scattering amplitude. Therefore, the only extra condition for the double phase representation to exist is condition (iv). The reality of $P_1(s,t,u)$ follows from the other conditions listed above.

We assume this extra condition (iv) for the following reasons. First, without condition (iv), the double phase representation becomes much more complicated than (4) and is likely to be no longer useful. Secondly, condition (iv) may very well be satisfied because the zeros of the amplitude could have some direct physical significance just as the poles do. In fact, we see, throughout the analysis of this paper, no indication that the double phase representation (4) may be too restrictive.

Aside from formal interest, the double phase representation (4) has practical usefulness. The usefulness of the phase representation in discussing high-energy behavior of elastic scattering was already pointed out.[1] In the previous work,[1] however, one could not discuss the question of whether or not the forward peak of high-energy elastic scattering shrinks.[2] This is because only the analyticity in energy can be exploited when the (single) phase representation (1) is used. In order to discuss the question of shrinkage, one must use the double phase representation (4). In fact, we show in this paper that the double phase representation (4) provides a straightforward explanation for no-shrinkage.

We start our analysis by discussing in Sec. 2 the analyticity and symmetry of the phase of $A(s,t,u)$ when this amplitude satisfies the conditions listed above. Because of condition (iv), the phase has no extra branch points in the momentum-transfer plane other than those which belong to $A(s,t,u)$. When there is crossing symmetry, $A(s,t,u)=\pm A(u,t,s)$, the phase become the same in the s- and u-physical regions.

We then derive in Sec. 3 explicit representations (14), (15), and (16) of $Q(s,t,u)$ in (4) in terms of the phase discussed in Sec. 2. The condition (v) is shown to be necessary in order for these representations of $Q(s,t,u)$ to be bounded by finite polynomials at infinity. This boundedness of $Q(s,t,u)$ also implies that the phase remains finite in the physical regions even in the limit of infinite energy. It is shown in the Appendix that the above boundedness of the physical phase and condition (v) are, in fact, sufficient for the boundedness of $Q(s,t,u)$ by finite polynomials.

We derive in Sec. 4 the asymptotic forms of the double phase representation (4) when some of the variables become infinite. We assume here that the amplitude exhibits asymptotically a power-law behavior in energy (usually called the Regge behavior).

We then examine in Sec. 5 the case when the forward peak of high-energy elastic scattering does not shrink.[2] It is shown that this case actually materializes when the phase in the crossed channel no longer diverges logarithmically at infinity in its momentum-transfer plane.

We summarize our analyses in Sec. 6. Besides, we discuss previous theoretical work concerning the question of shrinkage. We also list some of the specific predictions of our phase representation approach to high-energy elastic scattering.

2. PHASE OF SCATTERING AMPLITUDE

We discuss in this section the analyticity and symmetry of the phase of the scattering amplitude $A(s,t,u)$ when $A(s,t,u)$ satisfies conditions (i), (ii), (iii), (iv), and (v) listed in the previous section. We observe for this purpose that conditions (i) and (iv) imply that $A(s,t,u)$ is written in the form of (4) in which $Q(s,t,u)$ has no zeros or poles except at infinity.

The s-phase $\delta(s,t)$ [or $\delta(s,u)$] of $A(s,t,u)$ is defined in the s-physical region, where s is energy and t (or u) is momentum transfer, by

$$A(s+i\epsilon,\,t)=\pm\,|A(s+i\epsilon,\,t)|^{i\delta(s,t)}. \qquad (5)$$

This definition is the same as (2). We require that

[2] What we mean by the shrinking peak is explained at the beginning of Sec. 5. The possibility of no shrinkage was emphasized by the present authors, Y. Nambu and M. Sugawara, in Phys. Rev. Letters **10**, 304 (1963). See also the new Brookhaven data of K. J. Foley *et al.*, Phys. Rev. Letters **10**, 543 (1963).

$\delta(s,t)$ vanishes at $s=s_0$ and is continuous in s.[3] This definition of $\delta(s,t)$ can be stated also as

$$\delta(s,t)=\frac{1}{2i}\ln\left[\frac{A(s+i\epsilon,\,t)}{A(s-i\epsilon,\,t)}\right]=\frac{1}{2i}\ln\left[\frac{Q(s+i\epsilon,\,t)}{Q(s-i\epsilon,\,t)}\right]$$
$$=\left(\frac{1}{2i}\right)[\ln Q(s+i\epsilon,\,t)-\ln Q(s-i\epsilon,\,t)],\quad(6)$$

because the polynomials in (4) cancel each other in the above ratio and $Q(s,t,u)$ has no zeros or poles except at infinity.

Analyticity of $\delta(s,t)$ in t is now seen directly from (6). Since $Q(s,t,u)$ has no zeros or poles except at infinity, $\delta(s,t)$ is analytic in t everywhere except for the t and the u cut which belong to $A(s,t,u)$. The assumption that $A(s,t,u)$ has no essentially singular points implies that $\delta(s,t)$ has no poles. The divergence of $\delta(s,t)$ at infinity is at most logarithmic since $A(s,t,u)$ is bounded by finite polynomials at infinity. The reality in the sense that $\delta^*(s,t)=\delta(s,t^*)$ follows from the reality of $A(s,t,u)$.

The significance of condition (iv) is to be mentioned. Without condition (iv) we hardly see how all the zeros of $A(s,t,u)$ could cancel in the ratio of $A(s,t,u)$'s in (6). If there are any zeros which do not cancel in this ratio, these zeros become the extra branch points of $\delta(s,t)$ in t.

The above analyticity of $\delta(s,t)$ implies that $\delta(s,t)$ satisfies the once-subtracted dsipersion relation

$$\delta(s,t)=\delta(s,\,t=0)+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\rho(s,t')dt'}{t'(t'-t)}+\frac{t}{\pi}\int_{u_0}^{\infty}\frac{\rho(s,u')du'}{t'(u'-u)},\quad(7)$$

where $t'=a-s-u'$ in the last integral and the imaginary parts are given by

$$\rho(s,t)=\left(\frac{1}{2i}\right)[\delta(s,\,t+i\epsilon)-\delta(s,\,t-i\epsilon)]$$
$$=\frac{1}{(2i)^2}[\ln Q(s+i\epsilon,\,t+i\epsilon)-\ln Q(s+i\epsilon,\,t-i\epsilon)$$
$$-\ln Q(s-i\epsilon,\,t+i\epsilon)+\ln Q(s-i\epsilon,\,t-i\epsilon)],\ \text{etc.}\quad(8)$$

By definition, the ρ's are real and nonzero only when $s\geq s_0$ and $t\geq t_0$, etc. In fact, these inequalities give the exact domains in which the ρ's are nonzero. We remark that the double spectral functions in the Mandelstam representation are nonzero only in regions which are smaller than those defined by the above inequalities.

There is always a finite gap between two cuts in (7). The entire physical region in t appears on this gap. Therefore, the phase $\delta(s,t)$ remains real and finite in the physical region, even if the phase may diverge logarithmically at infinity. However, as $s\to\infty$, the u cut goes away to $-\infty$ in the t plane and the physical region in t also extends to $-\infty$. If the phase $\delta(s,t)$ remains finite in the physical region even in the limit of $s\to\infty$, then $\delta(s=\infty,\,t)$ satisfies the unsubtracted dispersion relation

$$\delta(s=\infty,\,t)=\delta(s=\infty,\,t=\infty)+\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\rho(s=\infty,\,t')dt'}{t'-t},\quad(9)$$

where $\delta(s=\infty,\,t=\infty)$ is a finite, real number.[4] It is shown in Sec. 3 that the phase must, in fact, be finite in the physical region in the limit of infinite energy in order for $Q(s,t,u)$ in (4) to be bounded by finite polynomials at infinity. In other words, the boundedness of the physical phase in the limit of infinite energy is a consequence of all the conditions listed in Sec. 1.

We add a few remarks. First, we do not continue $\delta(s,t)$ with respect to s. Throughout this paper, the first variable in the phase $\delta(s,t)$ is real and $s\geq s_0$, though the second variable is allowed to be complex. Secondly, $\delta(s,\,t=0)$ in (7) is the phase of the forward amplitude. If there is an optical theorem, the s dependence of $\delta(s,\,t=0)$ is fairly simple.[5] Thirdly, if one uses u, instead of t, as the momentum-transfer variable, the s phase is written as $\delta(s,u)$. Since $\delta(s,t)$ and $\delta(s,u)$ are the same phase expressed in terms of different variables, $\delta(s,u)$ satisfies the dispersion relation of the form of (7) with the same imaginary parts as those in (7). However, $\delta(s=\infty,\,t)$ and $\delta(s=\infty,\,u)$ are different functions. For example, u is $-\infty$ in $\delta(s=\infty,\,t)$ while u is finite in $\delta(s=\infty,\,u)$. Also, $\delta(s=\infty,\,u)$ has only the u cut, while $\delta(s=\infty,\,t)$ has only the t cut.

The t phase, $\delta(t,s)$ or $\delta(t,u)$ and the u phase, $\delta(u,s)$ or $\delta(u,t)$, are defined exactly the same way in the t- and u-physical regions, respectively. The corresponding imaginary parts are $\rho(t,s)$, $\rho(t,u)$ and $\rho(u,s)$, $\rho(u,t)$, respectively. All the previous analyses and remarks apply to these δ's and ρ's. Among six ρ's thus defined, only three are independent. This is because the definition (8) implies that

$$\rho(s,t)=\rho(t,s)\,,\quad \rho(t,u)=\rho(u,t)\,,\quad(10)$$
$$\rho(u,s)=\rho(s,u)\,,$$

where we mean that these pairs of functions are the same functions of respective variables, but do not mean that they are symmetric under the interchanges of respective variables. The three δ's are, however, independent of each other. For example, $\delta(s,t)$ and $\delta(t,s)$ are entirely different functions.[6]

[3] In this paper we assume no discontinuities in $\delta(s,t)$ since this is the phase of the scattering amplitude. The discontinuities cause simply technical complications, which are discussed in the second paper quoted in Ref. 1.

[4] These arguments are based upon the theorem concerning the limit of the analytic function at infinity proved by M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961), and also in the second paper quoted in Ref. 1.

[5] See, for example, the first paper quoted in Ref. 1.

[6] To avoid possible confusion, one may introduce subscripts I, II, and III to δ's and ρ's in order to indicate in which physical regions these δ's and ρ's are originally defined. If I, II, and III refer to the s-, t-, and u-physical regions, respectively, Eq. (10) now reads as $\rho_{\rm I}(s,t)=\rho_{\rm II}(t,s)$, etc. The remarks after Eq. (10) imply that $\rho_{\rm I}(s,t)\neq\rho_{\rm I}(t,s)$, etc. and $\delta_{\rm I}(s,t)\neq\delta_{\rm II}(t,s)$, etc. In terms of the above notation, Eq. (12) reads as $\delta_{\rm I}(s,t)=\delta_{\rm III}(s,t)$, and $\delta_{\rm II}(t,s)=\delta_{\rm II}(t,u)$ when $s=u$. Equation (13) then becomes $\rho_{\rm I}(s,t)=\rho_{\rm III}(s,t)$ and $\rho_{\rm I}(s,u)=\rho_{\rm I}(u,s)$.

We now discuss symmetry which δ's and ρ's may have when $A(s,t,u)$ has crossing symmetry. For the sake of definiteness, we assume that

$$A(s,t,u)=\pm A(u,t,s)\,, \tag{11}$$

where we mean that $A(s,t,u)$ at most changes sign under the interchange of s and u. One then derives directly from the definition (6) that

$$\delta(s,t)=\delta(u,t) \text{ for all } t\,, \quad \text{when} \quad s=u\geq s_0=u_0\,, \tag{12}$$
$$\delta(t,s)=\delta(t,u) \text{ for all } s=u\,, \quad \text{when} \quad t\geq t_0\,.$$

These relations imply that, when there is crossing symmetry between the s and u channels, the s and u phases are the same and the t phase is symmetric in the momentum-transfer plane. There is, however, no crossing symmetry in $\delta(s,u)$ with respect to the interchange of s and u even when $s\geq s_0$ and $u\geq u_0$. One can derive from (6) a relation $\delta(s,u)=\delta(u=s,\,s=u)$ when $s\geq s_0$ and $u\geq u_0$, which is $\delta_{\rm I}(s,u)=\delta_{\rm III}(s,u)$ according to the notation given in Ref. 6. This means simply that the s and u phases are the same and must not be confused with crossing symmetry in $\delta(s,u)$ under the interchange of s and u.

Concerning the ρ's, one obtains the following relations directly from the definition (8),

$$\rho(s,t)=\rho(u,t) \text{ for all } t\geq t_0\,, \quad \text{when} \quad s=u\geq s_0=u_0\,, \tag{13}$$

$\rho(s,u)$ is symmetric under the s, u interchange.

3. DOUBLE PHASE REPRESENTATION

We derive in this section the explicit expression for $Q(s,t,u)$ in terms of the phase defined in the previous section. The definition (4) implies that $Q(s,t,u)$ is analytic everywhere except for the three cuts of $A(s,t,u)$, has no zeros or poles except at infinity, and is bounded by finite polynomials at infinity. Thus, $\ln Q(s,t,u)$ is also analytic everywhere except for these three cuts and is bounded by logarithmic functions at infinity. Therefore, one can write down a double dispersion relation for $\ln Q(s,t,u)$, in which subtraction is known and spectral functions are either δ's in the single integrals because of (6) or ρ's in the double integrals because of (8). This double dispersion relation represents the explicit expression for $Q(s,t,u)$ in terms of the phase of $A(s,t,u)$. If one writes down[7] the double dispersion relation for $\ln Q(s,t,u)/su$, one obtains

$$Q(s,t,u)=\exp\left\{\frac{s}{\pi}\left[\int_{s_0}^{\infty}\frac{\delta(s',\,u=0)ds'}{s'(s'-s)}+\int_{t_0}^{\infty}\frac{\delta(t',\,u=0)dt'}{(a-t')(t'-t)}\right]+\frac{u}{\pi}\left[\int_{u_0}^{\infty}\frac{\delta(u',\,s=0)du'}{u'(u'-u)}+\int_{t_0}^{\infty}\frac{\delta(t',\,s=0)dt'}{(a-t')(t'-t)}\right]\right.$$
$$\left.+\frac{su}{\pi^2}\left[\iint_{s_0t_0}^{\infty}\frac{\rho(s',t')ds'dt'}{s'u'(s'-s)(t'-t)}+\iint_{u_0t_0}^{\infty}\frac{\rho(u',t')du'dt'}{s'u'(u'-u)(t'-t)}+\iint_{s_0u_0}^{\infty}\frac{\rho(s',u')ds'du'}{s'u'(s'-s)(u'-u)}\right]\right\}\,, \tag{14}$$

which is normalized as $Q(s=u=0)=1$. The factor su is necessary to make the contribution from infinity vanish. If one uses st or tu instead of su, one obtains the same as (14) except for the interchange of corresponding variables. All these expressions are equivalent.

The expression (14) can be expressed in terms of $\delta(s,t)$ and $\delta(u,t)$ as

$$Q(s,t,u)=\exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)ds'}{s'(s'-s)}+\frac{u}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',t)du'}{u'(u'-u)}\right\}$$
$$\times\exp\left\{\frac{(t-a)}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',t)du'}{u'(u'+t-a)}+\frac{(t-a)}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',\,s=0)dt'}{(t'-a)(t'-t)}-\frac{(t-a)}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',\,s=0)du'}{u'(u'+t-a)}\right\}\,. \tag{15}$$

This expression corresponds to the single phase representation (1) in which t is regarded as a parameter. In order to derive (15) from (14), one writes down the dispersion relations for $\delta(s,t)/u$ and $\delta(u,t)/s$ of the type of (7) and then expresses the double integrals in (14) in terms of the single integrals involving $\delta(s,t)$ and $\delta(u,t)$. One also needs the crossing relations (12) to obtain (15).

Expression (14) can also be expressed in terms of $\delta(s,u)$ and $\delta(t,u)$ as

$$Q(s,t,u)=\exp\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',u)ds'}{s'(s'-s)}+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',u)dt'}{t'(t'-t)}\right\}$$
$$\times\exp\left\{\frac{(u-a)}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',u)dt'}{t'(t'+u-a)}+\frac{u}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',\,s=0)du'}{u'(u'-u)}-\frac{u}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',\,s=0)dt'}{(t'-a)(t'-a+u)}\right\}\,. \tag{16}$$

[7] The derivation is similar to that given in the first paper cited in Ref. 4.

This expression is the single phase representation in which u is regarded as a parameter. One derives (16) from (14) using only the dispersion relations for $\delta(s,u)/u$ and $\delta(t,u)/su$. No crossing relations are necessary.

The single phase representation in which s is regarded as a parameter is the same as (16) with s and u interchanged. The expression (16) is different from (15) with t and u interchanged, only by a constant factor. This difference is merely due to the fact that the expression (15) assumes a normalization $Q(s=u=0)=1$ which is not invariant under the interchange of t and u.

The expressions (14), (15), and (16) all satisfy the requirement that $Q(s,t,u)$ is analytic except for the three cuts of $A(s,t,u)$ and has no zeros or poles except at infinity. Besides, these expressions are real in the sense that $Q^*(s,t,u)=Q(s^*,t^*,u^*)$ and also symmetric under the interchange of s and u because of (12) and (13) when there is crossing symmetry (11). Therefore, the polynomial $P_1(s,t,u)$ in (4) must be real in the same sense and crossing symmetry (11) of $A(s,t,u)$ must be taken over by $P_1(s,t,u)/P_2(s,t,u)$ in (4).

However, it is not immediately clear if these expressions (14), (15), and (16) are bounded by finite polynomials at infinity. One can show that this is actually the case when there is crossing symmetry (11) and the phases $\delta(s,t)$, etc., remain finite in the physical regions even in the limit of infinite energy. The last condition implies essentially that

$$\begin{gathered}\delta(s=\infty,t)\,,\;\delta(s=\infty,u)\,,\;\delta(t=\infty,s)\,,\;\delta(t=\infty,u)\,,\\ \delta(u=\infty,s)\,,\quad\text{and}\quad\delta(u=\infty,t)\text{ are all finite.}\end{gathered}\tag{17}$$

In order to see that the condition (17) is necessary, one examines (15) in the limit when $s\to\infty$ and t remains finite. The s-dependent integrals in (15) are split as

$$\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)ds'}{s'(s'-s)}=\frac{\delta(s=\infty,t)}{\pi}\ln\frac{s_0}{s_0-s}+\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)-\delta(s=\infty,t)}{s'(s'-s)}ds',\tag{18}$$

etc., where the second terms diverge as $s\to\infty$ only less strongly than logarithmically.[8] Therefore, it is necessary for $\delta(s=\infty,t)$ and $\delta(u=\infty,t)$ to be finite in order for (15) to be bounded in the above sense.[9] One obtains other conditions in (17) similarly from (16) and its s, u interchanged.

To see that crossing symmetry (11) is necessary, one recalls that (15) does not follow from (14) without the crossing relations (12). Without (12), one must replace the exponent of the second exponential factor in (15) by

$$\left\{\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)ds'}{s'(a-s'-t)}+\frac{s}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',u=0)dt'}{(a-t')(t'-t)}-\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',u=0)ds'}{s'(a-s'-t)}\right\}+\{s,\,u\text{ interchanged}\}.\tag{19}$$

Evidently these terms diverge linearly in s when $s\to\infty$ and t remains finite. This means that, without crossing symmetry (11), the expression (14) is no longer bounded by finite polynomials when $s\to\infty$ and t remains finite.

It is shown in Appendix that crossing symmetry (11) and the condition (17) are, in fact, sufficient for the expression (14) to be bounded by finite polynomials at infinity.

When crossing symmetry is, for example, $A(s,t,u)=\pm A(t,s,u)$ instead of (11), one writes down the double dispersion relation for $\ln Q(s,t,u)/st$ instead of $\ln Q(s,t,u)/su$, to obtain the double phase representation. Then all the preceding arguments hold without any change. In particular, the expressions (15) and (16) are correct regardless of which channels crossing symmetry of $A(s,t,u)$ applies to.

As a summary, conditions (i), (ii), (iii), and (iv) listed in Sec. 1 become consistent with each other and one expects the double phase representations (14), (15), (16), etc., as long as there is crossing symmetry of the type of (11) with respect to the interchange of some pair of variables and also the phase remains finite in the physical regions even in the limit of infinite energy. The last condition can also be stated as (41) in the Appendix.

The polynomials $P_1(s,t,u)$ and $P_2(s,t,u)$ in (4) can be made explicit when information is available concerning the zeros and poles of $A(s,t,u)$. As the simplest, yet very important example, we discuss here the polynomials of the $\pi^0+\pi^0\to\pi^0+\pi^0$ amplitude. According to the Mandelstam assumption, this amplitude has no poles and is symmetric with respect to the interchange of all pairs of variables. Thus, $P_2(s,t,u)$ can be chosen as unity and $P_1(s,t,u)$ becomes fully symmetric and real in the sense of (ii) in Sec. 1. It was found[1] that this amplitude has, in the s plane with $t=0$, either two zeros if $a_0+2a_2\geq 0$ or no zero if $a_0+2a_2<0$, where a_0 and a_2 are the S-wave scattering lengths in the channels with the total isospin 0 and 2, respectively. First, the above full symmetry requires that $P_1(s,t,u)$ is a linear combination of $s^n+t^n+u^n$ with $n=0,2,3,4,\cdots$. Then the above numbers of zeros in the s plane with $t=0$ imply that n can at most be 0, 2, and 3. One thus finds that $P_1(s,t,u)=c_0$ if $a_0+2a_2<0$ and $P_1(s,t,u)=c_0+c_2(s^2+t^2+u^3)+c_3(s^3+t^3+u^3)$ if $a_0+2a_2\geq 0$, where the constants c_0, c_2, and c_3 are all real because of the reality of $P_1(s,t,u)$.

4. ASYMPTOTIC BEHAVIOR

It was shown[1] that the phase representation (1) has a simple asymptotic behavior when $s\to\infty$. In particu-

[8] The argument is given in the first paper quoted in Ref. 1.

[9] It may appear offhand that only $\delta(s=\infty,t)+\delta(u=\infty,t)$ need be finite. However, if $\delta(s=\infty,t)$ is infinite, the separation in (18) is no longer correct and we do not know how to prove the boundedness of (15). This is why we assume that both $\delta(s=\infty,t)$ and $\delta(u=\infty,t)$ are finite.

lar, if the phase $\delta(s,t)$ satisfies the condition that

$$\int^{\infty} \frac{\delta(s,t)-\delta(s=\infty,t)}{s} ds \text{ converges}, \quad (20)$$

the second term in (18) no longer diverges as $s\to\infty$. Therefore, if the u phase $\delta(u,t)$ also satisfies the condition (20), the expression (15) of $Q(s,t,u)$ has the asymptotic form

$$Q(s,t,u) \underset{s\to\infty}{\longrightarrow} \propto \beta(t)\left(\frac{s_0}{s_0-s}\right)^{\delta(s=\infty,t)/\pi}\left(\frac{u_0}{u_0-u}\right)^{\delta(u=\infty,t)/\pi}. \quad (21)$$

One can show that $\beta(t)$ in (21) is real and analytic in t everywhere except for the t cut. For this, one simply recalls that $Q(s,t,u)$ is analytic in t everywhere except for the t cut in the limit of $s\to\infty$ and $\delta(s=\infty,t)$ and $\delta(u=\infty,t)$ are also analytic in t everywhere except for the t cut. In fact, one can derive from (15) the following expression

$$\beta(t)=\exp\left\{\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\delta(t')dt'}{t'(t'-t)}\right\}, \quad (22)$$

where

$$\delta(t)=\delta(t,s=0)-\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\rho(s',t)-\rho(s=\infty,t)}{s'}ds' -\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\rho(u',t)-\rho(u=\infty,t)}{u'}du' +\frac{1}{\pi}\int_{u_0}^{\infty}\left[\frac{1}{u'}+\frac{1}{a-t-u'}\right]\rho(u',t)du'. \quad (23)$$

To derive (22) and (23), one splits the integral in (18) further as

$$\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)ds'}{s'(s'-s)} =\frac{\delta(s=\infty,t)}{\pi}\ln\frac{s_0}{s_0-s}+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)-\delta(s=\infty,t)}{s'-s}ds' -\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)-\delta(s=\infty,t)}{s'}ds', \quad (24)$$

where the second term approaches zero as $s\to\infty$ when the condition (20) is satisfied.[1] One thus replaces the integrals in the first exponential factor in (15) by the corresponding third term in (24). One then rewrites all the integrals in the exponent of the resulting expression of (15), using the dispersion relations for $\delta(s,t)$ and $\delta(u,t)$ of the type of (7). One needs no crossing relations.

For the sake of completeness, we give a complete expression of $Q(s,t,u)$,

$$Q(s,t,u)=Q\beta(t)\gamma(s,t) \times\left(\frac{s_0}{s_0-s}\right)^{\delta(s=\infty,t)/\pi}\left(\frac{u_0}{u_0-u}\right)^{\delta(u=\infty,t)/\pi}, \quad (25)$$

which is exact as long as the s and u phases satisfy the condition (20). In (25), $\beta(t)$ is given by (22) and (23), and $\gamma(s,t)$ by

$$\gamma(s,t)=\exp\left\{\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',t)-\delta(s=\infty,t)}{s'-s}ds' +\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',t)-\delta(u=\infty,t)}{u'-u}du'\right\}. \quad (26)$$

This $\gamma(s,t)$ is due to the second term in (24) and, therefore, approaches unity as $s\to\infty$ when the s and u phases satisfy the condition (20). A real, positive constant Q in (25) is due to the normalization $\beta(0)=1$ and given by

$$Q\equiv\exp\left\{-\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\delta(s',t=0)-\delta(s=\infty,t=0)}{s'}ds' -\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',t=0)-\delta(u=\infty,t=0)}{u'}du' -\frac{a}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',s=0)dt'}{t'(t'-a)}+\frac{a}{\pi^2}\int_{u_0}^{\infty}\int_{t_0}\frac{\rho(u',t')du'dt'}{s't'u'} -\frac{a}{\pi^2}\int_{s_0}^{\infty}\int_{u_0}\frac{\rho(s',u')ds'du'}{s't'u'}\right\}. \quad (27)$$

One obtains (27) when one derives (22) and (23).

One can identify $\delta(t)$ of (23) by rewriting the dispersion relation for $\delta(t,s)/s$ as

$$\delta(t,s)=\delta(t,s=0) +\frac{\rho(s=\infty,t)}{\pi}\ln\frac{s_0}{s_0-s}+\frac{\rho(u=\infty,t)}{\pi}\ln\frac{u_0}{u_0-u} +\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\rho(s',t)-\rho(s=\infty,t)}{s'(s'-s)}ds' +\frac{u}{\pi}\int_{u_0}^{\infty}\frac{\rho(u',t)-\rho(u=\infty,t)}{u'(u'-u)}du' +\frac{1}{\pi}\int_{u_0}^{\infty}\left[\frac{1}{u'}+\frac{1}{a-t-u'}\right]\rho(u',t)\,du', \quad (28)$$

where the dispersion integrals are split into two terms exactly the same way as in (18). By comparing (23) with (28), one sees that $\delta(t)$ is the finite part of the t phase $\delta(t,s)$ at infinity in its momentum-transfer plane. In fact, if one assumes that the individual integrals in (23) are convergent, $\delta(t)$ can be identified as

$$\delta(t)=\lim_{s\to\infty}\left[\delta(t,s)-\frac{\rho(s=\infty,t)}{\pi}\ln\frac{s_0}{s_0-s}-\frac{\rho(u=\infty,t)}{\pi}\ln\frac{u_0}{u_0-u}\right]. \quad (29)$$

So far, one has rewritten only the expression (15). However, since no crossing relations are needed in all the preceding derivations, the expression (16) can also be written as

$$Q(s,t,u)=Q'\beta(u)\gamma(s,u)\times\left[\frac{s_0}{s_0-s}\right]^{\delta(s=\infty,u)/\pi}\left[\frac{t_0}{t_0-t}\right]^{\delta(t=\infty,u)/\pi}, \quad (30)$$

where $\beta(u)$ and $\gamma(s,u)$ are the same as (22), (23), and (26), respectively, except for the interchange of t and u. A real, positive constant Q' in (30) is not quite the t, u interchanged of (27), simply because (16) is not quite the t, u interchanged of (15) due to the normalization $Q(s=u=0)=1$ assumed in (15). Similarly one obtains the s, u interchanged of (30).

We summarize the asymptotic forms of the amplitude $A(s,t,u)$. In the forward direction where t is finite, one obtains from (21) that

$$A(s,t,u)\underset{s\to\infty}{\longrightarrow}\propto\beta(t)\left(\frac{s}{s_0}\right)^{\alpha(t)}e^{i\delta(s=\infty,t)},$$
$$\alpha(t)=n-[\delta(s=\infty,t)+\delta(u=\infty,t)]/\pi, \quad (31)$$

where an integer n is the difference between the total numbers of zeros and poles of $A(s,t,u)$ in the s plane when t is fixed. In the backward direction where u is finite, one obtains from (30) that

$$A(s,t,u)\underset{s\to\infty}{\longrightarrow}\propto\beta(u)\left(\frac{s}{s_0}\right)^{\alpha(u)}e^{i\delta(s=\infty,u)},$$
$$\alpha(u)=m-[\delta(s=\infty,u)+\delta(t=\infty,u)]/\pi, \quad (32)$$

where an integer m is the difference between the total numbers of zeros and poles of $A(s,t,u)$ in the s plane when u is fixed. The results (31) and (32) are correct as long as the phase satisfy the condition (20). The α's in (31) and (32) are both real and analytic except for a single cut which corresponds to the respective variable. The β's in (31) and (32) are the same as the β's in (25) and (30), respectively, except for real, finite polynomials in t and u, respectively. These polynomials are the polynomials which remain in the asymptotic forms of $P_1(s,t,u)/P_2(s,t,u)$ in (4) when $s\to\infty$ with t and u fixed, respectively. Thus, the β's in (31) and (32) are both real and analytic except for a finite number of poles and a single cut which corresponds to the respective variable.

In the case of the $\pi^0+\pi^0\to\pi^0+\pi^0$ amplitude discussed at the end of the previous section, the β's in (31) and (32) are exactly the same as those in (25) and (30), respectively. The asymptotic forms (31) and (32) are usually called the Regge behavior.

5. CASE OF NO SHRINKAGE

If $\alpha(t)$ in (31) does not vary with t in some region near $t=0$, the shape of the forward peak of high-energy elastic scattering depends only on t, and vice versa. One usually describes this situation by stating that the forward peak does not shrink.[2] Because of the analyticity of $\alpha(t)$, no shrinkage in this sense means that $\alpha(t)$ is constant not only in the above region of t but everywhere in the t plane. We discuss in this section some of the consequences of the requirement that $\alpha(t)$ is constant.

According to (31), $\alpha(t)$ consists of two phases. However, it is extremely unlikely for these phases to cancel exactly for all $t\geq t_0$, because these phases become the same if there is crossing symmetry (11) and are otherwise independent of each other. Thus, no shrinkage means that the two phases in (31) are individually constant in t. The dispersion relation (9) then implies that

$$\rho(s=\infty,t)=\rho(u=\infty,t)=0 \quad (33)$$

for all $t\geq t_0$. Conversely, there is no shrinkage if (33) is the case.

One can show that the condition (33) implies that the t phase does not diverge logarithmically at infinity in its momentum-transfer plane, and vice versa. If (33) is the case, the expression (28) of the dispersion relation for $\delta(t,s)$ indicates that the t phase no longer diverges logarithmically at infinity. Conversely, if the t phase is required to have no logarithmic divergence at infinity, the expression (28) implies either (33) or $\rho(s=\infty,t)=-\rho(u=\infty,t)$ for all $t\geq t_0$. The latter possibility is, however, extremely unlikely because these ρ's are the same if there is crossing symmetry (11) and are otherwise independent of each other. One thus sees that the case of no shrinkage corresponds to the case when the phase of the crossed channel becomes the least divergent at infinity in its momentum-transfer plane.

The remaining divergence in the t phase at infinity is due to the fourth and fifth terms in (28). These do not diverge if the integrals in (23) converge individually. This last condition is not only sufficiently weak in itself, but is very similar to the condition (20) which is assumed already in the power behavior (31). Therefore, it is likely that the t phase is bounded everywhere in its momentum-transfer plane in the case of no shrinkage. For the sake of simplicity, we assume for the rest of this section that the integrals in (23) converge individually and therefore the t phase is bounded everywhere in its momentum-transfer plane.

It then follows from (29) that

$$\delta(t)=\delta(t,\, s=\infty)\,, \tag{34}$$

meaning that $\delta(t)$ is the real, finite limit of the t phase at infinity in its momentum-transfer plane. The asymptotic form (31) can be written in this special case as

$$A(s,t,u) \underset{s\to\infty}{\longrightarrow} \propto is\beta(t)\,, \tag{35}$$

where $\beta(t)$ is given aside from real, finite polynomials by

$$\beta(t)=\exp\left\{\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',\, s=\infty)dt'}{t'(t'-t)}\right\}. \tag{36}$$

In (35), one has required that the total cross section approaches as $s\to\infty$ a finite, nonzero limit and that the forward amplitude becomes pure imaginary in the limit of $s\to\infty$.

According to (35), the shape of the forward peak of high-energy elastic scattering is determined by $\beta(t)$. The expression (36) implies that this shape is of a pure exponential form of t in a region near $t=0$, but approaches a power form of t as momentum transfer increases. One sees this most clearly if one applies the separation of the type of (18) to the integral in (36), to rewrite (36) as

$$\beta(t)=\left[\frac{t_0}{t_0-t}\right]^{\delta(t=\infty,\, s=\infty)/\pi}$$
$$\times\exp\left\{\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',\, s=\infty)-\delta(t=\infty,\, s=\infty)}{t'(t'-t)}dt'\right\}. \tag{37}$$

The exponential factor in (37) approaches a finite limit as $t\to\infty$ if the t phase also satisfies the condition (20). Moreover, $\delta(t=\infty,\, s=\infty)$ in (37) is equal to the forward phase $\delta(t=\infty,\, s=0)$ if the t phase in the limit of infinite energy is also independent of the momentum-transfer variable.

If one considers the $\pi^0+\pi^0\to\pi^0+\pi^0$ amplitude, one can obtain more consequences of no shrinkage because of high symmetry available in this amplitude. The optical theorem also applies to this amplitude, which implies[1] that $\delta(s=\infty,\, t=0)=\pi/2$ if $a_0+2a_2\geq 0$, and $\delta(s=\infty,\, t=0)=-\pi/2$ if $a_0+2a_2<0$.

One can then argue that the case of $a_0+2a_2<0$ is excluded. For this, one notices that $\delta(t=\infty,\, s=\infty)$ in (37) is equal to $\delta(s=\infty,\, t=0)=\pm\pi/2$ because of symmetry. One also recalls that the asymptotic form (35) with $\beta(t)$ given by (36) is exact for this amplitude. Then $\delta(s=\infty,\, t=0)=-\pi/2$ implies that the asymptotic form (35) behaves as s for small $|t|$ and as $|t|^{1/2}s$ for large $|t|$. This behavior is, however, not permissible physically. Quite similarly, one can argue also that the cubic term in $P_1(s,t,u)$ in the case of $a_0+2a_2\geq 0$ must be excluded.

Thus, the $\pi^0+\pi^0\to\pi^0+\pi^0$ amplitude must be written, in the case of no shrinkage, as

$$A(s,t,u)=[c_0+c_2(s^2+t^2+u^2)]Q(s,t,u)\,, \tag{38}$$

where c_0 and c_2 are real constants and $Q(s,t,u)$ is given by (14), (15) and (16). In this case, one must have $a_0+2a_2\geq 0$ and $\delta(s=\infty,\, t=0)=\pi/2$. This sign of a_0+2a_2 is consistent with the prevailing evidences. We remark that $\delta(s=\infty,\, t=0)=\pi/2$ implies that the forward peak of high-energy elastic scattering [proportional to $\beta^2(t)$] approaches a simple inverse power behavior of t for large momentum transfer. This could easily be checked experimentally.

6. SUMMARY AND CONCLUSION

We have shown in the previous sections how one finds and uses the double phase representation (4) for the elastic amplitude $A(s,t,u)$. This representation (4) is a generalization of the (single) phase representation (1). Similarity is obvious not only between the expressions (4) and (1) but also the assumptions underlying these representations which are listed in Sec. 1. In both representations, the phase defined by (2) and (5) must be finite in the physical regions even in the limit of infinite energy. The double phase representation (4) requires, in addition, crossing symmetry and an extra assumption concerning the zeros of $A(s,t,u)$. Therefore, the double phase representation (4) is considerably more restrictive than the single representation (1).

This extra assumption concerning the zeros makes the phase of $A(s,t,u)$ analytic everywhere in the momentum-transfer plane except for the branch cuts which belong to $A(s,t,u)$. Without this assumption, not only does the double phase representation become much more complicated than (4), but most of the analysis done in this paper becomes impossible to carry out. This is because the analysis consists of using the dispersion relation for the phase which otherwise involves unknown integrals corresponding to the extra branch points. If there is no crossing symmetry in $A(s,t,u)$, the double phase representation (4) with $Q(s,t,u)$ given by (14) diverges exponentially at infinity in the s or u plane with t fixed. Therefore, crossing symmetry is assumed throughout the analysis of this paper. The analysis is valid as long as $A(s,t,u)$ has either even or odd crossing symmetry with respect to the interchange of some pair of s, t, and u.

We add an additional remark in connection with the last statement. The double phase representation (4) is more restrictive than the Mandelstam representation in the sense that the former requires the extra assumption concerning the zeros. However, as was just stated, the double phase representation (4) does not necessarily require the boundedness of $A(s,t,u)$ by finite polynomials at infinity. On the other hand, the Mandelstam respresentation breaks down as soon as $A(s,t,u)$ is no longer bounded by finite polynomials at infinity. Therefore, the double phase representation (4) is more general in this sense than the latter.

It was shown previously[1] that the phase representation is very useful in discussing high-energy behavior of elastic scattering. In fact, we have derived in this paper the asymptotic forms of the amplitude which are simple power forms of energy, assuming the double phase representation (4) and also that the phase approaches the limit at infinite energy not too slowly. This last condition is expressed by (20). In the forward direction where t is finite, the amplitude approaches the expression (31). The expression (32) is the asymptotic form in the backward direction where u is finite. In these asymptotic forms, the α's and β's are both real and analytic everywhere except for a single cut (either the t or the u cut) and a finite number of poles which the β's may have. These α's and β's are all written in terms of the phase and the zeros and poles of the amplitude. The significance of this derivation of high-energy behavior is that the power behavior in energy is merely a consequence of the usual analyticity assumption.

We have discussed in particular the case when $\alpha(t)$ in (31) is constant. This is the case when the forward peak of high-energy elastic scattering does not shrink.[2] We found that the case of no shrinkage is the case when the crossed channel no longer diverges logarithmically at infinity in the momentum-transfer plane. The very simple analyticity of the phase in the momentum-transfer plane is already implied by assuming the double phase representation (4) (see the second paragraph of this section). Therefore, no shrinkage is actually the simplest situation one can expect from the point of view of the behavior of the phase of the amplitude in the momentum-transfer plane.

According to some of the previous works, however, no shrinkage is not consistent with analyticity and unitarity. Gribov[10] pointed out that the asymptotic form (35) cannot be consistent with the analyticity of $A(s,t,u)$ and the unitarity condition valid in the purely elastic region. He assumes that there is a purely elastic region in the t-physical region. This is correct in the case of pion-pion scattering. He then continues analytically this elastic unitarity condition with respect to the angular variables involved in the unitarity condition. He obtains this way the continued unitarity condition which depends essentially on the amplitude at infinite energy. He then shows that this continued unitarity condition contradicts the asymptotic form (35). The major difficulty in this proof lies in justifying the above continued unitarity condition. This continuation consists necessarily of using the Cauchy contour theorem with respect to the variable to be continued analytically. This means that the validity of this continued unitarity condition depends upon the divergence of the amplitude at infinity. Assuming that the amplitude is sufficiently well-behaved at infinity, one obtains the above continued unitarity condition. However, in the case when the amplitude behaves like (35), i.e., has a linear divergence in s at infinity, one cannot justify the above continued unitarity condition. Therefore, his proof breaks down in the case of actual interest, though it is valid, for example, in the case of usual potential scattering.

The conventional approach to high-energy scattering is to make use of analyticity in the angular-momentum plane of the partial-wave amplitude defined in the t-physical region. According to this approach, the asymptotic form (35) can most easily be realized by assuming a fixed pole in the angular-momentum plane, assuming also that the Sommerfeld-Watson transformation is valid. Recently, Oehme[11] has shown that the unitarity condition in the purely elastic region is incompatible with the existence of such fixed poles in the analytically continuable partial-wave amplitude. Because of the fact that Oehme works directly with the partial-wave amplitude, the continuation of the elastic unitarity condition is exact in his case. Therefore, one may regard the above Oehme's proof as a revision of the proof by Gribov.[10] However, it is assumed in this Oehme's proof that the asymptotic form (35) of the full amplitude $A(s,t,u)$ actually implies a fixed pole of the partial-wave amplitude in the angular-momentum plane. In fact, one cannot find any complete argument which justifies the above assumption. Therefore, Oehme's proof does not exclude the asymptotic form (35) either.

It is interesting to consider in this connection what kind of analyticity in the angular-momentum plane could possibly be the simplest consistent with the asymptotic form (35) and the elastic unitarity condition in the t-physical region. According to our preliminary work, essential singularity in the angular-momentum plane, for example, may be the case, though this question needs further study.

We have not discussed in this paper the possible limitations due to the unitarity condition valid in the purely elastic region which may exist in some of the physical regions involved. This is primarily because we do not know how to use rigorously the unitarity condition for the purpose of discussing high-energy behavior of elastic scattering. Even if we do not foresee any serious limitation, we may be overlooking some interesting consequences of the unitarity condition. The use of the unitarity condition in general is likely to be more complicated in the case of the phase representation than in the case of the usual dispersion relation. This may be one of the main disadvantages of the phase representations (1) and (4).

In view of the fact that the case of no shrinkage is of great current interest,[2] we finally list below a few of the predictions of our phase representation approach to high-energy elastic scattering. For the sake of simplicity, we assume in the following predictions that all the phases become constant with respect to momentum

[10] V. N. Gribov, Nucl. Phys. **22**, 249 (1961).

[11] R. Oehme, Phys. Rev. Letters **9**, 358 (1962).

transfer in the limit of infinite energy. This means that all the peaks in high-energy elastic scattering do not shrink.

(a) We expect both forward and backward peaks in any elastic scattering, either purely elastic or some sort of exchange. This is because any amplitude which has analyticity and crossing symmetry of some sort approaches the asymptotic form (31) in the forward direction and the asymptotic form (32) in the backward direction. The heights of these peaks depend upon the individual types of scattering. These peaks are the peaks in the differential cross sections plotted against momentum-transfer with energy fixed. The total scattering cross sections, either forward or backward, may very well approach zero in the limit of infinite energy. In fact, we expect only the forward cross section of purely elastic scattering to remain nonzero in the limit of infinite energy.

(b) The shape of these peaks is described by pure exponential functions of t, in the forward direction, and of u, in the backward direction, for small $|t|$ and $|u|$, respectively. As $|t|$ and $|u|$ become large, the functions describing these peaks approach simple power forms of t and u, respectively.

(c) If the spins and isospins of colliding particles become irrelevant in the high-energy region, purely elastic scattering among strongly interacting particles become similar to $\pi^0+\pi^0\rightarrow\pi^0+\pi^0$ scattering in the limit of infinite energy. Then the forward peak of purely elastic scattering is expected to behave as an inverse power of t for large momentum-transfer.

We conclude our discussion by comparing our phase representation approach with the conventional approach to high-energy elastic scattering. The conventional approach consists of assuming that all these peaks in high-energy elastic scattering are dominated by moving poles in the angular-momentum plane, which are, in turn, associated with the known particles and resonances observed in the lower energy region. From the point of view of our approach, these peaks in high-energy elastic scattering are merely direct manifestations of analyticity. Especially no shrinkage in some of the peaks means simply that some of the phases assume the gentlest behavior in the momentum-transfer plane. Therefore, no shrinkage is no surprise but what one expects naturally from our phase representation approach.

APPENDIX

We prove in this Appendix that $Q(s,t,u)$ given by (14) is bounded at infinity by finite polynomials if the condition (17) and the crossing relations (12) and (13) are satisfied. For this, we prove that the integrals in the exponent of (14) diverge at most logarithmically at infinity.

Because of (12), the single integrals in (14) can be combined as

$$\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\delta(s',u=0)ds'}{s'(s'-s)}+\frac{u}{\pi}\int_{u_0}^{\infty}\frac{\delta(u',s=0)du'}{u'(u'-u)}+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',s=0)dt'}{t'(t'-t)}-\frac{a}{\pi}\int_{t_0}^{\infty}\frac{\delta(t',s=0)dt'}{t'(t'-a)}. \quad (39)$$

The integrals in (39) are of the type of the integral in (18). Therefore, according to the argument below (18), all the integrals in (39) diverge at most logarithmically when the condition (17) is satisfied.

The double integrals in (14) can be written with the help of (13) in a symmetrical form

$$\frac{st}{\pi^2}\iint_{s_0t_0}^{\infty}\frac{\rho(s',t')ds'dt'}{s't'(s'-s)(t'-t)}+\cdots+\frac{s}{\pi^2}\iint_{s_0t_0}^{\infty}\left(\frac{1}{t'}+\frac{1}{u'}\right)\frac{\rho(s',t')ds'dt'}{s'(s'-s)}+\cdots+\frac{a}{\pi^2}\iint_{s_0t_0}^{\infty}\frac{\rho(s',t')ds'dt'}{s't'u'}, \quad (40)$$

where the dots stand for the terms which are the same as the preceding one except for the interchanges of appropriate variables.

Now, one observes that the condition (17) is equivalent to that of

$$\int^{\infty}\rho(s=\infty,t)dt/t,\quad \int^{\infty}\rho(s=\infty,u)du/u, \quad (41)$$

etc., are all convergent, which then implies that

$$\rho(s=\infty,u=\infty)=\rho(s=\infty,t=\infty)=\rho(u=\infty,t=\infty)=0. \quad (42)$$

One can then show that all the integrals in (40) diverge individually at most logarithmically. By applying the separation of the type of (18) twice, the first integral in (40) is split as

$$\frac{st}{\pi^2}\iint_{s_0t_0}^{\infty}\frac{\rho(s',t')ds'dt'}{s't'(s'-s)(t'-t)}=\left(\ln\frac{s_0}{s_0-s}\right)\frac{t}{\pi^2}\int_{t_0}^{\infty}\frac{\rho(s=\infty,t')dt'}{t'(t'-t)}+\left(\ln\frac{t_0}{t_0-t}\right)\frac{s}{\pi^2}\int_{s_0}^{\infty}\frac{\rho(s',t=\infty)ds'}{s'(s'-s)}+\frac{st}{\pi^2}\iint_{s_0t_0}^{\infty}\frac{\rho(s',t')-\rho(s=\infty,t')-\rho(s',t=\infty)}{s't'(s'-s)(t'-t)}ds'\,dt', \quad (43)$$

where one has used (42). Because of (41), all the integrals in (43) diverge at most logarithmically. The same separation makes the second integral in (40) split as

$$\frac{s}{\pi^2}\int\int_{s_0 t_0}^{\infty}\left[\frac{1}{t'}+\frac{1}{u'}\right]\frac{\rho(s',t')ds'dt'}{s'(s'-s)}=\frac{s}{\pi^2}\int_{s_0}^{\infty}\frac{ds'}{s'(s'-s)}\left[\int_{t_0}^{\infty}\left(\frac{1}{t'}+\frac{1}{a-s'-t'}\right)\rho(s',t')dt'\right]$$
$$=\left(\ln\frac{s_0}{s_0-s}\right)\frac{1}{\pi^2}\int_{t_0}^{\infty}\frac{\rho(s=\infty,t')dt'}{t'}+\frac{s}{\pi^2}\int_{s_0}^{\infty}\frac{ds'}{s'(s'-s)}\left[\int_{t_0}^{\infty}\frac{\rho(s',t')dt'}{a-s'-t'}+\int_{t_0}^{\infty}\frac{\rho(s',t')-\rho(s=\infty,t')}{t'}dt'\right], \quad (44)$$

where one has used (41). Because of (41), the integrals in the last expression in (44) also diverge at most logarithmically.

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Perturbation-Theory Rules for Computing the Self-Energy Operator in Quantum Statistical Mechanics*

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Convenient rules are given for the general term in the time-independent perturbation-theory expansion for the self-energy operator of quantum statistical mechanics. The rules are derived by starting from the usual formalism involving time-dependent Green's functions.

I. INTRODUCTION

A PERTURBATION theory for quantum statistical mechanics was developed by Peierls[1] in 1933. However, the general term in this theory was hard to characterize; furthermore, spurious terms, which are now known to cancel out, seemed to appear in the expression for the total number of particles. In 1958, Montroll and Ward[2] gave a perturbation theory in which the spurious terms were absent and the general term was described, but their formalism, involving an unnecessary expansion in powers of the fugacity, was exceedingly complicated. In recent years any number of formalisms have been proposed.[3] These are all essentially equivalent, varying only in details. The procedure of Glassgold, Heckrotte, and Watson involves a contour integration, that of Bloch and de Dominicis multiple temperature integrations, that of Luttinger and Ward infinite sums. Thouless,[4] however, has given a very convenient expression for the logarithm of the partition function.

To propose still another formalism would appear to be both inconsiderate and imprudent. Our motivation is that the rules we describe here are considerably simpler than any other prescription previously proposed. The rules are closely related to those given by Thouless,[4] but we shall work with the self-energy operator in terms of which one can find not only the partition function but also the single-particle excitations. Furthermore, it should be observed that the derivation of the rules is not restricted to the single-particle self-energy operator but, rather, is quite general. Thus, for example, one can easily use the method described here to obtain explicit time-independent rules for the space-time correlation function of any two physical observables.

The rules for calculating are given in Sec. II. These rules were first obtained intuitively[5] by the following

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