Hard-Sphere Boson Gas with Weak Attraction or Repulsion

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The coordinate-space variational treatment for the hard-sphere boson gas developed by Aviles and Iwamoto has been used to calculate the correction to the ground-state energy produced by a weak potential added outside the hard sphere. The first three terms in the low-density expansion are compared with the exact result of Wu. The first two terms agree well, while the third is poor.

I. INTRODUCTION

THE coordinate-space variational method of Aviles[1] and Iwamoto[2] will be used to calculate the ground-state energy per particle of an infinite, uniform, spinless boson gas, interacting by means of a hard-sphere potential of radius r_0, surrounded by a weak short-range attraction. (This is the physically interesting situation; the results can also be trivially applied to the case of a weak, short-range repulsion.) The trial function used is a product of pair functions

$$\Psi=\prod_{i<j} f(\mathbf{r}_i-\mathbf{r}_j). \tag{1}$$

II. CALCULATION OF THE ENERGY

It has been shown[1,2] that the energy per particle is

$$E=\frac{\hbar^2}{2m}\frac{\rho r_0^3}{2}\int d^3x\left[\nabla f(x)\cdot\nabla f(x)-f(x)\nabla^2 f(x)+\frac{2m}{\hbar^2}V(x)f^2(x)\right]G(x), \tag{2}$$

where $\mathbf{x}=\mathbf{r}/r_0$, ρ is the particle density, and $G(x)$ is the two-particle correlation function. The correlation function is expanded in a cluster series, formally identical to that familiar from the classical imperfect gas theory. Keeping "chain-connected" parts only, Aviles[1] and Iwamoto[2] summed a complete subset of cluster terms for the trial function

$$\begin{aligned} f(x)&=1-x^{-1}e^{-\epsilon(x-1)}, & x&\geqq 1,\\ &=0, & x&\leqq 1. \end{aligned} \tag{3}$$

An additional logarithmic term has been evaluated by the author,[3] and this correlation function will be used to compute the energy.

The potential to be used, simulating crudely an atom-atom potential, is

$$\begin{aligned} V(x)&=+\infty, & 0&\leqq x<1\\ &=-V_0, & 1&<x<b\\ &=0, & x&>b. \end{aligned} \tag{4}$$

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The first three terms in the energy, omitting V_0, were found to be[1-3]

$$E_0=4\pi\rho r_0\frac{\hbar^2}{2m}\left[1+\frac{5\sqrt{\alpha}}{3\sqrt{3}}+\frac{3\alpha}{4}\ln\alpha\right], \tag{5}$$

where $\alpha\equiv 8\pi\rho r_0^3$, and the variationally best value of $\epsilon=(\alpha/3)^{1/2}$ has been used. For this same value of ϵ,

$$G(x)\approx G_0(x)=1+\frac{2}{x}[e^{-\epsilon x}-e^{-2\epsilon x}]+\frac{3\epsilon}{16x}\left\{e^{-2\epsilon x}\left[\left(3\epsilon x-\frac{13}{2}\right)\ln(4\gamma\epsilon x)-3\epsilon x\right]+e^{2\epsilon x}\left(3\epsilon x+\frac{13}{2}\right)\mathrm{Ei}(-4\epsilon x)\right\}, \tag{6}$$

where γ is the Euler constant, $\mathrm{Ei}(z)$ is the exponential integral, and terms required to evaluate the energy up to the order given in Eq. (5) have been retained.[4] Since the potential vanishes for $x>b$, it is legitimate to expand $f^2(x)$ and $G_0(x)$ in a series in ϵ, retaining terms up to order $\epsilon^2\ln\epsilon$. This process yields

$$f^2(x)\approx(1-1/x)^2(1+2\epsilon), \tag{7a}$$

$$G_0(x)\approx 1+2\epsilon+6\epsilon^2\ln\epsilon. \tag{7b}$$

The attractive part of the potential energy per particle can be evaluated at once from Eq. (2) [with $\epsilon^2=\alpha/3$ as before],

$$\langle V\rangle=-4\pi\rho r_0\frac{\hbar^2}{2m}\delta\left[1+4\left(\frac{\alpha}{3}\right)^{1/2}+\alpha\ln\alpha\right], \tag{8}$$

where $\delta\equiv mr_0^2V_0(b-1)^3/3\hbar^2$, and higher terms in α are neglected. The total energy per particle is

$$E=4\pi\rho r_0\frac{\hbar^2}{2m}\left[(1-\delta)+\frac{5\sqrt{\alpha}}{3\sqrt{3}}\left(1-\frac{12}{5}\delta\right)+\tfrac{3}{4}\alpha\ln\alpha(1-\tfrac{4}{3}\delta)\right]. \tag{9}$$

[4] Equation (6) corresponds to Eq. (14a), Ref. 3, with the insertion of the γ which was erroneously omitted. This change does not affect any of the previously derived results.

III. COMPARISON WITH THE EXACT RESULTS

Wu[5] has shown that for the first three terms in the exact ground-state energy of the spinless boson gas the results are shape-independent, and should depend only on the scattering length a. For the potential of Eq. (4), it is well known[6] that

$$a=r_0+(b-1)r_0[1-\tan\beta r_0(b-1)/\beta r_0(b-1)],$$

where

$$\beta=(mV_0)^{1/2}/\hbar. \quad (10)$$

Since the present calculation is, in effect, a correction to E of first order in V_0, the scattering length can be approximated by

$$a\approx r_0[1-mr_0^2V_0(b-1)^3/3\hbar^2]=r_0(1-\delta). \quad (11)$$

Replacing r_0 by a throughout Eq. (5), and keeping terms linear in δ, one obtains

$$E'=4\pi\rho r_0\frac{\hbar^2}{2m}\left[(1-\delta)+\frac{5\sqrt{\alpha}}{3\sqrt{3}}\left(1-\frac{5}{2}\delta\right)+\tfrac{3}{4}\alpha\ln\alpha(1-4\delta)\right]. \quad (12)$$

The comparison with Eq. (9) shows exact agreement for the first term (which is trivial), 4% agreement for the second term, and very poor agreement for the logarithmic term.

The present results would constitute a consistent first-order perturbation calculation if the unperturbed function [Eqs. (1) and (3)] were an exact eigenfunction of the hard-sphere problem. The errors obtained here for the various terms are qualitatively what was to be expected from the degree of agreement previously found for the hard-sphere case without attraction.[3]

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Regge Poles and Complex Singularities*

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We investigate the analyticity in complex angular momentum for the case in which complex singularities are present in double-spectral representations of scattering amplitudes. The simple example we consider is Σ-Σ scattering. We show that Regge continuation exists and has the same qualitative characteristics as for the case of no complex singularities. In particular, we find no direct connection between the presence of complex singularities and the existence of the branch cuts in angular momentum.

I. INTRODUCTION

Several authors[1] have investigated the Regge behavior of S-matrix elements with normal thresholds. In these investigations, it has been generally assumed that Regge poles are the only singularities in the complex angular momentum plane. Recently, however, it has been suggested[2,3] that branch points, with associated branch cuts, may also be present. The very slow approach of nuclear cross sections to their ultimate constant values has been attributed by Udgaonkar and Gell-Mann[3] to the presence of branch cuts. Since nuclei are composite objects with prominent anomalous thresholds, it is appealing to assume that the

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