Contribution to the Theory of Coulomb Stripping*

A. DAR, A. DE-SHALIT AND A. S. REINER
Department of Nuclear Physics, The Weizmann Institute of Science, Rehovoth, Israel
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The theory of deuteron stripping for incident deuteron energies below the Coulomb barrier is presented for proton-neutron interaction of finite range. Reasonable approximations valid only for the conditions prevailing in the case of Coulomb stripping enable factorization of the transition amplitude in analytic form. The results account for observed regularities and explain the discrepancy between experiment and a theory based on zero-range nuclear forces. Stripping with associated Coulomb excitation is shown to be a competing process for sufficiently low deuteron energy. The reaction enables excitation of a new class of excited states. It should in particular yield information about the alleged multiplet structure of low-lying states of odd-A nuclei, of which the even-even neighbors show vibrational spectra. Angular distributions of outgoing protons resulting from stripping on targets with $J_i \geqslant 1$ may yield information about static quadrupole moments of those nuclei. The contribution of the polarizability of the deuteron to the differential cross sections is shown to be negligible.

I. INTRODUCTION

STRIPPING reactions have proven to be one of the most useful tools in the study of nuclear structure. The mechanism, a lowest order direct nuclear interaction between one of the nucleons of the deuteron with the target nucleus, is apparently widely valid, and the characteristic patterns for the angular distribution of (d,p) and (d,n) reactions are reasonably understood on the basis of this mechanism.[1]

Stripping reactions have been studied extensively at deuteron energies large compared to the Coulomb barrier. The deuteron as a whole will then be able to make relatively close contact with the target nucleus, thus, enabling a direct interaction.

For deuteron energies small compared to the Coulomb barrier one expects a decrease of the stripping efficiency, while at the same time the relative importance of Coulomb effects and the actual structure of the deuteron increases.

Stripping at energies smaller than the Coulomb barrier has been treated by Ter Martirosyan[2] and independently by Biedenharn *et al.*[3,4] and goes sometimes under the noncharacteristic name of Coulomb stripping. [This name would be more appropriate for the Oppenheimer-Phillips process,[5] where deuteron dissociation by the Coulomb field preceeds a typical nuclear interaction.] The above-mentioned authors have used an approximation which neglects the finite range of the proton-neutron force in the deuteron. Experimental results, however, show absolute cross sections which are a factor 4–5 bigger than those calculated the zero-range approximation.[6]

The cause of the discrepancy may be understood once it is realized that Coulomb stripping is a marginal direct interaction process, in the sense that the transition amplitude is proportional to the weak overlap of the tail of the wave function of the captured neutron with that of the incident deuteron. An approximation which neglects the finite size of the deuteron, forces the neutron to be close to the proton. It thus prevents the neutron from penetrating closer to the nucleus where its overlap with the capturing state is much better. We have here an indication why a zero-range approximation may underestimate the cross section.

Section II contains a treatment of Coulomb stripping for a conventional finite range proton-neutron interaction. It will be shown that reasonable approximations can be made in the case of Coulomb stripping, which would not be valid for normal stripping conditions. These approximations lead to a factorization of the transition amplitude, and one consequently obtains the cross section in an analytic form. This approach will be shown to remove the discrepancy between experimental results and the zero-range theory.

The increase in the relative importance of the Coulomb effects makes it necessary to investigate competing processes, which also lead to a final nucleus consisting of $A+1$ nucleons and a scattered proton. The most probable of this is presumably Coulomb excitation accompanying stripping and this process is described in Sec. III. The mechanism is of particular interest since a new class of states in the final nucleus can be reached. Indeed, the n-p interaction does not involve the target coordinates and leads only to final states containing a neutron in a single particle state coupled to the target in its ground state. The stripping reaction combined with Coulomb excitation may lead to states describing a single neutron coupled to a Coulomb-excited target. Such processes are, in principle, also possible in ordinary

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[1] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, New York, 1961).

[2] K. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **29**, 713 (1956) [translation: Soviet Phys.—JETP **2**, 620 (1956)].

[3] L. C. Biedenharn, K. Boyer, and M. Goldstein, Phys. Rev. **104**, 382 (1956).

[4] W. O. Barfield, B. M. Bacon, and L. C. Biedenharn, Phys. Rev. **125**, 964 (1962).

[5] J. Oppenheimer and M. Phillips, Phys. Rev. **48**, 500 (1935).

[6] J. R. Erskine, W. W. Buechner, and H. A. Enge, Phys. Rev. **128**, 720 (1962).

stripping reactions. Their cross section is, as we shall see, smaller than that for Coulomb stripping which in turn is less probable than normal stripping. The associated stripping is, therefore, generally masked when normal stripping can proceed, but is of relevance when Coulomb stripping is the dominant process.

Section IV discusses the effect of a finite polarizability of the deuteron. In spite of the loose structure of the deuteron, it turns out that for deuteron energies sufficiently below the Coulomb barrier, one may neglect the contribution of its polarizability to the differential cross section.

II. COULOMB STRIPPING

The Hamiltonian describing a (d,p) reaction can be written in the form

$$H=H_N(\xi)+T_p+T_n+V_{pn}(\mathbf{r},\boldsymbol{\sigma}_p,\boldsymbol{\sigma}_n)+V_{nN}+V_{pN}{}^{\mathrm{nC}}+V_{pN}{}^{\mathrm{C}}. \quad (1)$$

$H_N(\xi)$ is the target Hamiltonian written in internal coordinates ξ; T_p, T_n, and V_{pn} are the kinetic and interaction energies of the nucleons in the deuteron; V_{nN} and V_{pN} denote their interaction with the target nucleus and we have decomposed V_{pN} into its Coulomb part $V_{pN}{}^{\mathrm{C}}$ and the remainder $V_{pN}{}^{\mathrm{nC}}$ such that

$$V_{pN}=V_{pn}{}^{\mathrm{nC}}+V_{pn}{}^{\mathrm{C}} \quad \text{and} \quad V_{pN}{}^{\mathrm{C}}=\sum_{\text{protons}}\frac{e^2}{|\mathbf{r}_p-\mathbf{r}_i|}.$$

We also use the coordinates $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_p+\mathbf{r}_n)$.

We denote by $\Psi_i{}^{(+)}$ scattering states which are solutions of $(E-H)\Psi_i{}^{(+)}=0$. $\Psi_i{}^{(+)}$ describes a deuteron with energy $E_d=\hbar^2\mathbf{k}_d{}^2/4M$ incident on a target specified by $|\alpha_iJ_iM_i\rangle$ and outgoing scattered waves. α_i stands for all quantum numbers required to describe the target nucleus in its ground state in addition to its total angular momentum J_i and the magnetic quantum number M_i.

The final state of our system, $\Phi_f{}^{(-)}$, is a product wave function of $|\alpha_fJ_fM_f\rangle$ and that of a proton with asymptotic momentum $\mathbf{k}_p$. The proton is supposed to move in an effective field V_p which approximates V_{pN}. V_p usually consists of a nuclear optical potential V_{op} and of Ze^2/r_p, the dominant monopole part of $V_{pN}{}^{\mathrm{C}}$ for $r_p>R_N$, where R_N is the nuclear radius. Inside the nucleus the Coulomb potential is assumed to be due to a Saxon-Woods-type charge distribution.

The complete transition amplitude is then given by

$$f=\langle\Phi_{\mathbf{k}_p}{}^{(-)\alpha_fJ_fM_f}|V_{pn}+\sum_{\text{protons}}\frac{e^2}{|\mathbf{r}_p-\mathbf{r}_i|}-\frac{Ze^2}{r_p}+V_{pN}-V_{\mathrm{op}}|\Psi_{\mathbf{k}_d}{}^{(+)\alpha_iJ_iM_i}\rangle. \quad (2)$$

V_{nN} and V_{pN} are in the usual distorted-wave Born approximation replaced by an optical potential acting on the center of mass of the deuteron. $\Psi_i{}^{(+)}$ is then a solution belonging to the Hamiltonian

$$H=H_N(\xi)+T_p+T_n+V_{pn}(r)+Ze^2/R+V_{\mathrm{op}}(R), \quad (3)$$

where it is understood that Ze^2/R is omitted for $R<R_N$. For Coulomb stripping we expect the dominant contribution in (2) to come from the interaction region around $\bar{r}_p$, the classical turning point for a proton of a momentum $\mathbf{k}_p$. It is, therefore, reasonable to neglect, for $\bar{r}_p\gg R_N$, the difference $V_{pN}-V_{\mathrm{op}}$, the range of which is of order R_N.

The amplitude f may then be decomposed into two amplitudes $f=f^{(1)}+f^{(2)}$, which describe, respectively, stripping due to the proton-neutron interaction V_{pn}, and stripping with associated Coulomb excitation via

$$V^{\mathrm{Ce}}=\sum_{\text{protons}}\frac{e^2}{|\mathbf{r}_p-\mathbf{r}_i|}-\frac{Ze^2}{r_p}. \quad (2)$$

In the modified distorted-wave Born approximation, which we shall use for Coulomb stripping, $\Psi_i{}^{(+)}$ will be chosen to be an approximate solution of the Hamiltonian

$$H'=H_N(\xi)+T_p+T_n+V_{pn}+Ze^2/r_p. \quad (5)$$

The true monopole part of $V_{pN}{}^{\mathrm{C}}$ rather than Ze^2/R is restored in (5), and in using (2) we again neglect V_{nN} and $V_{pN}{}^{\mathrm{nC}}$ in comparison with V^{Ce}. Higher order terms in the Born series will lead to contributions to f which are at least one order higher in V_{pn} or V^{Ce}.

Let us first consider the Coulomb stripping amplitude $f^{(1)}$ which, on introducing the approximations described above, reads

$$f^{(1)}=\langle\Phi_{\mathbf{k}_p}{}^{(-)\alpha_fJ_fM_f}|V_{np}|\Psi_{\mathbf{k}_d}{}^{(+)\alpha_iJ_iM_i}\rangle, \quad (6)$$

with

$$\Phi_{\mathbf{k}_p}{}^{(-)\alpha_fJ_fM_f}=\varphi_{\alpha_fJ_fM_f}(\xi,\mathbf{r}_n,\boldsymbol{\sigma}_n)\psi_{\mathbf{k}_p}{}^{(-)}(\mathbf{r}_p,\boldsymbol{\sigma}_p). \quad (7)$$

$\varphi_{\alpha_fJ_fM_f}$ describes the final nucleus produced in the reaction while $\psi_{\mathbf{k}_p}{}^{(-)}$ consists of a proton spin wave function $\chi_{m_{sp}}(\boldsymbol{\sigma}_p)$, coupled to an ingoing Coulomb wave function

$$\psi_{\mathbf{k}_p}{}^{(-)}(\mathbf{r}_p)=\exp\left(-\frac{\pi}{2}\eta_p\right)\Gamma(1-i\eta_p)\exp(i\mathbf{k}_p\cdot\mathbf{r}_p)\times F(i\eta_p,1,-i(k_pr_p-\mathbf{k}_p\cdot\mathbf{r}_p)). \quad (8)$$

$$\eta_p=Ze^2m/\hbar^2k_p,$$

$\Psi_i{}^{(+)}$, a scattering solution of H' Eq. (5), can be written as a product of the wave function of the target $\varphi_{\alpha_iJ_iM_i}$ and of a deuteron wave scattered by a point Coulomb potential Ze^2/r_p. The latter wave function has the form

$$X_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_n,\mathbf{r}_p)\chi_{M_d}{}^{S_d}(\boldsymbol{\sigma}_n,\boldsymbol{\sigma}_p). \quad (9)$$

No coupling between orbital and spin parts is assumed while, moreover, for $\chi_{M_d}{}^{S_d}(\sigma_n,\sigma_p)$ a pure ${}^3S$ state is taken;

$$\chi_{M_d}=\sum m_{s_p},m_{s_n}(\tfrac{1}{2}m_{s_p}\tfrac{1}{2}m_{s_n}|1M_d)\chi m_{s_p}\chi m_{s_n}. \quad (10)$$

We now return to the amplitude $f^{(1)}$ and make the usual statement regarding the overlap integral of final and target nucleus present in Eq. (6). This function of

$\mathbf{r}_n$, $\boldsymbol{\sigma}_n$ will be assumed to represent a neutron with quantum numbers $\alpha_n l_n j_n m_n$ moving in a central field of the target.

$$\langle\varphi_f(\xi,\mathbf{r}_n,\boldsymbol{\sigma}_n)|\varphi_i(\xi)\rangle = \sum_{m_n}(j_n m_n J_i M_i|J_f M_f)\phi_{\alpha_n l_n j_n m_n}(\mathbf{r}_n,\boldsymbol{\sigma}_n). \quad (11)$$

In substituting (11) into (2) we then find

$$f^{(1)}=\sum_{m_n}(j_n m_n J_i M_i|J_f M_f) \times\langle\phi_{\alpha_n l_n j_n m_n}(\mathbf{r}_n,\boldsymbol{\sigma}_n)\psi_{k_p}^{(-)}(\mathbf{r}_p)\chi m_{s_p}(\boldsymbol{\sigma}_p) \times|V_{pn}|X_{k_d}^{(+)}(\mathbf{r}_n,\mathbf{r}_p)\chi_{M_d}{}^1(\boldsymbol{\sigma}_p,\boldsymbol{\sigma}_n)\rangle. \quad (12)$$

We first perform the spin summation

$$f^{(1)}=\sum m_{s_p}m_{s_n}m_{l_n m_n}(j_n m_n J_i M_i|J_f M_f)(\tfrac{1}{2}m_{s_p}\tfrac{1}{2}m_{s_n}|1M_d) \times(\tfrac{1}{2}m_{s_n}l_n m_{l_n}|j_n m_n)f_{\alpha_n l_n}m_{l_n}{}^{(1)}. \quad (13)$$

[Since χ_{M_d} is an eigenstate of $\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n$ with eigenvalue 1, one sees that for an interaction of the form $V_{pn}=V_{pn}{}^{(1)}+V_{pn}{}^{(2)}(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n)$, (12) yields the same results provided V_{pn} is replaced by $V_{pn}{}^{(1)}(r)+V_{pn}{}^{(2)}(r)$.] Equation (13) defines the partial amplitude $f_{\alpha_n l_n m_{l_n}}{}^{(1)}$

$$f_{\alpha_n l_n m_{l_n}}{}^{(1)}=\langle R_{\alpha_n l_n}(r_n)Y_{l_n m_{l_n}}(\Omega_n)\psi_{\mathbf{k}_p}{}^{(-)}(\mathbf{r}_p) \times|V_{pn}|X_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p,\mathbf{r}_n)\rangle. \quad (13')$$

For a further evaluation of $f^{(1)}$ we shall employ approximations which are exclusively valid for Coulomb stripping. All are based on the observation that the dominant contribution to f can actually be localized. Indeed, the Coulomb barrier prevents the proton from approaching the target nucleus, thus requiring the interaction area to be outside $\bar{r}_p$, the classical turning point for the proton. The exponential decrease of $R_{\alpha_n l_n}(r_n)$, the neutron radial wave function outside the centrifugal barrier, favors on the other hand interactions as close as possible to the nuclear surface. The effective interaction area resulting from these two tendencies is determined by the intrinsic deuteron wave function and the proton-neutron interaction. Both have a characteristic range of the order of the deuteron radius R_d, which is much smaller than $\bar{r}_p$, the classical distance of closest approach. The interaction region in (2) will, therefore, be limited to a region $R_d\ll\bar{r}_p$ in the vicinity of the classical distance of the closest approach. On this basis we now suggest the following approximations:

$$R_{\alpha_n l_n}(r_n)\sim A_{\alpha_n l_n}(E_n)\frac{e^{-k_n r_n}}{k_n r_n}, \quad (14)$$

where $k_n=(-2mE_n/\hbar^2)^{1/2}$ is related to E_n, the binding energy of the captured neutron.

Equation (14) expresses the fact that outside the centrifugal barrier [i.e., for $k_n r_n\gg l(l+1)/2$] the radial part of the captured neutron wave function may be replaced by its asymptotic behavior suitably normalized by $A_{\alpha_n l_n}(E_n)$.

$$Y_{l_n m_{l_n}}(\Omega_n)\sim Y_{l_n m_{l_n}}(\Omega_p). \quad (15)$$

Since $\bar{r}_p$ is big compared to the dimensions of the deuteron, (15) asserts that, seen from the nucleus, the neutron and the proton in the deuteron appear at approximately the same directions.

$$X_{k_d}{}^{(+)}(\mathbf{r}_n,\mathbf{r}_p)\sim\psi_{k_d}{}^{(+)}(\mathbf{r}_p)\chi(\mathbf{r}), \quad (16)$$

where

$$\psi_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p)=\exp\left(-\frac{\pi}{2}\eta_d\right)\Gamma(1+i\eta_d) \times\exp(i\mathbf{k}_d\cdot\mathbf{r}_p)F(-i\eta_d,1,i(k_d r_p-\mathbf{k}_d\cdot\mathbf{r}_p)) \quad (17)$$

and $\chi(\mathbf{r})$ is the spatial part of the deuteron ground-state wave function. $\psi_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p)$ is an outgoing wave function for the deuteron center of mass but with the argument R replaced by r_p. The deuteron is thus described as a rigid body which maintains a fixed shape as it traverses the Coulomb field. In the approximation frequently made one replaces $X_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p,\mathbf{r}_n)$ by $\psi_{\mathbf{k}_d}{}^{(+)}(\mathbf{R})\chi(\mathbf{r})$, implying that the most important factor determining the reaction is just the natural spread of the deuteron due to its low binding energy. This approximation neglects entirely the asymmetry of the proton and neutron in the deuteron with respect to the target Coulomb field. Our choice, on the other hand, describes a proton which is less likely to reach the nuclear surface than the neutron, which fact should be important under the conditions for Coulomb stripping.

In addition to the replacements mentioned above we shall use the expansion of $e^{-k_n r_n}/k_n r_n$ needed only for $r_p>r$

$$\frac{e^{-k_n r_n}}{k_n r_n}=\frac{e^{-k_n|\mathbf{r}_p-\mathbf{r}|}}{k_n|\mathbf{r}_p-\mathbf{r}|}=-4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}j_l(ik_n r)h_l(ik_n r_p) \times Y_{lm}{}^*(\Omega_r)Y_{lm}(\Omega_p), \quad (18)$$

where j_l, h_l are spherical Bessel and Hankel functions, respectively.

We now substitute (14)–(18) into (13). Since the deuteron has been assumed to be a pure ^3S state, only the $l=m=0$ term in (18) contributes. The partial amplitude for Coulomb stripping to a state with a captured neutron characterized by $\alpha_n l_n m_n$ appears now factorized and reads

$$f_{\alpha_n l_n m_{l_n}}{}^{(1)}=-A_{\alpha_n l_n}(E_n)\int h_0(ik_n r_p) \times Y_{l_n m_{l_n}}{}^*(\Omega_p)\psi_{\mathbf{k}_p}{}^{(-)*}(\mathbf{r}_p)\psi_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p)d\mathbf{r}_p \times(4\pi)^{1/2}\int_0^{\bar{r}_p}j_0(ik_n r)V_{np}(r)\chi_0(r)r^2dr, \quad (19)$$

where $\chi_0(r)$ is the radial part of the deuteron wave function.

Using the saddle-point approximation,[2] Eq. (19) can be written in the form

$$f_{\alpha_n l_n m_n}{}^{(1)}=A_{\alpha_n l_n}(E_n)Y_{l_n m_{l_n}}{}^*(\bar{\Omega}_p)I(\theta_p)F(k_n,\bar{r}_p), \quad (20)$$

where $\bar{\Omega}_p$ is the value of Ω_p at the saddle point and θ_p is the angle between $\mathbf{k}_d$ and $\mathbf{k}_p$. The orbital factor I equals

$$I(\theta_p)=\int \frac{e^{-k_n r_p}}{k_n r_p}\psi_{k_p}{}^{(-)*}(\mathbf{r}_p)\psi_{k_d}{}^{(+)}(\mathbf{r}_p)d\mathbf{r}_p, \quad (21)$$

while the deuteron "form factor" $F(k_n,\bar{r}_p)$ reads

$$F(k_n,\bar{r}_p)=-(4\pi)^{1/2}\int_0^{\bar{r}_p} j_0(ik_n r)V_{pn}(r)\chi_0(r)r^2dr. \quad (22)$$

In terms of the amplitude (20) the partial differential cross section for Coulomb stripping corresponding to (13) is given by

$$\frac{d\sigma^{(1)}}{d\Omega}=\frac{M^2}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{(2J_f+1)}{(2J_i+1)} \times|A_{\alpha_n l_n}(E_n)|^2|F(k_n,\bar{r}_p)|^2|I(\theta_p)|^2. \quad (23)$$

The orbital contribution $I(\theta_p)$ has been calculated by Ter Martirosyan[2] who used some techniques developed by Sommerfeld.[7] The result is

$$|I(\theta_p)|^2=\frac{64\pi^4\eta_p\eta_d}{k_n(e^{2\pi\eta_p}-1)(e^{2\pi\eta_d}-1)} \times\frac{\exp[2\eta_d(\pi-\phi_d)+2\eta_p\phi_p]}{\{(k_d-k_p)^2+k_n^2\}^2} \times\left|\frac{F(i\eta_p,\, i\eta_d,\, 1-\xi)}{1+\xi}\right|^2, \quad (24)$$

where

$$\phi_p=\arctan\frac{2k_nk_p}{k_d^2-k_p^2+k_n^2};\quad \phi_d=\arctan\frac{2k_nk_d}{k_d^2-k_p^2-k_n^2};$$

$$\eta=\frac{Ze^2m}{\hbar^2k};\quad \xi=\xi_0\sin\frac{\theta_p}{2};\quad \xi_0=\frac{4k_pk_d}{(\mathbf{k}_d-\mathbf{k}_p)^2+\mathbf{k}_n^2};$$

$$0\leqslant\phi_p,\ \phi_d\leqslant\pi.$$

$|I(\theta_p)|^2$ yields the angular distribution of the protons with respect to the direction of the incident deuterons. The factor is common to both the finite-range and zero-range theories. It has been shown in Ref. 2 that for η_p, $\eta_d\gg1$, $|I(\theta_p)|^2$ represents a Gaussian distribution around a backward angle which describes the experimental data reasonably well. [This result was also derived later by R. Lemmer[8] who used an elegant semiclassical argument.]

Next we turn to the form factor $F(k_n,\bar{r}_p)$ given by (22). This integral over the relative coordinate of the deuteron can be calculated for any given $V_{pn}(r)$ and associated $\chi_0(r)$.

[7] A. Sommerfeld, *Wellenmechanik* (Frederick Ungar Publishing Company, New York, 1953), Chap. 7.
[8] R. Lemmer, Nucl. Phys. **39**, 680 (1962).

In the zero-range limit $F(k_n,\bar{r}_p)$ is independent of k_n. For finite ranges, however, it is an increasing function of k_n, and for any given value of k_n, $F(k_n,\bar{r}_p)$ increases with the increase of the range of V_{pn}.

Let us take, for example, $V_{pn}(r)$ to be a Hulthén potential

$$V_{pn}(r)=-V_0\frac{e^{-\mu r}}{1-e^{-\mu r}}, \quad (25)$$

where $V_0=44.5$ is the depth parameter of the well in MeV and μ^{-1} is the range of the nuclear force ($\sim1.4\times10^{-13}$ cm). The corresponding deuteron wave function is

$$\chi_0(r)=\left[\frac{2\alpha(\alpha+\mu)(2\alpha+\mu)}{\mu^2}\right]^{1/2}\frac{e^{-\alpha r}(1-e^{-\mu r})}{r}. \quad (26)$$

α is related to ϵ_d, the binding energy of the deuteron, by $\hbar^2\alpha^2/M=\epsilon_d$.

On substituting (25), (26) into (22) one obtains

$$F(k_n)=-\left[\frac{2\alpha(\alpha+\mu)(2\alpha+\mu)}{\mu^2}\right]^{1/2}\frac{(4\pi)^{1/2}V_0}{(\alpha+\mu)^2-k_n^2}, \quad (27a)$$

where we have replaced the upper limit of integration $\bar{r}_p$ by ∞. This latter approximation does not markedly change $F(k_n,\bar{r}_p)$ since in actual cases $\alpha+\mu\gg k_n$. One may compare at this stage $F(k_n)$ with the corresponding form factor F_0 for the zero-range potential

$$F_0=-(\hbar^2/M)(8\pi\alpha)^{1/2}. \quad (27b)$$

We have plotted in Fig. 1, $|F(k_n)/F_0|^2$ as a function of the Q value of the reaction. This ratio increases with Q. For $Q=0$, $|F(k_n)/F_0|^2\simeq4.5$. Experimentally it was found by Erskine *et al.*[6] that the zero-range formula for the absolute differential cross section in the case of $Bi^{209}(d,p)Bi^{210}$, indeed, underestimates the experimental value for $Q=-0.2$ MeV by a factor 4.5 which increases slowly with Q.

The agreement between the calculated absolute cross section and its experimental value for $Bi^{209}(d,p)Bi^{210}$ might be somewhat accidental, but it is believed to

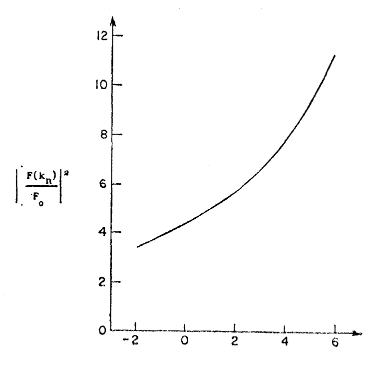

FIG. 1. Ratio of deuteron form factor for Coulomb stripping using a Hulthén potential and normalized zero-range potential as function of Q value.

demonstrate the importance of the finite range of V_{pn} in the description of Coulomb stripping potentials.

III. COULOMB STRIPPING ACCOMPANIED BY COULOMB EXCITATION

It was mentioned before that since the proton-neutron interaction does not involve the target coordinates, V_{pn} cannot lead to a final nucleus, where the captured neutron is coupled to an excited state of the target nucleus. In normal stripping such an excitation can in principle be caused by the term $V_{\rm op}-V_{pN}{}^{\rm nC}$ in the perturbation. In Coulomb stripping the deuteron is too far away from the nucleus to feel any appreciable nuclear force. There remains, however, the possibility to excite those states by the remaining long-range Coulomb perturbation $V^{\rm Ce}$, which for $r_p>r_i$ can be expanded into its multipole parts as follows:

$$V^{\rm Ce}=\sum_{i=1}^{Z}\frac{e^2}{|\mathbf{r}_p-\mathbf{r}_i|}-\frac{Ze^2}{r_p}=\sum_{i=1}^{Z}\sum_{\lambda=1}^{\infty}\sum_{\mu=-\lambda}^{+\lambda}\frac{4\pi}{2\lambda+1}e^2r_p^{-(\lambda+1)}r_i^{\lambda}\times Y_{\lambda\mu}(\Omega_i)Y_{\lambda\mu}^*(\Omega_p). \quad (28)$$

The reaction amplitude $f^{(2)}$ resulting from $V^{\rm Ce}$ will now contain instead of the overlap integral (11) the factor

$$\langle\varphi_{\alpha_fJ_fM_f}(\xi,\mathbf{r}_n,\boldsymbol{\sigma}_n)\sum_{j=1}^{Z}r_j^{\lambda}Y_{\lambda\mu}(\Omega_j)|\varphi_{\alpha_iJ_iM_i}(\xi)\rangle, \quad (29)$$

where the final nucleus is assumed to be described by a single neutron coupled to a Coulomb excited target nucleus, or

$$\varphi_{\alpha_fJ_fM_f}(\xi,\mathbf{r}_n,\boldsymbol{\sigma}_n)=\sum_{m_n,m_{sn},m_{sp}}(\tfrac{1}{2}m_{s_n}l_nm_{l_n}|j_nm_n)\times(j_nm_nJ_i'M_i'|J_fM_f)\chi m_{s_n}(\boldsymbol{\sigma}_n)\times R_{\alpha_nl_n}(r_n)Y_{l_nm_{l_n}}(\Omega_n)\varphi_{\alpha_i'J_i'M_i'}(\xi), \quad (30)$$

where $\varphi_{\alpha_i'J_i'M_i'}$ is the wave function of the Coulomb excited target nucleus. Equation (29) will, thus, contain the reduced matrix element for an electric transition of multipolarity λ. By means of

$$B_{i\to i'}(E\lambda)\equiv\sum_{\mu M_i'}|\langle\alpha_{i'}J_{i'}M_{i'}|e\sum_{j=1}^{Z}r_j^{\lambda}Y_{\lambda\mu}(\Omega_j)|\alpha_iJ_iM_i\rangle|^2=\frac{1}{2J_i+1}|\langle\alpha_{i'}J_{i'}\|e\sum_{j=1}^{Z}r_j^{\lambda}Y_{\lambda}(\Omega_j)\|\alpha_iJ_i\rangle|^2,$$

one expresses $f^{(2)}$ as

$$f^{(2)}(\lambda)=\frac{4\pi e}{2\lambda+1}\{B_{i'\to i}(E\lambda)\}^{1/2}\sum m_{s_n}m_{s_p}m_{l_nm_nM_i'\mu}(J_iM_i\lambda\mu|J_i'M_i')\times(\tfrac{1}{2}m_{s_p}\tfrac{1}{2}m_{s_n}|1M_d)(\tfrac{1}{2}m_{s_n}l_nm_{l_n}|j_nm_n)(j_nm_nJ_i'M_i'|J_fM_f)\times\int\int R_{\alpha_nl_n}(r_n)Y_{l_nm_{l_n}}^*(\Omega_n)\Psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r}_p)\Psi_{\mathbf{k}_d}^{(+)}(\mathbf{r}_p)Y_{\lambda\mu}^*(\Omega_p)\chi_0(r)r_p^{-(\lambda+1)}d\mathbf{r}_pd\mathbf{r}. \quad (31)$$

Let us call again the integral on the right-hand side of (31) a partial amplitude $f_{\alpha_nl_nm_{l_n}}{}^{(2)}(\lambda\mu)$. By virtue of the same argments as used in Sec. II to reduce the amplitude $f^{(1)}$, [Eq. (13)] into a product of two integrals [Eq. (19)], we obtain for $f_{\alpha_nl_nm_{l_n}}{}^{(2)}(\lambda\mu)$

$$f_{\alpha_nl_nm_{l_n}}{}^{(2)}(\lambda\mu)=-A_{\alpha_nl_n}(E_n)\int h_0(ik_nr_p)Y_{l_nm_{l_n}}^*(\Omega_p)Y_{\lambda\mu}^*(\Omega_p)\psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r}_p)\times\psi_{\mathbf{k}_d}^{(+)}(\mathbf{r}_p)r_p^{-\lambda-1}d\mathbf{r}_p(4\pi)^{1/2}\int_0^{\bar{r}_p}j_0(ik_nr)\chi_0(r)r^2dr. \quad (32)$$

The corresponding partial differential cross section is given by

$$\frac{d\sigma^{(2)}}{d\Omega}=\frac{M^2}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{16\pi^2e^2B_{i'\to i}(E\lambda)}{(2\lambda+1)^2(2J+1)}\sum_{\text{magnetic quantum numbers}}(\tfrac{1}{2}m_{s_n}l_nM_{l_n}|j_nm_n)(\tfrac{1}{2}m_{s_n}l_nm_{l_n}'|j_nm_n')\times(j_nm_nJ_i'M_i'|J_fM_f)(j_nm_n'J_i'M_i''|J_fM_f)(J_iM_i\lambda\mu|J_i'M_i')\times(J_iM_i\lambda\mu'|J_i'M_i'')f_{\alpha_nl_n}m_{l_n}{}^{(2)}(\lambda\mu)f_{\alpha_nl_n}m_{l_n'}{}^{(2)*}(\lambda\mu'). \quad (33)$$

On using the saddle-point approximation one may reduce Eq. (33) to the following form (see Appendix)

$$\frac{d\sigma^{(2)}}{d\Omega}=\frac{M^2}{(2\pi\hbar)}\frac{k_p}{k_d}\frac{(2J_f+1)(2l_n+1)(2j_n+1)(2J_i'+1)}{(2J_i+1)(2\lambda+1)}e^2B_{i'\to i}(E\lambda)\times|A_{\alpha_nl_n}(E_n)|^2|I_\lambda(\theta_p)|^2|G(k_n\bar{r}_p)|^2\begin{pmatrix}l_n & \frac{1}{2} & j_n\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}^2\sum_{\substack{n=\frac{1}{2},-\frac{1}{2}\\ m}}\begin{pmatrix}\lambda & J_i & J_i'\\ 0 & m & -m\end{pmatrix}^2\begin{pmatrix}j_n & J_i' & J_f\\ n & -m & m-n\end{pmatrix}^2. \quad (33')$$

We further give the result for the singlet case $l_n=0$:

$$\left(\frac{d\sigma^{(2)}}{d\Omega}\right)_{l_n=0}=\frac{M^2}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{(2J_f+1)}{(2J_i+1)}\frac{e^2B_{i'\to i}(E\lambda)}{(2\lambda+1)^2}|A_{\alpha_n l_n}(E_n)|^2|I_\lambda(\theta_p)|^2|G(k_n,\bar{r}_p)|^2. \tag{34}$$

$G(k_n,\bar{r}_p)$ is a deuteron form factor corresponding to (22) and reads

$$G(k_n,\bar{r}_p)=-(4\pi)^{1/2}\int_0^{\bar{r}_p} j_0(ik_nr)\chi_0(r)r^2dr. \tag{35}$$

The corresponding orbital factor becomes

$$I_\lambda(\theta_p)=\int h_0(ik_nr_p)\psi_{\mathbf{k}_p}^{(-)*}(\mathbf{r}_p)\times\psi_{\mathbf{k}_d}^{(+)}(\mathbf{r}_p)r_p^{-(\lambda+1)}d\mathbf{r}_p. \tag{36}$$

The integral is again sharply peaked around $r_p\simeq\bar{r}_p$ and can, therefore, be approximated by

$$I_\lambda(\theta_p)\simeq\bar{r}_p^{-(\lambda+1)}I(\theta_p), \tag{37}$$

where $I(\theta_p)$ is given by Eq. (24), the angular factor for the case of pure Coulomb stripping. To the extent that (36) holds, approximately the same angular distribution is predicted for Coulomb stripping with or without Coulomb excitation. One also notes that the cross sections for final states belonging to the same parent core remain proportional to $2J_f+1$.

The relative importance of the described process is determined by the ratio of the cross sections for Coulomb stripping with and without associated Coulomb excitation. We shall assume the same neutron reduced widths and binding energies in order to obtain an estimate for the ratio of these cross sections:

$$\rho=\frac{d\sigma^{(2)}}{d\Omega}\Big/\frac{d\sigma^{(1)}}{d\Omega}=\frac{(2\bar{J}_f+1)}{(2J_f+1)}\frac{e^2B_{i'\to i}(E\lambda)}{(2\lambda+1)^2r_p^{2\lambda+2}}\times\left|\frac{G(k_n,\bar{r}_p)}{F(k_n,\bar{r}_p)}\right|^2. \tag{38}$$

Here $\bar{J}_f$ refers to the case of stripping accompanied by Coulomb excitation. The deuteron form factor $G(k_n,\bar{r}_p)$ calculated for a Hulthén potential is approximately

$$G(k_n,\bar{r}_p)\simeq\left[\frac{2\alpha(\alpha+\mu)(2\alpha+\mu)}{4\pi\mu^2}\right]^{1/2}\frac{e^{(k_n-\alpha)\bar{r}_p}}{2k_n(k_n-\alpha)}. \tag{39}$$

For the ratio of form factors one then finds

$$\frac{G(k_n,\bar{r}_p)}{F(k_n,\mathbf{r}_p)}\simeq\frac{e^{(k_n-\alpha)\bar{r}_p}\{(\alpha+\mu)^2-k_n^2\}}{2k_n(k_n-\alpha)V_0}. \tag{40}$$

The ratio ρ is presumably most interesting for the common case of electric quadrupole excitation: For $k_n\sim2{,}7\alpha$, $\mu=3\alpha$, $V_0=44{,}5$ MeV and $E_d/Z=(1/15)$ (in MeV), $J_f=\frac{1}{2}$ and $\bar{J}_f=\frac{5}{2}$ we obtain

$$\rho\simeq0.6B_{i'\to i}(E2), \tag{41}$$

where $B(E2)$ is expressed in units of $e^2\times10^{-48}$ cm^4. Clearly, in order to obtain measurable values of ρ one needs not only a "collective" target but also rather low deuteron energies.

We first consider as targets the so called even-even vibrational nuclei. Little is known about their odd neighbors, but it seems that some of their low-lying states indeed come close to a description of a single particle coupled to an excited core.[9] For $B_{i'\to i}(E2)$ values $\simeq0.2$ and $l_n=0$, ρ may become as big as 0.1–0.2. It is clear that the relative importance of $f^{(2)}$ increases with Z/E_d but absolute cross sections may fall below measurable values for too large values of this parameter.

A word on rotational nuclei is in order here, since the largest $B_{i'\to i}(E2)$ values are found for those nuclei. However, low-lying states of odd rotational nuclei are considered to represent an intrinsic structure which rotates as a whole and not as an odd nucleon coupled to an even-even core. The latter states might occur at much higher energies for which Eq. (41) does not hold. It would nevertheless be interesting to test by Coulomb stripping the purity of the strong coupling model, in particular for product nuclei with spin $\frac{1}{2}$.

Before closing this section we wish to point at a possible interference between the amplitudes $f^{(1)}$ and $f^{(2)}$ in particular in the quadrupole case. If the target nucleus has a ground state with $J_i\geqslant1$ then $f^{(2)}$ contributes also to transitions caused by pure Coulomb stripping, namely, those leading to a state in which a neutron is coupled to the *ground* state of the target parent. $f^{(2)}$ will then contain the reduced static quadrupole moment $Q(E2)$ instead of $B(E2)$. The possible change in the cross section might yield information on static quadrupole moments.

IV. POLARIZATION OF THE DEUTERON

Since for the deuteron the centers of mass and charge do not coincide, the asymmetric Coulomb field acting on the deuteron may, in principle, polarize the deuteron. The ensuing modification of the deuteron wave function can be approximately calculated following either the variational method of Downs[10] or the perturbation method of Ramsey *et al.*[11] The approximation is based

[9] A. de-Shalit, Phys. Rev. **122**, 1530 (1962).
[10] B. W. Downs, Phys. Rev., **98**, 194 (1955).
[11] N. F. Ramsey, B. J. Malenka, and U. E. Kruse, Phys. Rev. **91**, 1162 (1953).

on the fact that for $E_d \ll Ze^2/R_N$ the electric field is changing slowly in the region of the classical turning point while the local velocity of the deuteron is small. This enables the use of an adiabatic approximation according to which the modified ground state of the deuteron is to first order given by

$$\chi(r,r_p)=\chi(r)+\xi\chi_1(r)\,, \tag{42}$$

where $\xi=Ze/r_p^2$ is the local electric field acting on the proton. $\chi_1(r)$ for a Serber interaction of the Hulthén type with a range parameter μ, is given by

$$\chi_1(r)=g(r)\cos\theta_p\,, \tag{43}$$

where

$$g(r)=\frac{eM}{\hbar^2}\left[\frac{2\alpha(\alpha+\mu)(2\alpha+\mu)}{4\pi\mu^2}\right]^{1/2}\left\{\frac{1}{4\alpha}re^{-\alpha r}-\frac{1}{\mu^2(2\alpha+\mu)^2}\times\left[\frac{2(1+\alpha r)}{r^2}e^{-\alpha r}-\frac{2}{r^2}e^{-(\alpha+\mu)r}-\frac{2(\alpha+\mu)}{r}e^{-(\alpha+\mu)r}-\mu(2\alpha+\mu)e^{-(\alpha+\mu)r}\right]\right\}, \tag{44}$$

and $\cos\theta_p=(\mathbf{k}_d\cdot\mathbf{k}_p)/k_dk_p$.

If we now take $\chi(\mathbf{r},\mathbf{r}_p)$ for the intrinsic deuteron wave function instead of $\chi(r)$ in (23) we obtain for the Coulomb stripping cross section

$$\frac{d\sigma^{(1)}}{d\Omega}=\frac{M^2}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{(2J_f+1)}{(2J_i+1)}\times\sum|f_{\alpha_nl_nm_{ln}}{}^{(1)}+\Delta f_{\alpha_nl_nm_{ln}}{}^{(1)}|^2. \tag{45}$$

The additional amplitude $\Delta f^{(1)}$ is given by

$$\Delta f_{\alpha_nl_nm_{ln}}{}^{(1)}=-4\pi A_{\alpha_nl_n}(E_n)Ze^2\times\int_0^{\bar{r}_p}j_1(ik_nr)V_{np}(r)g(r)r^2dr\times\int h_1(ik_nr_p)\cos\theta_pY_{l_nm_{ln}}{}^*(\Omega_p)\times\psi_{\mathbf{k}_p}{}^{(-)*}(\mathbf{r}_p)\psi_{\mathbf{k}_d}{}^{(+)}(\mathbf{r}_p)r_p{}^{-2}d\mathbf{r}_p. \tag{46}$$

We shall use the previous arguments to replace $h_1(ik_nr_p)$ by its asymptotic behavior $ie^{-k_nr_p}/k_nr_p$ and take $r_p{}^{-2}$ out of the integral at the point $r_p=\bar{r}_p$. In addition, we can assume the integrand to be peaked backwards, and may thus substitute for the slowly varying function $\cos\theta$ the value -1. The differential cross section, now taking into account the deuteron polarization, reduces then to the form

$$\frac{d\sigma^{(1)}}{d\Omega}=\frac{d\sigma^{(1)}}{d\Omega}\left\{1+\frac{\Delta F(k_n\bar{r}_p)}{F(k_n,\bar{r}_p)}\right\}^2, \tag{47}$$

where $(d\sigma^{(1)}/d\Omega)$ is given by (27) and F equals

$$i\Delta F(k_n\bar{r}_p)=-4\pi\frac{Ze}{\bar{r}_p{}^2}\int_0^{\bar{r}_p}j_1(ik_nr)V_{np}(r)g(r)r^2dr. \tag{48}$$

One first concludes that the polarization of the deuteron does not change the angular distribution of the protons. Further it turns out that the ratio $\Delta F/F$ for both Yukawa and Hulthén potentials is only of the order of a few percents. Therefore, also the absolute value of the cross section is not appreciably affected. It can be shown in the same way that this correction is not important in the case of an accompanying Coulomb excitation either.

V. CONCLUSION

We have presented above an approximate theory of Coulomb stripping. The underlying direct reaction mechanism is the same as in normal stripping. Since, however, in Coulomb stripping the effective interaction area is relatively far outside the nuclear surface, particular approximations could be introduced. These lead to an analytical expression for the cross section. The characteristic backward distribution of outgoing protons appears unaffected by the retention of a proton-neutron interaction of finite range. Stripping form factors on the other hand are appreciably modified and now account for discrepancies found on comparing experiments with a zero-range interaction picture.

We have to admit that there is still lacking a more satisfactory proof, showing that the approximations hold to order deuteron radius over the distance to classical turning point. Work in this direction is in progress.

The second process considered is Coulomb stripping associated with Coulomb excitation, leading to an interesting new class of excited final states describing a neutron coupled to a Coulomb excited target. It appears that under favorable circumstances the latter process may compete with Coulomb stripping.

We mention that appreciable cross sections can be obtained for this type of neutron transfer reactions if heavy ions are used instead of deuterons. The theory given applies to them as well.

It has further been shown that the polarizability of the deuteron is of minor importance. In particular, the Oppenheimer-Phillips processes can be neglected.

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APPENDIX

In order to prove Eq. (33′) from (33) we first write the relevant sum by means of $3-j$ symbols as follows:

$$S=(2j_n+1)(2J_f+1)(2J_i'+1)\sum_{\substack{\text{magnetic quantum}\\ \text{numbers}}}(-)^{m_n'-m_n+M_i''-M_i'}$$
$$\times\begin{pmatrix}\frac{1}{2} & l_n & j_n\\ m_{s_n} & m_{l_n} & -m_n\end{pmatrix}\begin{pmatrix}\frac{1}{2} & l_n & j_n\\ m_{s_n} & m_{l_n}' & -m_n'\end{pmatrix}\begin{pmatrix}j_n & J_i' & J_f\\ m_n & M_i' & -M_f\end{pmatrix}\begin{pmatrix}j_n & J_i' & J_f\\ m_n' & M_i' & -M_f\end{pmatrix}$$
$$\times\begin{pmatrix}J_i & \lambda & J_i'\\ M_i & \mu & -M_c'\end{pmatrix}\begin{pmatrix}J_i & \lambda & J_i'\\ M_i & \mu' & -M_i''\end{pmatrix}f_{\alpha_nl_nm_{l_n}}{}^{(2)}(\lambda\mu)f_{\alpha_nl_nm_{l_n}'}{}^{(2)*}(\lambda\mu'). \quad \text{(A1)}$$

We shall frequently apply the formula

$$\sum_{m_3}\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix}l_1 & l_2 & j_3\\ n_1 & n_2 & -m_3\end{pmatrix}=\sum_{l_3n_3}(-)^{j_3+l_3+m_1+n_1}(2l_3+1)$$
$$\times\begin{Bmatrix}j_1 & j_2 & j_3\\ l_1 & l_2 & l_3\end{Bmatrix}\begin{pmatrix}l_1 & j_2 & l_3\\ n_1 & m_2 & n_3\end{pmatrix}\begin{pmatrix}j_1 & l_2 & l_3\\ m_1 & n_2 & -n_3\end{pmatrix} \quad \text{(A2)}$$

and any modification thereof applying the orthogonality relation for the $3-j$ symbols.

By virtue of the symmetry relations of $3-j$ symbols and Eq. (A2) we can cast Eq. (A1) in the form

$$S=(2j_n+1)(2J_f+1)(2J_i'+1)\sum_{\substack{\text{magnetic quantum}\\ \text{numbers}}}(-)^{J_f-J_i-\frac{1}{2}+m_{l_n}'+\mu'+\alpha}(2k+1)\begin{pmatrix}l_n & l_n & k\\ m_{l_n} & -m_{l_n}' & -\alpha\end{pmatrix}\begin{pmatrix}\lambda & \lambda & k\\ \mu & -\mu' & \alpha\end{pmatrix}$$
$$\times\begin{Bmatrix}J_i' & J_n & J_f\\ j_n & J_i' & k\end{Bmatrix}\begin{Bmatrix}\lambda & J_i' & J_i\\ J_i' & \lambda & k\end{Bmatrix}\begin{Bmatrix}l_n & j_n & \frac{1}{2}\\ j_n & l_n & k\end{Bmatrix}f_{\alpha_nl_nm_{l_n}}{}^{(2)}(\lambda\mu)f_{\alpha_nl_nm_{l_n}'}{}^{(2)*}(\lambda\mu'). \quad \text{(A3)}$$

It is at this point that we use the saddle-point approximation, whose content is [compare Eq. (20)]

$$f_{\alpha_nl_nm_{ln}}{}^{(2)}(\lambda\mu)=-Y_{l_nm_n}(\bar{\Omega}_p)Y_{\lambda\mu}(\bar{\Omega}_p)A_{\alpha_nl_n}(E_n)I_\lambda(\theta_p)G(k_n,\bar{r}_p). \quad \text{(A4)}$$

We now use the coupling formula for spherical harmonics

$$\sum_{\mu'}\begin{pmatrix}l_n & \lambda & k'\\ m_{l_n} & \mu' & \alpha'\end{pmatrix}Y_{l_nm_{l_n}}(\bar{\Omega}_p)Y_{\lambda\mu'}(\bar{\Omega}_p)=(-)^{\alpha'+k'+l_n}\left[\frac{(2l_n+1)(2\lambda+1)}{4\pi(2k'+1)}\right]^{1/2}\begin{pmatrix}k' & l_n & \lambda\\ 0 & 2 & 0\end{pmatrix}Y_{k'-\alpha'}(\bar{\Omega}_p). \quad \text{(A5)}$$

A last relation, the addition formula for Y's of equal solid angles reads

$$\sum_{\alpha'}(-)^{\alpha'}Y_{k'\alpha'}(\bar{\Omega}_p)Y_{k'-\alpha'}(\bar{\Omega}_p)=(2k'+1)/4\pi. \quad \text{(A6)}$$

One now substitutes (A4) into (A3) and applies Eqs. (A5) and (A6) in order to obtain

$$S=(2j_n+1)(2J_f+1)(2J_i'+1)|A_{\alpha_nl_n}(E_n)|^2|I_\lambda(\theta_p)|^2|G(k_n,\bar{r}_p)|^2$$
$$\times\sum_{k,k'}(-)^{J_f-J_i-\frac{1}{2}+k+\lambda+l_n}\frac{(2k+1)(2k'+1)(2l_n+1)(2\lambda+1)}{16\pi^2}$$
$$\times\begin{pmatrix}k' & l_n & \lambda\\ 0 & 0 & 0\end{pmatrix}^2\begin{Bmatrix}\lambda & \lambda & k\\ l_n & l_n & k'\end{Bmatrix}\begin{Bmatrix}\lambda & \lambda & k\\ J_i' & J_i' & J_i\end{Bmatrix}\begin{Bmatrix}l_n & l_n & k\\ j_n & j_n & \frac{1}{2}\end{Bmatrix}\begin{Bmatrix}j_n & j_n & k\\ J_i' & J_i' & J_f\end{Bmatrix}. \quad \text{(A7)}$$

Repeated application of (A2) and symmetry relations finally leads to (33′).