Three-Body Problem with Separable Potentials. II. *n-d* Scattering*

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(Received 29 March 1963)

The three-body formalism, using separable potentials in s and p states of nucleon pairs, is set up under the conditions of full antisymmetrization of the three-nucleon wave function. The formalism is applied to the problem of n–d scattering, where the effects of "polarization" are fully taken into account. The amplitudes for quartet and doublet scattering are found to satisfy two- and four-coupled one-dimensional integral equations, respectively. The quartet scattering length is found to agree with the figure for the so-called experimental set I for this parameter. The doublet scattering length, on the other hand, is found to be much more sensitive to the details of the potentials, thus preventing a theoretical resolution of its ambiguity within the present formalism without tensor forces.

I. INTRODUCTION

SCATTERING of nucleons by deuterons has been used by many authors as a means of understanding two-body forces in greater detail than two-body scattering data can provide. For, as is generally believed, the validity of a two-body potential can be assessed only through a simultaneous study of scattering of two particles on and off the energy shell. A three-particle system is the smallest unit whose physical parameters depend on the two-body "potential" in a more or less sensitive fashion. In a recent paper by one of us[1] this idea was discussed in some detail in the context of a *bound* three-body problem—the triton. In the same way, the study of a three-body scattering state, e.g., an n-d system, provides a complementary tool for obtaining information on the validity of a given two-body potential.

A major obstacle to translating these ideas into a practical program of calculations has always been the need to resort to various approximations (with effects of unknown magnitudes) in dealing with three-body systems. For example, a standard approximation used in the calculation of n-d scattering is the so-called "no-polarization" approximation, in which the distortion of the deuteron structure due to the projectile is neglected. Mixed feelings have been expressed by various authors on this approximation,[2] the common belief being that a proper antisymmetrization of the complete three-body wave function makes the error due to the "no-polarization" approximation much smaller than if such antisymmetrization is not carried out. No accurate estimate of this error has, however, been made with the conventional static potentials (Yukawa, Gaussian, etc.) that have so far been used to calculate n-d scattering.

Presumably as a result of this and other approximations, it has so far not been possible to resolve theoretically the ambiguity between the two famous sets of scattering lengths, both of which seem to fit the experimental data on n-d scattering at low energies.[2]

$$\text{(I)}\quad a_{3/2}=6.4\pm0.3\text{ F},\quad a_{1/2}=0.7\pm0.3\text{ F};$$
$$\text{(II)}\quad a_{3/2}=2.6\pm0.3\text{ F},\quad a_{1/2}=8.3\pm0.3\text{ F}.$$

Thus, the calculations of Troesch and Verde,[3] Gordon,[4] Delves and Brown,[5] using various approximations (among them the no-polarization approximation) have all tended to favor set II. On the other hand, the calculations of Christian and Gammel[6] and Haas and Robertson[7] produced evidence in favor of set I. However, these last authors[7] who worked with a Yukawa interaction, were also able to show that their results were so sensitive to approximations, that the neglect of the long-range tail of the kernel of their integrodifferential equation could shift the values of the scattering lengths almost all the way to set II.

Studies of the effect of distortion of the deuteron, using variational procedures, have also led to widely different conclusions by various authors. Thus, while Sartori and Rubinow,[8] and Burke and Haas[9] found negligible effects, Effimov[10] observed an effect as high as 50% for $a_{1/2}$, and somewhat less for $a_{3/2}$.[11] All these authors worked with Gaussian potentials. It is entirely

* Supported in part by the National Science Foundation.

† On leave of absence from Delhi University, Delhi, India, during the year 1962–63.

[1] A. N. Mitra, Nucl. Phys. **32**, 529 (1962); referred to as A.

[2] See, for example, H. S. W. Massey, in *Proceedings of the International Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959*, edited by T. C. Griffith and E. A. Power (Pergamon Press Inc., New York, 1960), Vol. II, p. 345.

[3] A. Troesch and M. Verde, Helv. Phys. Acta **24**, 39 (1951).

[4] M. M. Gordon, Phys. Rev. **80**, 1111 (1950).

[5] L. M. Delves and D. Brown, Nucl. Phys. **11**, 432 (1959).

[6] R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).

[7] F. A. Haas and H. H. Robertson, Proc. Phys. Soc. (London) **A73**, 160 (1959).

[8] L. Sartori and S. I. Rubinow, Phys. Rev. **112**, 214 (1958).

[9] P. G. Burke and F. A. Haas, Proc. Roy. Soc. (London) **A252**, 177 (1959).

[10] V. N. Efimov, Zh. Eksperim. i Teor. Fiz. **35**, 137 (1958) [translation: Soviet Phys.—JETP **8**, 98 (1959)].

[11] It may be noted that for $a_{3/2}$ the only effect of distortion can be felt through an interaction in a triplet odd state, particularly $3p$, which is believed to be much weaker than a triplet even force.

possible that a potential with a longer tail could produce further ambiguities, as the authors of Ref. 7 have found (see last paragraph). As a matter of fact, the range of the potential considered might conceivably play an important role in the understanding of all these conflicting results.

There is very little purely threoretical material available which might help to resolve these ambiguities on scattering lengths. Davidov and Filipov[12] had suggested that since in the quartet state of *n-d*, the neutron cannot fall inside the deuteron (exclusion principle), the "effective potential" for the quartet *n-d* state should be (a repulsive) one of range *larger* than the deuteron radius of 4.3 F—an argument which allows only set I. A more formal method of analysis was given by Spruch and Rosenberg[13] who were able to obtain rigorous upper bounds on the scattering lengths of neutrons by nuclei for a specified total angular momentum J. A calculation on these lines by Sartori and Rubinow[14] produced a result for $a_{3/2}$ in accord with the one given in set I. However, since the theory of Ref. 13 is valid only for cases where there are no bound states of the compound system, it does not apply to the doublet *n-d* state and, hence, cannot *independently* resolve the ambiguity in $a_{1/2}$.

Such ambiguities certainly illustrate the fact that three-body parameters are rather sensitive to approximations, and that the interpretation of results is at least partly obscured by the effect of approximations made in the calculations. Since, however, the magnitude of the three-body problem with conventional potentials leaves little alternative to approximations, the other possibility is perhaps to look into the problem with simplified interactions which at least would not allow approximations to stand in the way of interpretation of the results. Such an approach, using the so-called separable potentials, was advocated by one of us[1] in connection with the problem of H^3, where it was shown that a three-body problem could be algebraically reduced in an *exact* fashion to an equivalent two-body problem, at which stage a simple numerical calculation would suffice. It is, therefore, clear that a similar simplification would obtain for the problem of *n-d* scattering if such potentials are used, so that the effects of various approximations (mentioned in the last few paragraphs) can be tested against the exact solution that this alternative description can provide.

The present investigation for *n-d* scattering is a direct continuation of the program for studying three-body systems which was started with the case of bound states.[1] It may be noted that the *n-d* system is one in which both bound (H^3) and scattering states are involved. The case of all three nucleons in scattering states (e.g., in photodisintegration of H^3) represents a third possible three-body system which can, in principle, provide additional information on the consistency of two-body forces. However, this case will not be considered in this paper.

The formulation of the problem of *n-d* scattering is made here with a complete antisymmetrization of the wave function, including the effect of isospin—an improvement over the treatment of Ref. 1 where isospin effects were neglected. The $N-N$ interaction considered here is assumed to operate in s- and p-states. The formalism with tensor forces becomes considerably involved when isospin effects are included, and as such will be the subject of a subsequent publication.

In Sec. 2, the coupled Schrödinger equations for the spatial parts of the three-nucleon wave function for the two cases of $S=\frac{3}{2}$ and $S=\frac{1}{2}$ are obtained through a generalization of Verde's[15] treatment so as to include potentials of unequal ranges for the different $N-N$ states. In Sec. 3, the explicit structures of the appropriate symmetry for these spatial wave functions are obtained through the use of the assumed separable interactions in s and p states. These structures are expressible in terms of certain single-variable functions (one for each variety of $N-N$ interaction assumed), which represent the "two-body wave functions" characteristic of the various channels through which the *n-d* scattering can take place. Numerical values of the quartet and doublet scattering lengths are presented in Sec. 4, along with a discussion of the "polarization effects" due to the various channels.

2. THREE-BODY FORMALISM WITH ARBITRARY POTENTIAL SHAPES

In this section we shall obtain the coupled Schrödinger equations for the spatial components of the three-body wave function for the two cases of $S=\frac{3}{2}$ and $S=\frac{1}{2}$, after eliminating their spin and isospin components, following the techniques of Verde's article[15] referred to as B. The corresponding equations of B will, however, be generalized to include arbitrary shapes of the potentials in the odd and even states of triplet and singlet interactions, instead of taking equal shapes for all of them. As will turn out in Sec. 3, this generalization does not bring about any additional complication in the structure of the equivalent two-body equations of the type obtained in A, and as such leaves a wider scope for studying three-body systems with realistic separable potentials which fit two-body data, than might be possible with the limitation of equal shapes for the different potentials. The notations of B for the various spin and isospin

[12] A. S. Davidov and F. G. Fillippov, Zh. Eksperim. i Teor. Fiz. **31**, 340 (1956) [translation: Soviet Phys.—JETP 4, 257 (1957)].

[13] L. Spruch and L. Rosenberg, in *Proceedings of the International Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959*, edited by T. C. Griffith and E. A. Power (Pergamon Press Inc., New York, 1960), Vol. II, p. 375.

[14] S. I. Rubinow and L. Sartori, see Ref. 13, in *Proceedings of the International Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959*, edited by T. C. Griffith and E. A. Power (Pergamon Press Inc., New York, 1960), Vol. II, p. 385.

[15] M. Verde, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 170; referred to as B.

functions will, however, be frequently used in this section. The complete two-body potential in s and p states, but without tensor or spin-orbit parts, is taken as (cf., B)

$$\begin{aligned}M(\mathbf{p}|V|\mathbf{p}')&=-\lambda_{31}g(p)g(p')P_\sigma^+P_\tau^--\lambda_{13}f(p)f(p')P_\sigma^-P_\tau^+\\&\quad-3\lambda_{11}v(p)v(p')\mathbf{p}\cdot\mathbf{p}'P_\sigma^-P_\tau^-\\&\quad-3\lambda_{33}u(p)u(p')\mathbf{p}\cdot\mathbf{p}'P_\sigma^+P_\tau^+.\end{aligned}\tag{2.1}$$

The shapes of these potentials will be discussed in Sec. 4. The projection operators $P_{\sigma,\tau}^{\pm}$ are as defined in B.

For the three-nucleon system let the momenta of the particles be $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, such that

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0,\tag{2.2}$$

$$2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j,\quad \mathbf{P}_i+\mathbf{P}_j=-\mathbf{P}_k,\tag{2.3}$$

and

$$(\mathbf{P}_i\mathbf{P}_j|V|\mathbf{P}_i'\mathbf{P}_j')=\delta^3(\mathbf{P}_k-\mathbf{P}_k')(\mathbf{p}_{ij}|V|\mathbf{p}_{ij}').\tag{2.4}$$

According to B, the wave function $\Psi(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3)$ has the following structures for the cases $S=\frac{3}{2}$ and $S=\frac{1}{2}$ respectively, of the $n-d$ system.

$$\Psi_{3/2}=(\psi'\zeta''-\psi''\zeta')\chi^s,\tag{2.5}$$

$$\Psi_{1/2}=\psi^a\xi^s-\psi^s\xi^a+\psi'\xi''-\psi''\xi',\tag{2.6}$$

where χ, ζ, ξ are the spin, isospin, and spin-isospin functions, respectively, of the three particles, with the symmetries indicated by their superscripts, in the notation of B.[16] Now the "master" equation for Ψ is

$$D(E)\Psi=-\sum_{ijk}MV(ij)\Psi_k(ij),\tag{2.7}$$

where

$$D(E)=\tfrac{1}{2}(P_1^2+P_2^2+P_3^2)-ME,\tag{2.8}$$

and the following operator notation is used for the right-hand side:

$$V(ij)\Psi_k(ij)=\int(\mathbf{P}_i\mathbf{P}_j|V|\mathbf{P}_i'\mathbf{P}_j')\times\Psi(P_i',P_j',P_k)d\mathbf{P}_i'd\mathbf{P}_j'.\tag{2.9}$$

Ψ_k is expressed here as a function of $\mathbf{P}_i$ and $\mathbf{P}_j$ alone, via (2.3). The next step is to substitute the forms (2.5) or (2.6) for Ψ in (2.7) and eliminate the spin and isospin functions to obtain the coupled integral equations for the spatial functions in the two respective cases of $S=\frac{3}{2}$ and $S=\frac{1}{2}$. To write the resulting equations in a compact form, the following notation is introduced.

[16] ψ^s and ψ^a are, respectively, the totally symmetric and totally antisymmetric spatial parts of the wave function; ψ' and ψ'' are the spatial functions of the so-called "mixed symmetry" [2,1], ψ' being antisymmetric with respect to the basis momenta $\mathbf{P}_2$ and $\mathbf{P}_3$, and ψ'' symmetric in them. The superscripts on the spin and isospin functions have identical significance (in the basis representation provided by particles 2 and 3). The spin-isospin functions ξ' and ξ'' are expressible in terms of the pure spin and isospin functions χ and ζ as $\xi'=2^{-1/2}(\chi'\zeta''+\chi''\zeta')$ and $\xi''=2^{-1/2}(\chi'\zeta'-\chi''\zeta'')$.

Let the operator $\Lambda_{31}(ij)$ be the first term of $-MV(ij)$ defined by (2.1), so that

$$(\mathbf{p}_{ij}|\Lambda_{31}|\mathbf{p}_{ij}')=\lambda_{31}g(p_{ij})g(p_{ij}'),\tag{2.10}$$

with similar definitions for Λ_{13}, Λ_{11}, and Λ_{33}. Further, let the operators Λ^s, Λ', and Λ'' (for each of Λ_{31}, Λ_{13}, Λ_{11}, Λ_{33}) be defined as follows:

$$\begin{aligned}\Lambda^s&=\Lambda(12)+\Lambda(31)+\Lambda(23),\\\Lambda'&=\tfrac{1}{2}3^{1/2}(\Lambda(12)-\Lambda(31)),\\\Lambda''&=-\Lambda(23)+\tfrac{1}{2}(\Lambda(12)+\Lambda(31)).\end{aligned}\tag{2.11}$$

The effect of Λ^s on a particular spatial wave function ψ will be understood to mean the following:

$$\Lambda^s\psi=\Lambda(23)\psi_1(23)+\Lambda(31)\psi_2(31)+\Lambda(12)\psi_1(12),\tag{2.12}$$

where each term of (2.12) is interpreted in the sense of Eq. (2.9). Similar definitions hold for $\Lambda'\psi$ and $\Lambda''\psi$. Using these notations, elimination of the spin and isospin components from both sides of Eq. (2.7) leads finally to the following coupled equations for the two cases of $S=\frac{3}{2}$ and $S=\frac{1}{2}$, respectively.

(a) $S=\frac{3}{2}$

$$D(E)\psi'=\tfrac{1}{2}(\Lambda_{31}{}^s+\Lambda_{33}{}^s)\psi'+\tfrac{1}{2}(\Lambda_{31}'-\Lambda_{33}')\psi''+\tfrac{1}{2}(\Lambda_{31}''-\Lambda_{33}'')\psi',\tag{2.13a}$$

$$D(E)\psi''=\tfrac{1}{2}(\Lambda_{31}{}^s+\Lambda_{33}{}^s)\psi''+\tfrac{1}{2}(\Lambda_{31}'-\Lambda_{33}')\psi'-\tfrac{1}{2}(\Lambda_{31}''-\Lambda_{33}'')\psi'';\tag{2.13b}$$

(b) $S=\frac{1}{2}$

$$D(E)\psi^a=\tfrac{1}{2}(\Lambda_{11}{}^s+\Lambda_{33}{}^s)\psi^a+\tfrac{1}{2}(\Lambda_{11}'\psi''-\Lambda_{11}''\psi')-\tfrac{1}{2}(\Lambda_{33}'\psi''-\Lambda_{33}''\psi'),\tag{2.14}$$

$$D(E)\psi^s=\tfrac{1}{2}(\Lambda_{31}{}^s+\Lambda_{13}{}^s)\psi^s+\tfrac{1}{2}(\Lambda_{31}'\psi'+\Lambda_{31}''\psi'')-\tfrac{1}{2}(\Lambda_{13}'\psi'+\Lambda_{13}''\psi'');\tag{2.15}$$

$$\begin{aligned}D(E)\psi'&=\tfrac{1}{2}(\Lambda_{31}'-\Lambda_{13}')\psi^s+\tfrac{1}{2}(\Lambda_{33}''-\Lambda_{11}'')\psi^a\\&\quad+\tfrac{1}{4}(\Lambda_{31}{}^s+\Lambda_{13}{}^s+\Lambda_{11}{}^s+\Lambda_{33}{}^s)\psi'\\&\quad+\tfrac{1}{4}(\Lambda_{31}''+\Lambda_{13}''-\Lambda_{11}''-\Lambda_{33}'')\psi'\\&\quad+\tfrac{1}{4}(\Lambda_{31}'+\Lambda_{13}'-\Lambda_{11}'-\Lambda_{33}')\psi'',\end{aligned}\tag{2.16a}$$

$$\begin{aligned}D(E)\psi''&=\tfrac{1}{2}(\Lambda_{31}''-\Lambda_{13}'')\psi^s+\tfrac{1}{2}(\Lambda_{11}'-\Lambda_{33}')\psi^a\\&\quad+\tfrac{1}{4}(\Lambda_{31}{}^s+\Lambda_{13}{}^s+\Lambda_{11}{}^s+\Lambda_{33}{}^s)\psi''\\&\quad+\tfrac{1}{4}(\Lambda_{31}'+\Lambda_{13}'-\Lambda_{11}'-\Lambda_{33}')\psi'\\&\quad-\tfrac{1}{4}(\Lambda_{31}''+\Lambda_{13}''-\Lambda_{11}''-\Lambda_{33}'')\psi''.\end{aligned}\tag{2.16b}$$

These forms bring out explicitly the permutation symmetry structures of the various spatial wave functions that are involved in each of the quartet and doublet cases, and represent the appropriate generalizations of Eqs. (2.7) and (2.8) of B for the case of unequal potential shapes. From these equations it is now possible to read off the algebraic structures of the various ψ's by using the contents of Eqs. (2.9)–(2.12), so as to maintain the correct symmetries implied by their notations. This reduction is carried out in Sec. 3 separately for the two cases of $S=\frac{3}{2}$ and $S=\frac{1}{2}$.

3. REDUCTION TO EQUIVALENT TWO-BODY EQUATIONS

We discuss first the case of $S=\frac{3}{2}$. From Eqs. (2.13) it is easy to deduce that the following functional forms of ψ' and ψ'' are the only ones compatible with their correct symmetry, as well as the interaction (2.1).

$$D(E)\psi' = \tfrac{1}{2}\sqrt{3}(g(p_{12})G(P_3)-g(p_{31})G(P_2)) + \sqrt{3}u(p_{23})(\mathbf{p}_{23}\cdot\mathbf{P}_1)H(P_1)-\tfrac{1}{2}\sqrt{3}(u(p_{12})\mathbf{p}_{12}\cdot\mathbf{P}_3H(P_3) + u(p_{31})\mathbf{p}_{31}\cdot\mathbf{P}_2H(P_2)), \quad (3.1)$$

$$D(E)\psi'' = -g(p_{23})G(P_1)+\tfrac{1}{2}(g(p_{12})G(P_3)+g(p_{31})G(P_2)) + \tfrac{3}{2}(u(p_{12})\mathbf{p}_{12}\cdot\mathbf{P}_3H(P_3)-u(p_{31})\mathbf{p}_{31}\cdot\mathbf{P}_2H(P_2)). \quad (3.2)$$

We have introduced only two scalar functions G and H, one for each type of interaction allowed for the $S=\frac{3}{2}$ state, viz., Λ_{31} and Λ_{33}. Substituting these forms back into Eqs. (2.13), the functions G and H are found to be given self consistently by the following equations:

$$G(P_1)=-\lambda_{31}\int d\mathbf{p}_{23}'g(p_{23}')\psi_1''(23'), \quad (3.3)$$

$$\mathbf{p}_{23}\cdot\mathbf{P}_1H(P_1)=\sqrt{3}\lambda_{33}\int d\mathbf{p}_{23}'u(p_{23}')\mathbf{p}_{23}\cdot\mathbf{p}_{23}'\psi_1'(23'), \quad (3.4)$$

where $\psi_1'(23')$ and $\psi_1''(23')$ are the forms of (3.1) and (3.2) when $\mathbf{P}_1$ is eliminated by the use of Eq. (2.2), and the arguments $(23')$ are abbreviations for the momenta $\mathbf{P}_2'$, $\mathbf{P}_3'$. Substitution of Eqs. (3.1) and (3.2) in (3.3) and (3.4) leads finally to the equations

$$(\lambda_{31}^{-1}-h_{31}(P))G(P) = -\int d\mathbf{q}g(\mathbf{q}+\tfrac{1}{2}\mathbf{P})g(\mathbf{P}+\tfrac{1}{2}\mathbf{q})D^{-1}(\mathbf{P},\mathbf{q})G(q) - 3\int d\mathbf{q}g(\mathbf{q}+\tfrac{1}{2}\mathbf{P})\mathbf{q}\cdot(\mathbf{P}+\tfrac{1}{2}\mathbf{q}) \times u(\mathbf{P}+\tfrac{1}{2}\mathbf{q})D^{-1}(\mathbf{P},\mathbf{q})H(q), \quad (3.5)$$

$$(\lambda_{33}^{-1}-h_{33}(P))PH(P) = -3\int d\mathbf{q}D^{-1}(\mathbf{P},\mathbf{q})u(\mathbf{q}+\tfrac{1}{2}\mathbf{P})\hat{P}\cdot(\mathbf{q}+\tfrac{1}{2}\mathbf{P})g(\mathbf{P}+\tfrac{1}{2}\mathbf{q})G(q) + 3\int d\mathbf{q}D^{-1}(\mathbf{P},\mathbf{q})u(\mathbf{q}+\tfrac{1}{2}\mathbf{P})\hat{P}\cdot(\mathbf{q}+\tfrac{1}{2}\mathbf{P}) \times\hat{q}\cdot(\mathbf{P}+\tfrac{1}{2}\mathbf{q})u(\mathbf{P}+\tfrac{1}{2}\mathbf{q})qH(q), \quad (3.6)$$

where

$$h_{31}(P)=\int d\mathbf{q}g^2(q)(\tfrac{3}{4}P^2+q^2-EM)^{-1}, \quad (3.7)$$

$$h_{33}(P)=\int d\mathbf{q}u^2(q)q^2(\tfrac{3}{4}P^2+q^2-EM)^{-1}, \quad (3.8)$$

and

$$D(\mathbf{P},\mathbf{q})=P^2+q^2+\mathbf{P}\cdot\mathbf{q}-EM. \quad (3.9)$$

Equations (3.5) and (3.6) admit of a simple interpretation. $G(P)$ describes the wave function of the neutron with respect to the actual deuteron state, viz., $(np)\ {}^3S_1$. On the other hand, $H(P)$ represents the polarization effect due to the ${}^3P$ interaction, viz., the wave function of the proton with respect to the $(2n)\ {}^3P$ state (as a result of the replacement of the proton in H^2 by the incident neutron). The exchange of the target neutron by the incident one is, of course, automatically taken care of by the function $G(P)$, due to total antisymmetrization of the wave function. Neglect of $H(P)$ would, therefore, amount merely to a neglect of the virtual effect of the channel $p+(2n)\ {}^3P$ on $n+d$ scattering. We note in this connection that the triplet odd potential is generally believed to be much weaker than the others,[17–19] and further, that for a calculation of the (zero-energy) scattering length from Eqs. (3.5) and (3.6), the centrifugal barrier ($l=1$) is expected to reduce the virtual effect of $p+(2n)\ {}^3P$ on $n+d$ scattering considerably. With this understanding we shall ignore this particular polarization effect by setting $H=0$ in Eq. (3.5), for purposes of numerical calculations to be discussed in Sec. 4.

The case of $S=\frac{1}{2}$ is more involved than $S=\frac{3}{2}$, in that all the four interactions Λ_{31}, Λ_{13}, Λ_{11}, and Λ_{33} appear in it. By analogy with $S=\frac{3}{2}$, we now expect four independent functions to appear in the coupled integral equations, but in view of what has been said above for Λ_{33}, we may drop this term from the beginning. We do not, however, take this liberty with Λ_{11}, since this force, which is much less understood, may or may not be small. We, therefore, work only with the interactions Λ_{31}, Λ_{13}, and Λ_{11}, and obtain the following structures for the various ψ's through Eqs. (2.15) and (2.16).

$$D(E)\psi^a=\sum_{ijk}v(p_{ij})\mathbf{p}_{ij}\cdot\mathbf{P}_kH_a(P_k), \quad (3.10)$$

$$D(E)\psi^s=\sum_{ijk}(g(p_{ij})G_s(P_k)-f(p_{ij})F_s(P_k)), \quad (3.11)$$

$$D(E)\psi' = \tfrac{1}{2}\sqrt{3}(g(p_{12})G(P_3)-g(p_{13})G(P_2)) + (g\rightarrow f, G\rightarrow F)+\sqrt{3}v(p_{23})\mathbf{p}_{23}\cdot\mathbf{P}_1H(P_1) - \tfrac{1}{2}\sqrt{3}(v(p_{12})\mathbf{p}_{12}\cdot\mathbf{P}_3H(P_3) + v(p_{31})\mathbf{p}_{31}\cdot\mathbf{P}_2H(P_2)), \quad (3.12)$$

$$D(E)\psi'' = -g(p_{23})G(P_1)+\tfrac{1}{2}(g(p_{12})G(P_3)+g(p_{31})G(P_2)) + (g\rightarrow f, G\rightarrow F)+\tfrac{3}{2}(v(p_{12})\mathbf{p}_{12}\cdot\mathbf{P}_3H(P_3) - v(p_{31})\mathbf{p}_{31}\cdot\mathbf{P}_2H(P_2)). \quad (3.13)$$

The symbols $(g\rightarrow f, G\rightarrow F)$ in Eqs. (3.12) and (3.13) signify the presence of additional terms of identical

[17] This is true, e.g., of the Gammel-Thaler potential (Ref. 18). and the same was found to be the case by Mitra and Naqvi (Ref. 19), using separable potentials in the $T=1$ state.

[18] J. Gammel and R. Thaler, Phys. Rev. **107**, 291 (1957).

[19] A. N. Mitra and J. H. Naqvi, Nucl. Phys. **25**, 307 (1961).

structure, with the replacements indicated. We have here introduced two sets of functions G_s, F_s, H_a and G, F, H.[20] However, we expect these two sets to be related, in so far as the permutation symmetries of all the four ψ's are coupled in an essential manner. Indeed, by substituting Eqs. (3.10)–(3.13) back into Eqs. (2.14)–(2.16), and proceeding as before, we find in the same notation as in Eqs. (3.3) and (3.4),

$$G_s(P_1)\equiv G(P_1)$$

$$=\tfrac{1}{2}\lambda_{31}\int d\mathbf{p}_{23}'g(p_{23}')(\psi_1{}^s(23')-\psi_1''(23')), \tag{3.14}$$

$$F_s(P_1)\equiv F(P_1)$$

$$=-\tfrac{1}{2}\lambda_{13}\int d\mathbf{p}_{23}'f(p_{23}')(\psi_1{}^s(23')+\psi_1''(23')),$$

$$\mathbf{p}_{23}\cdot\mathbf{P}_1H_a(P_1)\equiv\sqrt{3}\mathbf{p}_{23}\cdot\mathbf{P}_1H(P_1) \tag{3.15}$$

$$=\tfrac{3}{2}\lambda_{11}\int d\mathbf{p}_{23}'v(p_{23}')\mathbf{p}_{23}\cdot\mathbf{p}_{23}'(\psi_1'(23')+\psi_1{}^a(23')). \tag{3.16}$$

Thus, as expected, there are only *three* independent functions (using the interactions Λ_{31}, Λ_{13}, and Λ_{11}), giving strong support to the conjecture that there is just one single-variable function for each variety of interaction that appears in the complete separable potential. Moreover, the assumption of different shapes for the different varieties of interaction leaves the basic structure of just as simple or as complicated, as the assumption of equal shapes would give. To obtain the coupled integral equations for G, F, H in a compact form, we introduce the following alternative notations:

$$G(P),F(P),PH(P)=\{G_\alpha(P)\},\quad \alpha=1,2,3; \tag{3.17}$$

$$\lambda_{31},\lambda_{13},\lambda_{11}=\{\lambda_\alpha\},\quad \alpha=1,2,3; \tag{3.18}$$

$$h_{31}(P),h_{13}(P),h_{11}(P)=\{h_\alpha(P)\},\quad \alpha=1,2,3, \tag{3.19}$$

where

$$h_{13}(P)=\int d\mathbf{q}f^2(q)(\tfrac{3}{4}P^2+q^2-ME)^{-1}, \tag{3.20}$$

$$h_{11}(P)=\int d\mathbf{q}v^2(q)q^2(\tfrac{3}{4}P^2+q^2-ME)^{-1}. \tag{3.21}$$

Using these notations, the coupled integral equations are

$$[\lambda_\alpha{}^{-1}-h_\alpha(P)]G_\alpha(P)$$

$$=\sum_\beta\int d\mathbf{q}D^{-1}(\mathbf{P},\mathbf{q})K_{\alpha\beta}(\mathbf{P},\mathbf{q})G_\beta(q), \tag{3.22}$$

where $\{K_{\alpha\beta}(\mathbf{P},\mathbf{q})\}$ is the matrix

$$\begin{pmatrix} \tfrac{1}{2}g(\xi)g(\eta) & -\tfrac{3}{2}g(\xi)f(\eta) & -\tfrac{3}{2}g(\xi)\hat{q}\cdot\boldsymbol{\eta}v(\eta) \\ -\tfrac{3}{2}f(\xi)g(\eta) & \tfrac{1}{2}f(\xi)f(\eta) & -\tfrac{3}{2}f(\xi)\hat{q}\cdot\boldsymbol{\eta}v(\eta) \\ -\tfrac{3}{2}v(\xi)\hat{P}\cdot\boldsymbol{\xi}g(\eta) & -\tfrac{3}{2}v(\xi)\hat{P}\cdot\boldsymbol{\xi}f(\eta) & -\tfrac{3}{2}v(\xi)(\hat{P}\cdot\boldsymbol{\xi})(\hat{q}\cdot\boldsymbol{\eta})v(\eta) \end{pmatrix}, \tag{3.23}$$

with

$$\boldsymbol{\xi}=\mathbf{q}+\tfrac{1}{2}\mathbf{P},\quad \boldsymbol{\eta}=\mathbf{P}+\tfrac{1}{2}\mathbf{q}. \tag{3.24}$$

The functions G_1, G_2, G_3 are, respectively, the "wave functions" for the channels $n+(np)\ {}^3S_1$, $p+(2n)\ {}^1S_0$, and $n+(np)\ {}^1P_1$, showing explicitly how the effects of "polarization" are incorporated in our formalism.

Finally, we have to solve Eqs. (3.5) and (3.22) for the quartet and doublet scattering lengths at zero energy, respectively, using the boundary conditions represented by the process of n-d scattering through the various channels involved, according to the above interpretations for the functions G_α. For $S=\frac{3}{2}$, we set in Eq. (3.5) with $H=0$,

$$G(P)=(2\pi)^3\delta^3(\mathbf{P}-\mathbf{k})-4\pi a_{3/2}(P)\ (P^2-k^2-i\epsilon)^{-1}, \tag{3.25}$$

where

$$ME=\tfrac{3}{4}K^2-\alpha^2,\quad (\alpha^2/M=2.226\text{ MeV}), \tag{3.26}$$

so that $a_{3/2}(0)$ is the exact quartet scattering length.

[20] The same functional notation for the two cases of $S=\frac{3}{2}$, $\frac{1}{2}$ need not cause confusion, since these cases do not overlap at any stage.

Similarly, in Eq. (3.22) for $S=\frac{1}{2}$ we set

$$G_\alpha(P)=(2\pi)^3\delta^3(\mathbf{P}-\mathbf{k})\delta_{\alpha1}-4\pi a_\alpha(P)\times(P^2-k^2-i\epsilon)^{-1}, \tag{3.27}$$

where $\delta_{\alpha1}$ is a Kronecker delta, and $a_1(0)$ represents the exact doublet scattering length. Substituting the forms (3.25) and (3.27) in (3.5) and (3.22), respectively, the following equations are finally obtained:

$$4\pi k_{31}(P)a_{3/2}(P)$$

$$=(2\pi)^3g(\tfrac{1}{2}P)g(P)(P^2+\alpha^2)^{-1}-4\pi\int d\mathbf{q}q^{-2}D^{-1}(\mathbf{P},\mathbf{q})\times g(\mathbf{q}+\tfrac{1}{2}\mathbf{P})g(\mathbf{P}+\tfrac{1}{2}\mathbf{q})a_{3/2}(q); \tag{3.28}$$

$$4\pi k_\alpha(P)a_\alpha(P)$$

$$=-(2\pi)^3K_{\alpha1}(\mathbf{P},0)(P^2+\alpha^2)^{-1}+\sum_\beta 4\pi\int d\mathbf{q}q^{-2}D^{-1}(\mathbf{P},\mathbf{q})K_{\alpha\beta}(\mathbf{P},\mathbf{q})a_\beta(q); \tag{3.29}$$

where

$$k_\alpha(P)=[\lambda_\alpha{}^{-1}-h_\alpha(P)](P^2-k^2)^{-1}, \tag{3.30}$$

TABLE I. Calculated ^1P phase shifts (in deg) YLAM data of Hull *et al.*

Energy (MeV)	10	20	30	40	50	80	100
Calc.	−1.5	−3.8	−6.1	−8.4	−10.5	−16.4	−19.8
YLAM	−2.5	−5.0	−7.3	−9.1	−10.6	−14.1	−15

and the notations implied by Eqs. (3.18) and (3.19) are understood here.

4. NUMERICAL RESULTS AND DISCUSSION

We are now in a position to discuss the numerical solutions of Eqs. (3.28) and (3.29) for the two scattering lengths. For this the potential shapes are taken as follows:

$$g(p), f(p), v(p)=(\beta_i^2+p^2)^{-1}; \quad (i=1, 2, 3). \quad (4.1)$$

$g(p)$ and $f(p)$ become the Yamaguchi potential[21] for the $^3S_1$ and $^1S_0$ states by taking

$$\beta_1=6.255\alpha, \quad \lambda_{31}=33.44\alpha^3; \quad (4.2)$$

$$\beta_2=6.255\alpha, \quad \lambda_{13}=23.37\alpha^3. \quad (4.3)$$

A second set of values considered in this section for the evaluation of $a_{3/2}$ is one due to Naqvi,[22] viz.,

$$\beta_1=5.8\alpha, \quad \lambda_{31}=23.2\alpha^3. \quad (4.4)$$

This is only a part of the total $T=0$ interaction—the contribution from the s state of the $2N$ system. As for the function $v(p)$ for the ^1P potential, the following parameters were found to fit Hull *et al.*'s YLAM data[23] rather well up to 100 MeV, as Table I shows.

$$\beta_3=6.255\alpha, \quad \lambda_{11}=5.067\alpha. \quad (4.5)$$

Since a positive λ_{11} greater than β_3/π^2 in the above model indicates a $^1P_1$ bound state which is clearly unphysical, a second fit was also obtained for

$$\beta_3=6.255\alpha, \quad \lambda_{11}=-50.6\alpha, \quad (4.6)$$

but the latter value of λ_{11} was not very sensitive to the data. However, as we shall see below, the effect of either of these two sets on $a_{1/2}$ was extremely small, and we shall not discuss this (academic) question of a repulsive or attractive ^1P interaction any further in the present context. We would like to repeat, however, that either of the sets (4.5) or (4.6) for λ_{11} indicate that the $^1P_1$ interaction is appreciably stronger than the ^3P interaction[15,16] and, hence, *a fortiori*, this fact gave us added confidence in our neglect of the ^3P interaction on the integral equations for $a_{3/2}(P)$ and $a_{1/2}(P)$.

TABLE II. Calculated values of $a_{3/2}$ in 10^{-13} cm.

Case	I	II	III	IV
$a_{3/2}$	5.91	5.3	6.32	5.7

Calculation of $a_{3/2}$ from Eq. (3.28) was carried out exactly, as well as under the approximation described in Sec. 4 of A, using by turn the interactions (4.2) and (4.4). The results are shown in Table II, where columns I and II represent, respectively, the exact and approximate values using (4.2), and columns III and IV the corresponding values using (4.4).

These values indicate, first, that the approximation considered in A is rather good for zero energy scattering, though it gives a somewhat smaller figure (by about 10%). Secondly, a comparison of columns I and III shows that the triplet s-state part, like (4.4), of a $T=0$ potential which also includes a tensor force, gives a somewhat higher figure, viz, 6.32 F, for $a_{3/2}$ than an effective central s-state interaction like (4.2), which gives 5.91 F. The increase in the quartet scattering length by a "more realistic" two-body interaction like (4.4) is certainly welcome in the present context in so far as it tends to bring its value substantially closer to the experimental value of 6.4 F (corresponding to Set I). While such a small difference like 6–7% should not, offhand, warrant the physical conclusion that a $T=0$ force with a tensor part gives a better representation of the quartet scattering length, than an effective $T=0$ s-state force, we would like to believe the effect to be qualitatively significant for the following reason. Apart from the effect of P waves (which we shall see from the case of $a_{1/2}$ to be negligible), the only other effect that could possibly influence the value of $a_{3/2}$ is that due to the actual tensor part of the $T=0$ force, which is responsible for $S-D$ interference in n-d scattering, and which has so far not been considered in our analysis. Now the kernel of Eq. (3.28) is seen to be *repulsive*, as already conjectured in Ref. 12. Therefore, the effect of the (hitherto neglected) tensor force would be to produce merely *continuum* D waves for the n-d system and *not* bound D waves, so that at zero energy the centrifugal barrier is expected to be very effective in preventing these D waves from affecting S-wave n-d scattering.[24] Under such conditions, it is very hard to visualize how the tensor force by itself could completely neutralize a 6-7% increase in the value of $a_{3/2}$, due to the potential (4.4), compared with (4.2),[25] though a more quantitative estimate is certainly in order. In any event, our result seems strongly to indicate a value of $a_{3/2}$ quite consistent with set I, in agreement with the findings of Ref. 14, which also support the theory of Spruch and Rosenberg.[13]

[21] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[22] J. H. Naqvi, Nucl. Phys. **36**, 578 (1962).
[23] N. H. Hull, K. E. Lassila, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **122**, 1606 (1961).

[24] Such a conclusion would be unwarranted for an *attractive* n-d kernel, in which case the effect of the (almost) *bound* D waves could be quite important. One of the authors (ANM) is indebted to Professor N. Austern for bringing this point to his attention.
[25] Our subsequent analysis of $a^{1/2}$ shows that continuum P waves have negligible (~0.3%) effect on s-wave scattering.

TABLE III. Calculated values of $a_{1/2}$ in 10^{-13} cm.

Case	I	II	III	IV	V
$a_{1/2}$	9.25	11.74	11.69	11.77	25.5

The case of $a_{1/2}$ is much more complicated because of several effects, and hence, seems to be much less clear. The calculation of this parameter was carried out, using the potentials (4.2), (4.3), and (4.5) and/or (4.6). The results are shown in Table III, where column I represents the exact value using only the potentials (4.2) and (4.3), and column II the corresponding value under the approximation of Sec. 4 of A. Columns III and IV represent the values of $a_{1/2}$ under the approximation of A, when the potentials (4.5) or (4.6), respectively, are added to (4.2) and (4.3). Table III shows, once again, that the approximation of A is quite good for zero-energy scattering. Further, the effect of "polarization" due to the channel $n+(np)$ $^1P_1$, on the $S=\frac{1}{2}$ $n+d$ scattering is very small, under the assumption of *both* attractive and repulsive interactions in the $^1P_1$ state of (np). Unfortunately, these values are nowhere near the corresponding set I value of $a_{1/2}$[2]; rather, the figure in column I is not far from the set II[2] value of this parameter! The "polarization effect" of the $p+(2n)$ $^1S_0$ channel was estimated as follows. Taking the first of Eq. (3.29) and setting $G_2=0$ (in addition to G_3), the equation for G_1 was solved with the potential (4.2), and a value of 25.5 F for $a_{1/2}$ was obtained, as shown in Column V of Table III. This shows that the polarization effect no doubt works in the right direction, but it is hard to make a precise estimate (percentage-wise) of its importance in the present context, since the quantities involved are still very large ($\sim$10 F) and of course far removed from the region of 1 F which is the expected order of magnitude for $a_{1/2}$ on the basis of the experimental set I value. It is possible, of course, that the ranges of the assumed singlet and triplet forces can play a much more important role[7] for $a_{1/2}$ than for $a_{3/2}$. A more important possibility lies in the recognition of the role of the tensor force for $a_{1/2}$, for the following reason. The large values of $a_{1/2}$ in Table III indicate very little net attraction and therefore a "just bound state," compared with the actual H^3. To bring down this parameter from such large values would, therefore, need additional attraction, most likely to be provided by the hitherto neglected tensor force in producing (almost) bound D waves for the n-d system. It may be noted that unlike the previous case of $a_{3/2}$ the *bound* n-d D waves in the present case can significantly affect S-wave n-d scattering in the doublet state. It is, of course, not clear off-hand as to what extent the inclusion of the tensor force can bring down the value of $a_{1/2}$ by providing additional attraction in the S wave of the n-d system. The formulation of the tensor force using full antisymmetrization in the present scope of the three-body investigation is still under way, and the questions as to the (1) role of the tensor force and (2) effect of the range of the $N-N$ interaction, on the doublet scattering length, are expected to be answered more specifically in due course.

A mechanism for detecting the di-neutron through a threshold effect in n-d scattering was suggested recently.[26] If the di-neutron exists, then it has been suggested that a detection of this threshold effect can have a bearing on the ambiguity in the doublet scattering length.[27] However, since such a threshold effect is expected to be small and has so far not been detected experimentally, we prefer not to discuss this question any further in this paper.

ACKNOWLEDGMENT

One of the authors (ANM) is indebted to Professor N. Austern for his interest and some valuable comments.

[26] See, for example, L. Fonda, Suppl. Nuovo Cimento **20**, 116 (1961).
[27] R. Alzetta *et al.* (unpublished).