An explicit representation of these operators is given by

$$A=e^{\alpha},$$
$$B=\lambda d/d\alpha.$$

Then

$$\exp\left(e^{\alpha}+\lambda\frac{d}{d\alpha}\right)=\exp\left(e^{f(\alpha)}\lambda\frac{d}{d\alpha}e^{-f(\alpha)}\right),$$

where

$$f(\alpha)=-\frac{1}{\lambda}e^{\alpha}.$$

Thus,

$$e^{A+B}=e^{f(\alpha)}e^{\lambda d/d\alpha}e^{-f(\alpha)},$$
$$=e^{f(\alpha)}e^{-f(\alpha+\lambda)}e^{\lambda d/d\alpha},$$
$$=\left[\exp\left(A\frac{e^{\lambda}-1}{\lambda}\right)\right]e^{B}.$$

The adjoint of the above yields the alternative representation:

$$e^{A+B}=e^{B}\exp\left(A\frac{1-e^{-\lambda}}{\lambda}\right).$$

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Two Vacuum Poles and Pion-Nucleon Scattering*

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A general expression is given for the pion-nucleon non-charge-exchange scattering amplitude for arbitrary energy and small momentum transfer on the assumption that only the vacuum pole P and the second vacuum pole P' exist in the upper half J plane. We derive sum rules for non-spin-flip and spin-flip amplitudes and use them, combined with the analysis of the high-energy π-N cross sections in terms of Regge poles, to investigate the behavior of P and P' trajectories near $t\approx 0$. For this purpose the importance of a precise measurement of the low-energy partial-wave phase shifts is emphasized. A sum rule for the S-wave pion-nucleon non-charge-exchange scattering length can be satisfied with $\alpha_{P'}\approx 0.5$.

I. INTRODUCTION

THERE have been many attempts to investigate the low-energy S-, P-, and D-wave pion-nucleon scattering based on the dispersion relations.[1–3] The charge-exchange scattering amplitude was successfully explained by Bowcock, Cottingham, and Lurié[2] by incorporating the $I=1$ pion-pion interaction into the analysis of CGLN.[1] However, the above method cannot be applied directly for the non-charge-exchange amplitude because the dispersion integrals diverge.

The aforementioned divergence problem which is related to the subtractions in the Mandelstam representation was greatly clarified by the Regge pole assumption[4] that all poles of the strong-interaction S matrix move in the complex J plane as a function of energy and that these poles control the asymptotic behavior. In a previous paper,[5] hereafter referred to as I, a sum rule was derived for the S-wave pion-nucleon non-charge-exchange scattering length, starting from the assumption that the amplitude can be written as the sum of two terms, the vacuum-Regge pole term which diverges at infinite energy and the remaining term which converges at infinity and satisfies an unsubtracted dispersion relation. This assumption led to a discrepancy between the observed and the calculated scattering lengths. Therefore, it was concluded that there should be another vacuum-Regge trajectory P' with $\alpha_{P'}(0)\sim 0.5$.[6] Existence of such a pole is also favored in the analysis of high-energy p-p and $\bar{p}$-p scattering,[7,8] high-energy π-p and K-p scattering.[7]

The purpose of the present paper is twofold: (a) to generalize the previous sum rule for pion-nucleon non-charge-exchange scattering, to hold for arbitrary s and small t (we assume, as in I, that only P and P' trajectories exist in the upper half J plane for t near zero); (b) to

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[1] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957), hereafter referred to as CGLN.
[2] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo Cimento **16**, 918 (1960); **19**, 142 (1961).
[3] For detailed references, see A. Takahashi, Progr. Theoret. Phys. (Kyoto) **27**, 665 (1962).
[4] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962). This assumption predicts a logarithmic shrinking of the p-p diffraction pattern with increasing energy. Such an effect has been observed experimentally [A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 108, 111 (1962)]. Moreover, the occurrence of Regge poles in the relativistic S matrix has been shown by Gribov, Domokos, Mandelstam, and Eden using the Mandelstam representation and elastic unitarity. See reference 7.
[5] K. Igi, Phys. Rev. Letters **9**, 76 (1962).
[6] In the previous paper I, it was concluded that there should be another vacuum trajectory in the region $1>\alpha(0)>0$. However, the notation of calling it as ABC pole has caused some confusion. It should have been noted by P' as introduced in the references 7 and 8. Detailed analysis for $\alpha_{P'}(0)$ is given in Appendix A.
[7] S. D. Drell, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962).
[8] F. Hadjioannou, R. J. N. Phillips, and W. Rarita, Phys. Rev. Letters **9**, 183 (1962); Y. Hara, Progr. Theoret. Phys. (Kyoto) **28**, 711 (1962).

apply these generalized sum rules in order to obtain the behavior of $\alpha_P(t)$, $\beta_P(t)$, $\alpha_{P'}(t)$, and $\beta_{P'}(t)$.

II. KINEMATIC CONSIDERATION

We shall begin by defining the necessary variables. Let the four-vector momenta of the pions be q_1 and q_2, and those of the antinucleon and nucleon be p_1 and p_2, respectively (Fig. 1). Define the Mandelstam variables[9]

$$t=-(q_1+q_2)^2=4(q^2+1)=4(p^2+m^2), \quad (2.1a)$$

$$s=-(p_1-q_1)^2=-p^2-q^2+2pq\cos\theta_3, \quad (2.1b)$$

$$\bar{s}=-(p_1-q_2)^2=-p^2-q^2-2pq\cos\theta_3, \quad (2.1c)$$

where q and p are the magnitudes of the pion and nucleon momenta, and $\cos\theta_3=p_2\cdot q_2/pq$, all in the barycentric system. In addition we define a new variable

$$\nu\equiv-(qm/p)\cos\theta_3,$$

$$=-\frac{s-m^2-1+(t/2)}{(t/2)-2m^2}m, \quad (2.2a)$$

which reduces to the incident pion energy ν_L in the πN laboratory system at $t=0$. The relation between ν and ν_L is

$$\nu=\frac{2m\nu_L+(t/2)}{2m^2-(t/2)}m. \quad (2.2b)$$

We shall next choose a new πN amplitude which is more convenient for the present purposes. Consider the πN amplitude which is the analytic continuation of the $\pi\pi\to N\bar{N}$ amplitude of Singh and Udgaonkar and has the form[10]

$$A^{(+)}=-\frac{8\pi i}{p^2}\left(\frac{p}{q}\right)^{1/2}\sum_J(J+\tfrac{1}{2})\times\left\{\frac{m\cos\theta_3}{[J(J+1)]^{1/2}}P_J'(\cos\theta_3)S_{-J}^{(+)}-\frac{\sqrt{t}}{2}P_J(\cos\theta_3)S_{+J}^{(+)}\right\}, \quad (2.3)$$

$$B^{(+)}=-\frac{8\pi i}{pq}\left(\frac{p}{q}\right)^{1/2}\sum_J\frac{(J+\frac{1}{2})}{[J(J+1)]^{1/2}}P_J'(\cos\theta_3)S_{-J}^{(+)}, \quad (2.4)$$

where $S_{\pm}^{(+)}$ is an S-matrix element for $\pi+\pi\to N+\bar{N}$ and the subscripts + and − refer to a nucleon and antinucleon having the same or opposite helicity. Let us define the following amplitude:

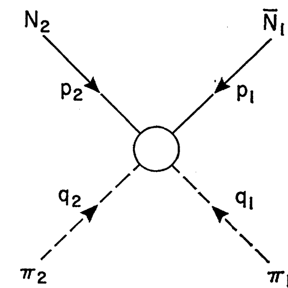

FIG. 1. The four-line diagram.

$$F^{(+)}(\nu,t)$$

$$\equiv\frac{1}{4\pi}\left[A^{(+)}(\nu,t)-\frac{qm}{p}\cos\theta_3B^{(+)}(\nu,t)\right], \quad (2.5a)$$

$$=\frac{1}{4\pi}\left[A^{(+)}(\nu,t)+\frac{s-m^2-1+(t/2)}{2m^2-(t/2)}mB^{(+)}(\nu,t)\right], \quad (2.5b)$$

$$=\frac{1}{4\pi}\left[\frac{4t^{1/2}\pi i}{p^2}\left(\frac{p}{q}\right)^{1/2}\sum_J(J+\tfrac{1}{2})P_J(\cos\theta_3)S_{+J}^{(+)}\right]. \quad (2.5c)$$

Then this function does not contain $P_J'(\cos\theta_3)$, so that the residues of the Regge pole contributions can be related to the πN total cross section at high energies.

III. A MODIFIED DISPERSION RELATION

Let us separate $F^{(+)}(\nu,t)$ into the P and P' Regge terms which give divergent behaviors as $\nu\to\infty$ and the remaining term $\bar{F}^{(+)}(\nu,t)$ which vanishes at infinity since we have assumed that only P and P' trajectories exist in the upper half J plane. To do this we write

$$F^{(+)}(\nu,t)=F_P(\nu,t)+F_{P'}(\nu,t)+\bar{F}^{(+)}(\nu,t), \quad (3.1)$$

where

$$F_P(\nu,t)=-\beta_P(t)\frac{P_{\alpha_P(t)}(\cos\theta_3)+P_{\alpha_P(t)}(-\cos\theta_3)}{\sin\pi\alpha_P(t)}, \quad (3,2)$$

and

$$F_{P'}(\nu,t)=-\beta_{P'}(t)\frac{P_{\alpha_{P'}(t)}(\cos\theta_3)+P_{\alpha_{P'}(t)}(-\cos\theta_3)}{\sin\pi\alpha_{P'}(t)}. \quad (3.3)$$

Then the dispersion relation for $\bar{F}^{(+)}(\nu,t)$ can be written for fixed t without subtraction:

$$\bar{F}^{(+)}(\nu,t)=B(t)+\frac{1}{\pi}\int_{\nu_{\min}}^{\infty}d\nu'\,\mathrm{Im}\bar{F}^{(+)}(\nu',t)\left[\frac{1}{\nu'-\nu}+\frac{1}{\nu'+\nu}\right]. \quad (3.4)$$

Here

$$B(t)=\frac{1}{4\pi}\frac{g_r^2}{2m}\left[\frac{1}{\nu_0-\nu_L}+\frac{1}{\nu_0+\nu_L+(t/2m)}\right]\left[\nu_0+\frac{t}{4m}\right], \quad (3.5)$$

$$\nu_0=-1/2m, \quad (3.6)$$

and

$$\nu_{\min}=\frac{1+(t/4m)}{1-(t/4m^2)}. \quad (3.7)$$

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[9] Notation: We use the metric such that $p\cdot q=\mathbf{p}\cdot\mathbf{q}-p_0q_0$. Hereafter we also use the pion mass unit.

[10] V. Singh and B. M. Udgaonkar, Phys. Rev. **123**, 1487 (1961).

From Eqs. (3.1) through (3.3) we find that

$$\mathrm{Im}\bar{F}^{(+)}(\nu',t)=\mathrm{Im}F^{(+)}(\nu',t)-\beta_P(t)P_{\alpha_P(t)}\left(\frac{p}{qm}\nu'\right)-\beta_{P'}(t)P_{\alpha_{P'}(t)}\left(\frac{p}{qm}\nu'\right). \tag{3.8}$$

Making use of (3.1), (3.4), and (3.8), separating out the singular term coming from the low-energy integral, we get

$$F^{(+)}(\nu,t)-F_P(\nu,t)-F_{P'}(\nu,t)$$

$$=B(t)+\frac{1}{\pi}\int_{\nu_{\min}}^{\infty}d\nu'\,\mathrm{Im}F^{(+)}(\nu',t)\left[\frac{1}{\nu'-\nu}+\frac{1}{\nu'+\nu}\right]-\frac{2}{\pi}\int_{qm/p}^{\infty}d\nu'\left[\beta_P(t)\frac{P_{\alpha_P(t)}[(p/qm)\nu']}{\nu'}+\beta_{P'}(t)\frac{P_{\alpha_{P'}(t)}[(p/qm)\nu']}{\nu'}\right]$$

$$-\frac{1}{\pi}\int_1^{\infty}dx'\,\beta_P(t)x\frac{P_{\alpha_P(t)}(x')}{x'}\left[\frac{1}{x'-x}-\frac{1}{x'+x}\right]-\frac{1}{\pi}\int_1^{\infty}dx'\,\beta_{P'}(t)x\frac{P_{\alpha_{P'}(t)}(x')}{x'}\left[\frac{1}{x'-x}-\frac{1}{x'+x}\right], \tag{3.9}$$

where $x=(p/qm)\nu$. In Eq. (3.9) the convergence of the integrals at high energy is assured. $F_P(\nu,t)$ and $F_{P'}(\nu,t)$ on the left-hand side and likewise the third and fourth integrals on the right-hand side have logarithmic singularities at $t=0$. However, using

$$\frac{x}{x'}\left[\frac{1}{x'-x}-\frac{1}{x'+x}\right]=\frac{1}{x'-x}+\frac{1}{x'+x}-\frac{2}{x'},$$

and[4]

$$P_\alpha(x)+P_\alpha(-x)-2P_\alpha(0)=-\frac{\sin\pi\alpha}{\pi}\int_1^{\infty}dx'\,P_\alpha(x')\left(\frac{1}{x'-x}+\frac{1}{x'+x}-\frac{2}{x'}\right),$$

we can rewrite the third and fourth integrals as

$$-F_{P(\mathrm{or}\,P')}(\nu,t)-\frac{2\beta_{P(P')}(t)}{\sin\pi\alpha_{P(P')}(t)}P_{\alpha_{P(P')}(t)}(0); \tag{3.10}$$

therefore, singular terms on both sides cancel. Using Eq. (2.2b) in the first integral, and the formula

$$-\frac{2\beta}{\sin\pi\alpha}P_\alpha(0)=\frac{\beta}{\pi^{3/2}}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right), \tag{3.11}$$

we obtain

$$F^{(+)}(\nu_L,t)=B(t)+\frac{1}{\pi}\int_1^{\infty}d\nu_L'\,\mathrm{Im}F^{(+)}(\nu_L',t)\left[\frac{1}{\nu_L'-\nu_L}+\frac{1}{\nu_L'+\nu_L+(t/2m)}\right]$$

$$-\frac{2}{\pi}\int_{qm/p}^{\infty}d\nu'\left[\beta_P(t)\frac{P_{\alpha_P(t)}[(p/qm)\nu']}{\nu'}+\beta_{P'}(t)\frac{P_{\alpha_{P'}(t)}[(p/qm)\nu']}{\nu'}\right]$$

$$+\frac{\beta_P(t)}{\pi^{3/2}}\Gamma\left(\frac{\alpha_P(t)+1}{2}\right)\Gamma\left(-\frac{\alpha_P(t)}{2}\right)+\frac{\beta_{P'}(t)}{\pi^{3/2}}\Gamma\left(\frac{\alpha_{P'}(t)+1}{2}\right)\Gamma\left(-\frac{\alpha_{P'}(t)}{2}\right). \tag{3.12}$$

IV. GENERALIZED SUM RULES

In this section we shall derive generalized sum rules for the non-spin-flip amplitude $f_1(\nu_L,t)$ and spin-flip amplitude $f_2(\nu_L,t)$ of CGLN, using the modified dispersion relation (3.12). These will enable us to investigate the behavior of $\alpha_P(t)$, $\beta_P(t)$, $\alpha_{P'}(t)$ and $\beta_{P'}(t)$ near $t\approx 0$.

First we shall relate $F^{(+)}(\nu_L,t)$ to the amplitudes f_1 and f_2. Using Eq. (2.5b), and Eqs. (3.5) and (3.6) of CGLN, namely,

$$f_1=\left(\frac{E+m}{2W}\right)\left(\frac{A+(W-m)B}{4\pi}\right), \tag{4.1}$$

and

$$f_2=\left(\frac{E-m}{2W}\right)\left(\frac{-A+(W+m)B}{4\pi}\right), \tag{4.2}$$

we obtain

$$\frac{2W}{E+m}f_1^{(+)}(\nu_L,0)=F^{(+)}(\nu_L,0)-\frac{1}{4\pi}\frac{k^2}{2m}B^{(+)}(\nu_L,0), \quad (4.3)$$

and

$$\frac{2W}{E-m}f_2^{(+)}(\nu_L,0)=-F^{(+)}(\nu_L,0)+\frac{1}{4\pi}\left[2W+\frac{k^2}{2m}\right]B^{(+)}(\nu_L,0), \quad (4.4)$$

where k^2 is the c.m. pion momentum.

In addition, we get

$$\frac{2W}{E+m}f_1^{(+)\prime}(\nu_L,0)=F^{(+)\prime}(\nu_L,0)-\frac{1}{4\pi}\frac{k^2}{2m}B^{(+)\prime}(\nu_L,0)-\frac{1}{4\pi}\frac{W^2+m^2-1}{8m^3}B^{(+)}(\nu_L,0), \quad (4.5)$$

and

$$\frac{2W}{E-m}f_2^{(+)\prime}(\nu_L,0)=-F^{(+)\prime}(\nu_L,0)+\frac{1}{4\pi}\left(2W+\frac{k^2}{2m}\right)B^{(+)\prime}(\nu_L,0)+\frac{1}{4\pi}\frac{W^2+m^2-1}{8m^3}B^{(+)}(\nu_L,0). \quad (4.6)$$

The prime here stands for differentiation with respect to t, and the expression for $F^{(+)}(\nu_L,0)$ was already given in I. The explicit expressions for $B^{(+)}(\nu_L,0)$, $B^{(+)\prime}(\nu_L,0)$, and $F^{(+)\prime}(\nu_L,0)$ are as follows:

$$B^{(+)}(\nu_L,0)=\frac{g_r^2}{2m}\left(\frac{1}{\nu_0-\nu_L}-\frac{1}{\nu_0+\nu_L}\right)+\frac{2}{3}\frac{P}{\pi}\int_1^\infty d\nu_L'\,k'\sigma_{\frac{3}{2}(P\frac{3}{2})}\left(\frac{3}{E'+m}-\frac{1}{E'-m}\right)\left(\frac{1}{\nu_L'-\nu_L}-\frac{1}{\nu_L'+\nu_L}\right), \quad (4.7)$$

and

$$B^{(+)\prime}(\nu_L,0)=\frac{g_r^2}{(2m)^2}\frac{1}{(\nu_0+\nu_L)^2}+\frac{P}{\pi}\int_1^\infty d\nu_L'\frac{1}{k'}\frac{\sigma_{\frac{3}{2}(P\frac{3}{2})}}{E'+m}\left(\frac{1}{\nu_L'-\nu_L}-\frac{1}{\nu_L'+\nu_L}\right)$$
$$+\frac{1}{\pi}\frac{1}{3m}\int_1^\infty d\nu_L'\,k'\sigma_{\frac{3}{2}(P\frac{3}{2})}\left(\frac{3}{E'+m}-\frac{1}{E'-m}\right)\frac{1}{(\nu_L'+\nu_L)^2}. \quad (4.8)$$

Here g_r^2 is the rationalized, renormalized pseudoscalar coupling constant. Experimentally $g_r^2/4\pi\approx 14$. We expect to use $B^{(+)}(\nu_L,0)$ only for small ν_L, i.e., ν_L less than the 33 resonance energy. Hence, we kept only the $P\frac{3}{2}$ state since the convergence of the integrals in Eqs. (4.7) and (4.8) is fast for small ν_L.

Differentiating Eq. (3.12) with respect to t, we get

$$F^{(+)\prime}(\nu_L,0)=\frac{f^2}{2}\left(\frac{1}{\nu_0-\nu_L}+\frac{1}{\nu_0+\nu_L}+\frac{1/m}{(\nu_0+\nu_L)^2}\right)$$
$$+\frac{1}{\pi}\int_1^M d\nu_L'\left[\operatorname{Im}F^{(+)\prime}(\nu_L',0)\left(\frac{1}{\nu_L'-\nu_L}+\frac{1}{\nu_L'+\nu_L}\right)-\frac{1}{8\pi m}\frac{(\nu_L'^2-1)^{1/2}\sigma_{\text{tot}}^{(+)}(\nu_L')}{(\nu_L'+\nu_L)^2}\right]+G(P,P'), \quad (4.9)$$

where $G(P,P')$ depends on $\alpha_P(0)$, $\alpha_P'(0)$, $\beta_P(0)$, $\beta_P'(0)$, $\alpha_{P'}(0)$, $\alpha_{P'}'(0)$, $\beta_{P'}(0)$, and $\beta_{P'}'(0)$, and $f^2\approx 0.08$. Here in a practical problem we can choose the upper limit of the second integral M to be the energy where the Regge behavior is already dominant. $\operatorname{Im}F^{(+)}(\nu_L,t)$ can be expressed in terms of partial-wave cross sections.[1]

Therefore,

$$\operatorname{Im}F^{(+)\prime}(\nu_L',0)=\frac{1}{4\pi}\left[\frac{d}{dt}\operatorname{Im}A^{(+)}(\nu_L',t)\Big|_{t=0}+\nu_L\frac{d}{dt}\operatorname{Im}B^{(+)}(\nu_L',t)\Big|_{t=0}\right]+\frac{1}{4\pi}\frac{W^2+m^2-1}{8m^3}\operatorname{Im}B^{(+)}(\nu_L',0)$$
$$\cong\frac{1}{4\pi}\left\{\frac{W'+m+\nu_L'}{E'+m}\left[\frac{1}{2k'}(2\sigma_{\frac{3}{2}(P\frac{3}{2})}-\sigma_{\frac{1}{2}(F\frac{5}{2})})+\frac{13}{k'}\sigma_{\frac{3}{2}[F(7/2)]}\right]\right.$$
$$-\frac{W'-m-\nu_L'}{E'-m}\left[\frac{1}{2k'}[\sigma_{\frac{1}{2}(D\frac{3}{2})}+5(\sigma_{\frac{1}{2}(F\frac{5}{2})}-2\sigma_{\frac{3}{2}[F(7/2)]})]\right]$$
$$+\frac{W'^2+m^2-1}{8m^3}\frac{k'}{E'+m}\left[2\sigma_{\frac{3}{2}(P\frac{3}{2})}-\tfrac{1}{3}\sigma_{\frac{1}{2}(D\frac{3}{2})}-\sigma_{\frac{1}{2}(F\frac{5}{2})}+\frac{20}{3}\sigma_{\frac{3}{2}[F(7/2)]}\right]$$
$$\left.+\frac{W'^2+m^2-1}{8m^3}\frac{k'}{E'-m}(-\tfrac{2}{3}\sigma_{\frac{3}{2}(P\frac{3}{2})}+\sigma_{\frac{1}{2}(D\frac{3}{2})}+2\sigma_{\frac{1}{2}(F\frac{5}{2})}-4\sigma_{\frac{3}{2}[F(7/2)]})\right\}, \quad (4.10)$$

where we took into account the $P\frac{3}{2}$, $D\frac{3}{2}$, $F\frac{5}{2}$, and $F\frac{7}{2}$ channels, which have resonances, for convenience. In practice, inclusion of the lower energy resonances, $P\frac{3}{2}$, $D\frac{3}{2}$, and $F\frac{5}{2}$ will be sufficient.

In Eq. (4.9),

$$G(P,P')=-\frac{2}{\pi}\left\{\frac{1}{8}\left(1-\frac{1}{m^2}\right)[\beta_P(0)+\beta_{P'}(0)]+\beta_P'(0)\int_1^M d\nu'\frac{P_{\alpha_P(0)}(\nu')}{\nu'}+\beta_{P'}'(0)\int_1^M d\nu'\frac{P_{\alpha_{P'}(0)}(\nu')}{\nu'}\right.$$
$$\left.+\beta_P(0)\int_1^M d\nu'\left[\frac{d}{dt}P_{\alpha_P(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'+\beta_{P'}(0)\int_1^M d\nu'\left[\frac{d}{dt}P_{\alpha_{P'}(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'\right\}$$
$$+\frac{1}{\pi^{3/2}}\left[\beta_P'(0)\Gamma\left(\frac{\alpha_P(0)+1}{2}\right)\Gamma\left(-\frac{\alpha_P(0)}{2}\right)+\frac{\beta_P(0)\alpha_P'(0)}{2}\Gamma\left(\frac{\alpha_P(0)+1}{2}\right)\Gamma\left(-\frac{\alpha_P(0)}{2}\right)\psi\left(\frac{\alpha_P(0)+1}{2}\right)\right.$$
$$\left.-\frac{\beta_P(0)\alpha_P'(0)}{2}\Gamma\left(\frac{\alpha_P(0)+1}{2}\right)\Gamma\left(-\frac{\alpha_P(0)}{2}\right)\psi\left(-\frac{\alpha_P(0)}{2}\right)\right]+(P\rightarrow P'). \quad (4.11a)$$

$\alpha_P(0)$, $\alpha_{P'}(0)$, $\beta_P(0)$ and $\beta_{P'}(0)$ are known quantities, having already been determined in I and Appendix A. In addition it is known experimentally[11] that $\alpha_P'(0)=1/50\mu^2$. This leaves only $\alpha_{P'}'(0)$, $\beta_P'(0)$, and $\beta_{P'}'(0)$ to be determined.[12]

With the set of values, $\alpha_P(0)=1$, $\alpha_P'(0)=1/50$, $\beta_P(0)=\sigma_{\text{tot}}^{(+)}(\infty)/4\pi\sim1/4\pi$, $\beta_{P'}(0)\sim0.21\,(\bar{\beta}_{P'}(0)\sim2.40)$, and $\alpha_{P'}(0)\sim0.5$, we get

$$G(P,P')=-0.22-0.05\int_1^M d\nu'\left[\frac{d}{dt}P_{\alpha_P(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'-0.13\int_1^M d\nu'\left[\frac{d}{dt}P_{\alpha_{P'}(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'+0.45\alpha_{P'}'(0)$$
$$-\left[0.64+\frac{2}{\pi}(M-1)\right]\beta_P'(0)-\left[1.08+\frac{2}{\pi}\int_1^M d\nu'\frac{P_{0.5}(\nu')}{\nu'}\right]\beta_{P'}'(0). \quad (4.11b)$$

If we take $M=14.3$ (which corresponds to 2 BeV), (4.11b) reduces to

$$-0.39+0.45\alpha_{P'}'(0)-9.11\beta_P'(0)-4.33\beta_{P'}'(0), \quad (4.11c)$$

since

$$\int_1^{14.3} d\nu'\frac{P_{0.5}(\nu')}{\nu'}=5.11,$$
$$\int_1^{14.3} d\nu'\left[\frac{d}{dt}P_{\alpha_{P'}(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'=0.46,$$

and

$$\int_1^{14.3} d\nu'\left[\frac{d}{dt}P_{\alpha_P(t)}\left(\frac{p}{qm}\nu'\right)\bigg|_{t=0}\right]\bigg/\nu'=2.26.$$

Therefore, Eqs. (4.5) and (4.6) with Eqs. (4.7), (4.8), (4.9), (4.10), and (4.11a,b,c) have the general form as follows:

$$f_{1(2)}^{(+)\prime}(\nu_L,0)=\text{Born term}$$
$$+\text{integral involving partial-wave cross sections}$$
$$+G(P,P'). \quad (4.12)$$

The left-hand side of Eq. (4.12) can easily be calculated, for small ν_L, by the low partial-wave phase-shift expansion[13]:

$$f_1^{(+)\prime}(\nu_L,0)=\frac{3}{2k^2}f_{P\frac{3}{2}}^{(+)}+\frac{15}{2k^2}f_{D\frac{5}{2}}^{(+)}+\cdots, \quad (4.13a)$$

and

$$f_2^{(+)\prime}(\nu_L,0)=\frac{3}{2k^2}(f_{D\frac{3}{2}}^{(+)}-f_{D\frac{5}{2}}^{(+)})+\cdots. \quad (4.14a)$$

In the low-energy region

$$f_1^{(+)\prime}(\nu_L,0)\cong\frac{3}{2k^2}f_{P\frac{3}{2}}^{(+)}, \quad (4.13b)$$

and

$$f_2^{(+)\prime}(\nu_L,0)\cong0, \quad (4.14b)$$

since $f_D\ll f_S$, f_P.

Therefore, we can investigate the behavior of P and P' trajectories near $t\approx0$ by requiring that the set of solutions obtained from the analysis of the high-energy π-N cross sections in terms of P and P' Regge poles, should satisfy the generalized sum rule for $f_1^{(+)\prime}(\nu_L,0)$ or $f_2^{(+)\prime}(\nu_L,0)$. This would further increase the accuracy of our final results.

[11] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); **8**, 41 (1962).

[12] Analysis would for instance enable us to check the conjecture of Squires and Wong (private communication) that $\beta(pq)^{-\alpha}$ might vary linearly between $t=0$ and $t=-50\mu^2$.

[13] Note that it is possible to make a direct comparison of Regge poles with an experiment without using partial-wave analysis. Because using Eqs. (3.12) and (3.13), $F^{(+)}(\nu_L,t)$ can be related to the c.m. cross section through Eq. (2.17) of CGLN according to which

$$d\sigma/d\Omega=\Sigma\,|\langle f|f_1+(\sigma\cdot k_2\sigma\cdot k_1/k_2k_1)f_2|i\rangle|^2.$$

The sum rules for the partial waves can also be obtained by relating $f_1^{(+)}(\nu_L,0)$, $f_2^{(+)}(\nu_L,0)$, $f_1^{(+)\prime}(\nu_L,0)$, $f_2^{(+)\prime}(\nu_L,0)$ to them through the following relations which are Eqs. (3.12), (3.13), and (3.14) of CGLN:

$$f_S(\nu_L)=f_1(\nu_L,0)-2k^2f_1'(\nu_L,0)+\sim D \text{ waves}, \quad (4.15)$$

$$f_{P\frac{1}{2}}(\nu_L)-f_{P\frac{3}{2}}(\nu_L)=f_2(\nu_L,0)-2k^2f_2'(\nu_L,0)+\sim F \text{ waves}, \quad (4.16)$$

$$-\frac{6}{k^2}f_{P\frac{3}{2}}(\nu_L)=-4f_1'(\nu_L,0)+8k^2f_1''(\nu_L,0)+\sim F \text{ waves}, \quad (4.17)$$

and so on.

For $\nu_L=1$, Eq. (4.16) gives

$$\lim_{E\to m}\frac{2W}{E-m}(f_{P\frac{1}{2}}{}^{(+)}-f_{P\frac{3}{2}}{}^{(+)}) =\lim_{k\to 0}\frac{4m(m+1)}{k^2}(f_{P\frac{1}{2}}{}^{(+)}-f_{P\frac{3}{2}}{}^{(+)}) =-F^{(+)}(\nu_L=1,\,0)+\frac{m+1}{2\pi}B^{(+)}(\nu_L=1,\,0). \quad (4.18)$$

By making use of

$$\lim_{k\to 0}\frac{f_{P\frac{1}{2}}{}^{(+)}-f_{P\frac{3}{2}}{}^{(+)}}{k^2}\equiv a_{P\frac{1}{2}}{}^{(+)}-a_{P\frac{3}{2}}{}^{(+)},$$

we get

$$a_{P\frac{1}{2}}{}^{(+)}-a_{P\frac{3}{2}}{}^{(+)}=-\frac{a^{(+)}}{4m^2}+\frac{1}{8\pi m}B^{(+)}(\nu_L=1,\,0) =-0.203\pm 0.015. \quad (4.19)$$

Here we have used

$$a^{(+)}=0.0013\pm 0.0036,^{14}$$

$$g_r^2/4\pi=14\pm 1,$$

and kept only $P\frac{3}{2}$ state as a rescattering term to $B^{(+)}(\nu_L=1,\,0)$ since the contribution from $D\frac{3}{2}$ and $F\frac{5}{2}$ states turns out to be less than 1% of the Born term. So we can predict that

$$a_{P\frac{1}{2}}{}^{(+)}-a_{P\frac{3}{2}}{}^{(+)}=-0.203\pm 0.015.$$

The corresponding experimental value is -0.16 ± 0.03.[14]

V. CONCLUDING REMARKS

As is discussed in the Appendix B, the subtraction problem in the Mandelstam representation was clarified from the Regge asymptotic behavior. The S-wave (+) amplitude scattering length is closely connected to the high-energy limit behavior through P and P' trajectories in the crossed channel. So if the dynamical approach becomes possible to get P and P' trajectories near $t=0$ (as was proposed by Chew[15] and Balázs[16]), then the $S(+)$ scattering length will also be obtained dynamically. In Sec. IV it was proposed to use sum rules, combined with the analysis of the high-energy π-N cross sections in terms of Regge poles, to investigate the behavior of P and P' trajectories near $t\approx 0$.

To be concrete, a sum rule for the S-wave (+) amplitude scattering length enables us to choose a set of values $\alpha_{P'}(0)$, $\beta_{P'}(0)$, and $\sigma_{\text{tot}}{}^{(+)}(\infty)$. Together with the above values and $\alpha_P'(0)\approx(1/50)(1/\mu^2)$, the generalized sum rule for $f_1'(\nu_L,0)$ or $f_2'(\nu_L,0)$ makes it possible to investigate $\alpha_{P'}'(0)$, $\beta_P'(0)$, and $\beta_{P'}'(0)$.

The necessary experiment for that purpose is (i) to get "total" partial-wave cross sections up to the energy that the Regge asymptotic behavior is already achieved (see 4.10); (ii) to get the low-energy phase shift precisely (for example, $P\frac{3}{2}$ phase shifts), see (4.13a,b).

We hope that more extensive and accurate data not only on the total cross sections at high energies but also on the low-energy region will soon be available in order to make it possible to investigate the P and P' Regge poles more precisely.

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APPENDIX A. ESTIMATION OF PARAMETERS FOR THE P'

In a previous paper I, we have derived a sum rule for the S-wave pion-nucleon non-charge-exchange scattering length, starting from the assumption that only P and P' exist in the upper half J plane:

$$\left(1+\frac{1}{m}\right)a^{(+)}=-\frac{f^2}{m}\frac{1}{1-1/4m^2} +\frac{\Gamma(\alpha_{P'}+1)\Gamma[(\alpha_{P'}+1)/2]\Gamma(-\alpha_{P'}/2)}{4\pi^2 2^{\alpha_{P'}}\Gamma(\alpha_{P'}+\frac{1}{2})}\bar{\beta}_{P'} +\frac{1}{2\pi^2}\int_1^{143.3}dk'\,[\sigma_{\text{tot}}{}^{(+)}(k')-\sigma_{\text{tot}}{}^{(+)}(\infty)] -\frac{1}{2\pi^2}\frac{\pi^{1/2}\Gamma(\alpha_{P'}+1)}{2^{\alpha_{P'}}\Gamma(\alpha_{P'}+\frac{1}{2})}\int_1^{143.3}d\nu'\,\frac{P_{\alpha_{P'}}(\nu')}{\nu'}\bar{\beta}_{P'}, \quad (A1)$$

[14] J. Hamilton and W. S. Woolcock, Phys. Rev. **118**, 291 (1960); S. W. Barnes, B. Rose, G. Giacomelli, J. Ring, K. Miyake, and K. Kinsey, *ibid.* **117**, 226 (1960).

[15] G. F. Chew (to be published).

[16] L. Balázs, University of California Radiation Laboratory Report 10157, 1962 (unpublished).

where we assumed that the Regge asymptotic behavior is already achieved at 20 BeV/c (=143.3 in units of pion mass).

In this Appendix A, we test the above sum rule (A1) by inserting parameters for the P' deduced from high-energy π^+p and π^-p total cross section.

The high-energy πp total cross section between 4.5 BeV/c and 20 BeV/c[17] was fitted with the following formula by Udgaonkar[18]:

$$\sigma_{\text{tot}}^{(+)}(\nu)=\sigma_{\text{tot}}^{(+)}(\infty)+\bar{\beta}_{P'}\nu^{-(1-\alpha_{P'})}, \qquad \text{(A2)}$$

where

$$\sigma_{\text{tot}}^{(+)}(\nu)=\tfrac{1}{2}[\sigma_{\text{tot}}^{\pi^+p}(\nu)+\sigma_{\text{tot}}^{\pi^-p}(\nu)]. \qquad \text{(A3)}$$

The cross section at infinite energy $\sigma_{\text{tot}}^{(+)}(\infty)$ and the coefficient $\bar{\beta}_{P'}$ are given in Table I for different values of $\alpha_{P'}$.

TABLE I. Good χ^2 fits to the πp data, 4.5–20 BeV/c. $\sigma_{\text{tot}}^{(+)}(\nu)=\sigma_{\text{tot}}^{(+)}(\infty)+\bar{\beta}_{P'}\nu^{-(1-\alpha_{P'})}$. Errors of $\bar{\beta}_{P'}$ are about 15%. If a P'' is taken into account $[\sigma_{\text{tot}}^{(+)}(\nu)=\sigma_{\text{tot}}^{(+)}(\infty)+\bar{\beta}_{P'}\nu^{-(1-\alpha_{P'})}+\bar{\beta}_{P''}\nu^{-(1-\alpha_{P''})}]$, the value $\bar{\beta}_{P'}$ becomes slightly smaller. In the future, this should be taken into account.

$\alpha_{P'}$	$\sigma_{\text{tot}}^{(+)}(\infty)$ (mb)	$\bar{\beta}_{P'}$ (μ units)
0.1	23.2	7.15
0.2	22.8	5.31
0.3	22.3	4.00
0.36	21.9	3.40
0.4	21.6	3.05
0.44	21.4	2.72
0.48	20.9	2.48
0.5	20.67	2.40

With these sets for $\alpha_{P'}$, $\bar{\beta}_{P'}$, and $\sigma_{\text{tot}}^{(+)}(\infty)$, we shall test our sum rule (A1). For convenience, let us introduce the following quantities:

$$\frac{\Gamma(\alpha_{P'}+1)\Gamma[(\alpha_{P'}+1)/2]\Gamma(-\alpha_{P'}/2)}{4\pi^2 2^{\alpha_{P'}}\Gamma(\alpha_{P'}+\frac{1}{2})}\bar{\beta}_{P'}\equiv I_1, \qquad \text{(A4)}$$

$$\frac{1}{2\pi^2}\int_1^{143.3} dk'\,[\sigma_{\text{tot}}^{(+)}(k')-\sigma_{\text{tot}}^{(+)}(\infty)]\equiv I_2, \qquad \text{(A5)}$$

$$-\frac{1}{2\pi^2}\frac{\pi^{1/2}\Gamma(\alpha_{P'}+1)}{2^{\alpha_{P'}}\Gamma(\alpha_{P'}+\frac{1}{2})}\int_1^{143.3} d\nu'\,\frac{P_{\alpha_{P'}}(\nu')}{\nu'}\bar{\beta}_{P'}\equiv I_3. \qquad \text{(A6)}$$

I_1, I_2, and I_3 are evaluated in Table II, for various values of $\alpha_{P'}$, using sets of parameters $\alpha_{P'}$, $\beta_{P'}$ and $\sigma_{\text{tot}}^{(+)}(\infty)$ given in Table I. To calculate I_2 the following data were used: the πp total cross section data tabulated by Sokolov *et al.* and Barashenkov *et al.* up to 1.6 BeV/c, the data by the Moyer group between 1.6 and 4.5 BeV/c, and the data by Von Dardel *et al.* between 4.5 and 20 BeV/c (see references 12–15, in I).

TABLE II. Values of integrals I_1, I_2, and I_3.

$\alpha_{P'}$	I_1	I_2	I_3	$I_1+I_2+I_3$
0.1	-3.25 ± 0.48	1.58 ± 0.21	-2.12 ± 0.32	-3.78 ± 0.83
0.2	-1.32 ± 0.19	1.72 ± 0.21	-2.22 ± 0.33	-1.82 ± 0.56
0.3	-0.65 ± 0.10	1.90 ± 0.21	-2.34 ± 0.35	-1.09 ± 0.50
0.36	-0.45 ± 0.07	2.05 ± 0.21	-2.35 ± 0.35	-0.75 ± 0.47
0.4	-0.37 ± 0.06	2.16 ± 0.21	-2.45 ± 0.37	-0.66 ± 0.47
0.44	-0.29 ± 0.04	2.23 ± 0.21	-2.47 ± 0.37	-0.53 ± 0.46
0.48	-0.24 ± 0.04	2.41 ± 0.21	-2.57 ± 0.39	-0.40 ± 0.47
0.5	-0.23 ± 0.03	2.48 ± 0.21	-2.66 ± 0.40	-0.41 ± 0.47

The numerical value of $(1+1/m)a^{(+)}$ is 0.0015 ±0.0041; that of $-(f^2/m)(1-1/4m^2)^{-1}$ is -0.012 ±0.001. Thus, it becomes possible that our sum rule (A1) can hold near the set of parameters $\alpha_{P'}\approx0.5$, $\bar{\beta}_{P'}\approx2.4$, and $\sigma_{\text{tot}}^{(+)}(\infty)\approx20.67$ mb, even though the experimental error is not still small. It should be noted that in a future analysis, a P'' trajectory may also be included in the high-energy formula (A2). This will reduce the value $\bar{\beta}_{P'}$ and, thus, (A2) will hold with the value $\alpha_{P'}$ slightly smaller than 0.5.

APPENDIX B. THE SUBTRACTION PROBLEM IN THE MANDELSTAM REPRESENTATION

First, we should like to discuss the subtraction problem in the Mandelstam representation for $A^{(\pm)}$ and $B^{(\pm)}$ amplitudes from the Regge asymptotic point of view.

At large s' and for $t<0$, we are in the physical region for the s' reaction. Then $\text{Im}A^{(+)}(s',t)$ and $\text{Im}B^{(+)}(s',t)$ will be controlled by the top-level Pomeranchuk pole in the crossed channels as follows:

$$\text{Im}A^{(+)}(s',t)\to s'^{\alpha_P(t)}\leqq s' \quad \text{for} \quad t\leqq0, \qquad \text{(B1)}$$

and

$$\text{Im}B^{(+)}(s',t)\to s'^{\alpha_P(t)-1}\leqq\text{const} \quad \text{for} \quad t\leqq0. \qquad \text{(B2)}$$

Similarly,

$$\text{Im}A^{(-)}(s',t)\to s'^{\alpha_\rho(t)}<s' \quad \text{for} \quad t\leqq0, \qquad \text{(B3)}$$

and

$$\text{Im}B^{(-)}(s',t)\to s'^{\alpha_\rho(t)-1}<\text{const} \quad \text{for} \quad t\leqq0. \qquad \text{(B4)}$$

On the other hand, the dispersion relations without subtraction for fixed t are

$$A^{(+)}(s,t)=\frac{1}{\pi}\int ds'\,\text{Im}A^{(+)}(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right), \qquad \text{(B5)}$$

[17] G. Von Dardel, R. Mermod, P. A. Piroué, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters 7, 127 (1961); G. Von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, *ibid.* 8, 173 (1962).

[18] B. M. Udgaonkar (private communication).

$$B^{(+)}(s,t)=g_r^2\left(\frac{1}{m^2-s}+\frac{1}{m^2-\bar{s}}\right)+\frac{1}{\pi}\int ds'\,\mathrm{Im}B^{(+)}(s',t)\left(\frac{1}{s'-s}-\frac{1}{s'-\bar{s}}\right), \quad \text{(B6)}$$

$$A^{(-)}(s,t)=\frac{1}{\pi}\int ds'\,\mathrm{Im}A^{(-)}(s',t)\left(\frac{1}{s'-s}-\frac{1}{s'-\bar{s}}\right), \quad \text{(B7)}$$

and

$$B^{(-)}(s,t)=g_r^2\left(\frac{1}{m^2-s}+\frac{1}{m^2-\bar{s}}\right)+\frac{1}{\pi}\int ds'\,\mathrm{Im}B^{(-)}(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-\bar{s}}\right). \quad \text{(B8)}$$

Comparing these equations, it becomes clear that the subtraction is necessary only for the $A^{(+)}$ amplitude. This is the reason why the charge-exchange scattering amplitude was successfully explained.[2]

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Evaluation of the Van Hove Correlation Functions for Certain Physical Systems*

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The space and time Fourier transforms of the Van Hove correlation function are evaluated for the cases of coherent scattering from simple crystals and, in a "quantum hydrodynamics" approximation, from liquid HeII. A compact approximate expression for the one-phonon part of the crystal correlation function transform is given, and the contribution of the two-phonon term is considered. A new method of obtaining quantum-mechanical corrections to the classical expression for the Van Hove self-correlation function is discussed.

I. INTRODUCTION

IT has been shown that the energy-transfer-dependent differential cross section for the coherent scattering of cold neutrons[1] or gamma rays[2] from an assembly of N identical atoms is given by

$$\frac{d^2\sigma}{d\Omega d\epsilon}=N\frac{d\sigma_A}{d\Omega}Z(\mathbf{q},\epsilon),$$

where

$$Z(\mathbf{q},\epsilon)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,\exp(-i\epsilon t)\Gamma(\mathbf{q},t) \quad (1)$$

and

$$\Gamma(\mathbf{q},t)\equiv N^{-1}\left\langle\sum_{j=1}^{N}\exp[-i\mathbf{q}\cdot\mathbf{r}_j(0)]\sum_{j'=1}^{N}\exp[i\mathbf{q}\cdot\mathbf{r}_{j'}(t)]\right\rangle_T. \quad (2)$$

Here $d\sigma_A/d\Omega$ is the appropriate scattering cross section for a single atom, $\mathbf{q}$ is the momentum transfer of the scattered particle, ϵ is the initial energy of the scattered particle minus its final energy, and $\mathbf{r}_j(t)$ is the Heisenberg position operator for the jth atom at time t. The operator $\langle\ \rangle_T$ denotes an ensemble average over the states of the target system at constant temperature T; thus we have

$$\langle O\rangle_T=\mathrm{Tr}[\exp(-2\beta H)O]/\mathrm{Tr}[\exp(-2\beta H)], \quad (3)$$

where O is any Heisenberg operator pertaining to the system, H is the system Hamiltonian, and

$$\beta\equiv1/2K_BT,$$

where K_B is the Boltzmann constant. Unless otherwise indicated, units with $\hbar=1$ will be used throughout this paper.

The evaluation of these functions and their counterparts for incoherent scattering has been undertaken by several authors[1–6]; the work of Van Hove[1] and Visscher[3] on crystals and of Vineyard,[4] Schofield,[5] and especially Rahman, Singwi, and Sjölander[6] on nearly classical fluids is of special interest here. We derive improved approximate expressions for $Z(\mathbf{q},\epsilon)$ and its three- and four-dimensional Fourier transforms for the cases of liquid HeII, idealized crystal lattices, and nearly classical fluids.

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