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Quantum Statistics of Masers and Attenuators

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We have studied the statistical properties of a single mode of a radiation field in interaction with a very large number of material quantum systems. The quantum systems are assumed to be in internal thermal equilibrium at some positive or negative temperature. Under appropriate conditions we find linear solutions to Heisenberg's equations of motion for the field. We then make use of the quantum characteristic function to evaluate the statistical properties of the field during the interaction. A general conclusion which may be drawn is as follows. If at some initial instant the field can be resolved into the sum of the Gaussian zero-point field and an independent "input" field, then at later times the total field may be resolved into the sum of three independent fields; the unaltered zero-point field, the amplified or attenuated input field, and a Gaussian thermal or spontaneous emission field.

I. INTRODUCTION

IN a recent paper[1] we have made a quantum analysis of the noise properties of parametric amplifiers. In this paper we apply similar mathematical methods to maser-type amplifiers or attenuators.

Many authors[2–21] have previously dealt with problems similar to that which is our present concern; namely, the interaction of a single quantum oscillator (in our case a radiation field oscillator) with a large number of loosely coupled quantum systems which are in thermal equilibrium among themselves at some positive or negative temperature. Our model, a generalization of that used, for example, by Senitsky,[8] is rather venerable, as are the assumptions which lead to the linearization of Heisenberg's equations.

Our main conclusion, that the field spontaneously generated by the quantum systems always has the statistical properties of additive Gaussian noise, agrees with results obtained by Schwinger[20] and Wells.[21] Our concern has been to keep the quantum-mechanical treatment of the problem in the simplest possible form.

II. THE MODEL

We consider the radiation field of a single mode of a lossless cavity, resonant at frequency ω. At some initial time, say at $t=t_1$, this radiation field is weakly coupled to a very large number of nearly independent quantum systems. The quantum systems will, for convenience, be referred to as "atoms" and their properties will be kept as general as possible consistent with simplicity in the mathematics. The "atoms" might be harmonic oscillators, magnetic particles with half-integer angular momenta, etc.

Let us specify the properties of the atoms. Each atom is assumed to have a set of M equally spaced energy levels, where M may be finite or infinite. The energy separation between adjacent levels of the jth atom is taken as $\hbar\omega_j$. It is assumed that there are sufficiently numerous atoms with energies in the neighborhood of any ω_j so that they may be represented by a continuous density function, $\sigma(\omega_j)$. Only atoms for which $\omega_j \approx \omega$ couple to the radiation field to an appreciable extent.

The atoms are assumed to be weakly coupled to each other, and they are also assumed to be in weak thermal contact with a "heat" bath at temperature T. They, therefore, have a Boltzmann distribution corresponding to the temperature T at time t_1.

The couplings among the atoms, and between the atoms and the heat bath, are assumed to be so weak that they can be ignored during the interaction time. Further, the number of atoms is assumed so large, and their individual interaction with the field so weak that the statistical distribution of the atoms is not changed appreciably during the interaction time. These assump-

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tions are made to insure that the amplification remains unsaturated during the interaction, i.e., that the amplification remains linear. There are of course other ways of obtaining linearity, but presumably they will lead to no new phenomena.

The Hamiltonian for the radiation field in the cavity before coupling to the atoms is taken as

$$H_a=\hbar\omega[a^\dagger(t)a(t)+\tfrac{1}{2}]=\tfrac{1}{2}[p^2(t)+\omega^2q^2(t)], \tag{1}$$

where $a(t)$ and $a^\dagger(t)$ are the Heisenberg annihilation and creation operators, respectively, for photons in the radiation field. The Heisenberg operators $p(t)$ and $q(t)$ are proportional to the electric and magnetic fields, respectively, and are related to $a(t)$ and $a^\dagger(t)$ by

$$\begin{aligned} p(t)&=i(\hbar\omega/2)^{1/2}[a^\dagger(t)-a(t)],\\ q(t)&=(\hbar/2\omega)^{1/2}[a^\dagger(t)+a(t)]. \end{aligned} \tag{2}$$

We shall adopt the notation that a Heisenberg operator, when it is evaluated at time $t=t_1$ (when the interaction is turned on) will be written with its time argument omitted; e.g., $a^\dagger(t_1)\equiv a^\dagger$.

When a basic set of eigenstates of the field is needed we will usually use the eigenstates of H_a. These eigenstates are represented by the kets

$$|n\rangle \quad (n=0,\,1,\,2,\,3,\cdots),$$

where

$$a^\dagger|n\rangle=(n+1)^{1/2}|n+1\rangle;\quad a|n\rangle=n^{1/2}|n-1\rangle;\quad a^\dagger a|n\rangle=n|n\rangle. \tag{3}$$

n is the number of quanta in state $|n\rangle$.

The Hamiltonian for the atoms before coupling to the radiation field is taken as

$$H_b=\textstyle\sum_j \hbar\omega_j\mathcal{H}_j(t), \tag{4}$$

where the sum is taken over all N atoms. $\mathcal{H}_j$ is the "number" operator for the jth atom. The energy eigenstates of the jth atom are represented by the kets $|m_j\rangle$, where

$$\mathcal{H}_j|m_j\rangle=m_j|m_j\rangle. \tag{5}$$

m_j has integral values from 0 to $M-1$.

We next define $b_j^+(t)$ and $b_j^-(t)$, respectively, as raising and lowering operators for the atoms so that

$$b_j^+|m_j\rangle=\lambda_{j,m+1}|m_j+1\rangle;\quad b_j^-|m_j\rangle=\lambda_{j,m}|m_j-1\rangle. \tag{6}$$

We choose the phases of the states $|m_j\rangle$ so that $\lambda_{j,m}$ is real. Also b_j^+ is the Hermitian adjoint of b_j^-. In order that there be just M levels for the jth atom, we require that

$$\lambda_{j,M}=\lambda_{j,0}=0,$$

or

$$b_j^+|(M-1)_j\rangle=b_j^-|0_j\rangle=0,$$

where $|0_j\rangle$ is the ground state of the atom. There are certain cases for which $\mathcal{H}_j$ can be related to the b_j's. For example, if the "atoms" are harmonic oscillators, then

$$\mathcal{H}_j=b_j^+b_j^-,$$

and

$$\lambda_{j,m}=(m_j)^{1/2}.$$

However, there need be no specific relationship between $\mathcal{H}_j$ and the b_j's.

Finally, we must postulate an interaction Hamiltonian. In order to conserve energy in first order, it must have the form

$$H_{ba}=\hbar\textstyle\sum_j \kappa_j[b_j^+(t)a(t)+b_j^-(t)a^\dagger(t)], \tag{7}$$

where κ_j is the coupling coefficient between the jth atom and the field. With this form of the interaction, destruction of a photon accompanies promotion of an atom to the next higher state, while creation of a photon accompanies the demotion of an atom. In order that the interaction be linear, the κ_j's must be sufficiently small that it is unlikely for any particular atom to change its state during the interaction time.

The total Hamiltonian is then

$$H=H_a+H_b+H_{ba}. \tag{8}$$

III. THE COMMUTATION RELATIONS AND EQUATIONS OF MOTION

The commutation relations for the field operators are constants of the Heisenberg equations of motion. They are

$$[a(t),a^\dagger(t)]=1;\quad [a(t),a(t)]=[a^\dagger(t),a^\dagger(t)]=0;\quad [q(t),p(t)]=i\hbar. \tag{9}$$

It follows from (5) and (6) that

$$[b_j^\pm(t),\mathcal{H}_k(t)]=\mp b_j^\pm(t)\delta_{jk}, \tag{10}$$

and that

$$\langle m_j|[b_j^-,b_k^+]|m_j\rangle=(\lambda_{j,m+1}^2-\lambda_{j,m}^2)\delta_{jk}. \tag{11}$$

Operators relating to different atoms of course commute.

From these commutation relations and the Hamiltonian (8) it follows that the Heisenberg equations of motion are

$$i\frac{da(t)}{dt}=\frac{1}{\hbar}[a(t),H]=\omega a(t)+\sum\kappa_jb_j^-(t), \tag{12a}$$

$$i\frac{db_j^-(t)}{dt}=\omega_jb_j^-(t)+\kappa_j[b_j^-(t),b_j^+(t)]a(t), \tag{12b}$$

$$\begin{aligned} i\frac{d}{dt}[b_j^-(t),b_j^+(t)] &=\kappa_j\{[[b_j^-(t),b_j^+(t)],b_j^+(t)]a(t)\\ &\quad+[[b_j^-(t),b_j^+(t)],b_j^-(t)]a^\dagger(t)\}, \end{aligned} \tag{12c}$$

together with the adjoints of (12a) and (12b).

These equations of motion are, in general, nonlinear. An exception arises for the case in which the "atoms" are harmonic oscillators, for in this case $[b_j^-(t),b_j^+(t)]\equiv 1$.

The general case, however, becomes linear for values of κ_j sufficiently small that we may replace the operators $[b_j^-,b_j^+]$ by constant numbers. To see that this replacement is reasonable, we note that since [from (6)]

$$[b_j^-,b_j^+]|m_j\rangle=(\lambda_{j,m+1}^2-\lambda_{j,m}^2)|m_j\rangle,$$

it is apparent that an appreciable change in the operator would result only from a change in the distribution of the atoms among their various energy levels. However, for sufficiently small κ_j, the chance of any particular atom making a transition becomes negligibly small, and so we see that $[b_j^-,b_j^+]$ stays constant. Mathematically, we see from (12c) that the change in $[b_j^-,b_j^+]$ is of first order in κ_j, and so enters (12b) in second order in κ_j. It is not so easy to obtain a physical picture of why replacement of $[b_j^-,b_j^+]$ by a number rather than by a constant operator is reasonable. It amounts to ignoring all commutators of $[b_j^-,b_j^+]$ with the operators b_j^- and b_j^+ as they occur in the solutions to the equations of motion. However, direct expansion of the formal solution of the equation of motion, i.e.,

$$a(t)=\exp(-iHt/\hbar)a\exp(iHt/\hbar),$$

in a power series in the coupling constants κ_j shows that all such commutators enter to a negligible order of κ_j.

Further simplification of (12b) results if we immediately average it over a reasonably large group of atoms which have the same frequency ω_j and the same coupling constant. This is permissible since the equations are now linear. If we do this, the constant $[b_j^-,b_j^+]$ is replaced by its expectation value, viz.,

$$\langle[b_j^-,b_j^+]\rangle\equiv\sum_{m_j}P(m_j)\langle m_j|[b_j^-,b_j^+]|m_j\rangle,$$

where $P(m_j)$ is the probability of finding an atom in the state $|m_j\rangle$. Using (11), this may be written as

$$\begin{aligned}\langle[b_j^-,b_j^+]\rangle&=\sum_{m_j}P(m_j)(\lambda_{j,m+1}^2-\lambda_{j,m}^2)\\&=\sum_{m_j}[P(m_j-1)-P(m_j)]\lambda_{j,m}^2,\end{aligned}$$

where we have shifted the index of the first sum down by one. Since the atoms have a Boltzmann distribution, $P(m_j-1)=\tau_jP(m_j)$ where

$$\tau_j=\exp(\hbar\omega_j/kT). \tag{13a}$$

Note, therefore, that

$$P(m_j)=\tau_j^{-m_j}/\sum_{m_j}\tau_j^{-m_j}. \tag{13b}$$

Thus, we have

$$\langle[b_j^-,b_j^+]\rangle=(\tau_j-1)\Lambda_j, \tag{14a}$$

where

$$\Lambda_j\equiv\sum_{m_j}\lambda_{j,m}^2P(m_j). \tag{14b}$$

With these approximations (12) becomes

$$\begin{aligned}i\frac{da(t)}{dt}&=\omega a(t)+\textstyle\sum_j\kappa_jb_j^-(t),\\ i\frac{db_j^-(t)}{dt}&=\omega_jb_j^-(t)+\kappa_j(\tau_j-1)\Lambda_ja(t).\end{aligned}\tag{15}$$

The harmonic oscillator case is obtained by letting $(\tau_j-1)\Lambda_j$ equal unity.

The solution of these linearized equations has the form

$$a(t)=u(t-t_1)a+\textstyle\sum_j v_j(t-t_1)b_j^-. \tag{16}$$

$u(t)$ and $v_j(t)$ are found in Appendix A and are given by

$$\begin{aligned}u(t)&=\exp[-i\omega t+(\mu/2)(1-\tau)\Lambda t],\\ v_j(t)&=-i\kappa_j\exp(-i\kappa_jt)\\&\quad\times\left\{\frac{1-\exp[i(\omega_j-\omega)t+\frac{1}{2}\mu(1-\tau)\Lambda t]}{i(\omega-\omega_j)-\frac{1}{2}\mu(1-\tau)\Lambda}\right\},\end{aligned}\tag{17}$$

where τ and Λ are τ_j and Λ_j evaluated at $\omega=\omega_j$ and μ is a constant defined by

$$\mu=2\pi\kappa^2\sigma, \tag{18}$$

and where κ and σ are κ_j and $\sigma(\omega_j)$ evaluated at $\omega_j=\omega$. It will be shown that $|u(t)|^2$ is the power gain of the field resulting from the interaction. From (17) and (13) we, therefore, see that if $\tau>1$ $(T>0)$ the field is attenuated, while if $\tau<1$ $(T<0)$ the field is amplified. Note that when the atoms are harmonic oscillators, then $\Lambda(\tau-1)=1$. For this case T must always be positive, and only attenuation can result from the interaction.

The commutation relation $[a(t),a^\dagger(t)]=1$ together with (14a) and (16) leads to the relation

$$|u(t-t_1)|^2+\textstyle\sum_j|v_j(t-t_1)|^2\Lambda_j(\tau_j-1)=1, \tag{19}$$

between $u(t)$ and the $v_j(t)$. This relation will be useful in simplifying later results.

IV. THE QUANTUM CHARACTERISTIC FUNCTION

As shown in reference 1 the statistical properties of the radiation field at any time t may be obtained from the quantum characteristic functions, defined by

$$\begin{aligned}C_q(\xi,t)&\equiv\langle\exp[i\xi q(t)]\rangle=\mathrm{Tr}\{\rho\exp[i\xi q(t)]\},\\ C_p(\xi,t)&\equiv\langle\exp[i\xi p(t)]\rangle=\mathrm{Tr}\{\rho\exp[i\xi p(t)]\},\end{aligned}$$

where ρ is the density matrix for the complete system, evaluated at $t=t_1$, and ξ is a real parameter. It was demonstrated in reference 1 that the expectation value of the nth moment of $p(t)$ or $q(t)$ is given by the nth derivative of C_p or C_q with respect to $i\xi$' evaluated at $\xi=0$. Furthermore, the Fourier transforms of C_p and C_q give the probability distributions for $p(t)$ and $q,(t)$

respectively; i.e.,

$$\langle p'(t)|\rho|p'(t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_p(\xi,t)\exp[-i\xi p'(t)]d\xi,$$

$$\langle q'(t)|\rho|q'(t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_q(\xi,t)\exp[-i\xi q'(t)]d\xi.$$

Both C_q and C_p have the general form

$$C(\xi,t)=\mathrm{Tr}\{\rho\exp[i\xi(\alpha^*a^\dagger(t)+\alpha a(t))]\},\qquad(20)$$

where comparison with (2) shows that

$$C_p=C[\alpha=-i(\hbar\omega/2)^{1/2}\equiv\alpha_p],$$
$$C_q=C[\alpha=(\hbar/2\omega)^{1/2}\equiv\alpha_q].$$

From the solutions of the Heisenberg equations of motion, and knowledge of the initial density matrix for the system, we can evaluate the characteristic function. Let us now do this.

The first step is to transform the operator in (20) into normal product form, in which all annihilation operators appear to the right of all creation operators. The identity[22]

$$e^{i\xi[\alpha^*a^\dagger(t)+\alpha a(t)]}=e^{-\xi^2|\alpha|^2/2}e^{i\xi\alpha^*a^\dagger(t)}e^{i\xi\alpha a(t)}\qquad(21)$$

is a direct result of the commutation relations (9).

Next, we note that before the atoms are coupled to the field, the density matrix of the system factors as a direct product

$$\rho=\rho_a\prod_{j=1}^{N}\rho_j,\qquad(22)$$

where ρ_a is the density matrix for the field, and ρ_j is the density matrix for the jth atom. Since the atoms have a Boltzmann distribution before the coupling is turned on, ρ_j is a diagonal matrix in the energy representation, with matrix elements

$$\langle m_j|\rho_j|m_j\rangle=P(m_j)=\tau_j^{-m_j}/\textstyle\sum_{m_j}\tau_j^{-m_j},\qquad(23)$$

where we have used (13). The density matrix for the field is determined by the assumed input conditions; we may, however, express it in terms of an appropriate input wave function by the relation

$$\rho_a=|\psi_a\rangle\langle\psi_a|.\qquad(24)$$

Note that the wave functions and density matrices are constants in the Heisenberg picture, which we are using throughout this work, and are determined from the initial conditions of the problem.

Using the solutions of the equations of motion, (16) along with (21), (22), (23), and (24), we find that we can rewrite (20) in the form

$$C(\xi,t)=\exp[-(\xi^2/2)|\alpha|^2]A(\xi,t)B(\xi,t),\qquad(25)$$

where

$$A(\xi,t)=\mathrm{Tr}\{\rho_a e^{i\xi\alpha^*u^*(t-t_1)a^\dagger}e^{i\xi\alpha u(t-t_1)a}\}$$
$$=\langle\psi_a|e^{i\xi\alpha^*u^*(t-t_1)a^\dagger}e^{i\xi\alpha u(t-t_1)a}|\psi_a\rangle,\qquad(26a)$$

and

$$B(\xi,t)=\prod_{j=1}^{N}\sum_{m_j=0}^{M}P(m_j)\times\langle m_j|e^{i\xi\alpha^*v_j^*(t-t_1)b_j^+}e^{i\xi\alpha v_j(t-t_1)b_j^-}|m_j\rangle.\qquad(26b)$$

The important result here is that in (25) the characteristic function has been separated into a constant factor, which will be related to the zero-point field, a second factor which contains operators and wave functions relating only to the input field, and a third factor whose operators and wave functions depend only on the initial atomic states. The second factor $A(\xi,t)$ contains the amplification properties of the interaction, while the third $B(\xi,t)$ contains the noise properties. Product characteristic functions of this sort are obtained classically from the sum of statistically independent fields.[23] Thus, we can consider that each of the three terms of (25) represents an individual statistically independent field: The sum of the three fields forms the total field.

Consider the first term,

$$\exp[-(\xi^2/2)|\alpha|^2].$$

This has the classical form of the characteristic function of Gaussian noise. Its average energy is

$$\tfrac{1}{2}[\langle p^2\rangle+\omega^2\langle q^2\rangle]=\tfrac{1}{2}[|\alpha_p|^2+\omega^2|\alpha_q|^2]=\tfrac{1}{2}\hbar\omega.$$

We can thus identify this field with the zero-point field. Note that this term is invariant under the amplification or attenuation processes considered here.

Let us now evaluate $B(\xi,t)$. To do this we note from (17) that $v_j(t-t_1)$ is a small quantity, of order κ_j, so that we can expand the exponentials of (26b), keeping only the first order terms. Thus

$$B(\xi,t)\prod_{j=1}^{N}\sum_{m_j=0}^{M}P(m_j)\times\langle m_j|(1+i\xi\alpha^*v_j^*b_j^+)(1+i\xi\alpha v_jb_j^-)|m_j\rangle.$$

The terms linear in b_j^- and b_j^+ have no diagonal matrix elements, and from (6), we have

$$\langle m_j|b_j^+b_j^-|m_j\rangle=\lambda_{j,m}^2.$$

Thus, taking the indicated matrix elements, we obtain

$$B(\xi,t)=\prod_{j=1}^{N}\sum_{m_j=0}^{M}P(m_j)[1-\xi^2|\alpha|^2|v_j(t-t_1)|^2\lambda_{j,m}^2].$$

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[22] This is a special case of the identity,

$$e^Ae^B=e^{\frac{1}{2}[A,B]}e^{(A+B)}=e^{[A,B]}e^Be^A,$$

which holds when the commutator $[A,B]$ commutes with A and B. For a proof see A. Messiah *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. 1, p. 442.

[23] W. R. Bennett, Proc. I.R.E. 44, 604 (1956).

Using (14b) and the relation $\sum_{m_j=0}^{M} P(m_j)=1$, we have

$$B(\xi,t)=\prod_{j=1}^{N}[1-\xi^2|\alpha|^2|v_j(t-t_1)|^2\Lambda_j].$$

Next, since the v_j's are small quantities, we can put this expression in exponential form, i.e.,

$$B(\xi,t)=\exp\{-\xi^2|\alpha|^2\sum_j|v_j(t-t_1)|^2\Lambda_j\}. \tag{27}$$

Lastly, from (17) we note that $v_j(t-t_1)$ has a strong maximum in the vicinity of $\omega_j=\omega$, since the factor $\mu(1-\tau)\Lambda/2$ is small compared to ω (so long as the gain per cycle is small compared to unity).

Since Λ_j and τ_j vary comparatively slowly with ω_j, we may reasonably replace them by Λ and τ (evaluated at $\omega_j=\omega$), respectively, in (19) and (27). Then, substitution of (19) in (27) yields as a final result

$$B(\xi,t)=\exp\{-\xi^2|\alpha|^2(1-G)/(\tau-1)\}, \tag{28}$$

where $\tau=\exp(\hbar\omega/kT)$, and we have substituted

$$G\equiv|u(t-t_1)|^2=\exp\{\mu(1-\tau)\Lambda(t-t_1)\}.$$

Here we are taking explicit recognition of the fact that G is the power gain of the input field resulting from the interaction, a fact which will be shown below. (It might be noted that the above approximations to obtain B are unnecessary when the atoms are harmonic oscillators.)

Like the zero-point field term, $B(\xi,t)$ has the classical form of the characteristic function of Gaussian noise. Note also that none of the detailed properties of the atoms remain in B. It is dependent only on the gain produced from the interaction and on the temperature of the atoms. The average energy of this field, $\frac{1}{2}\{\langle p^2\rangle+\omega^2\langle q^2\rangle\}$, equals

$$\hbar\omega\left(\frac{1-G}{\tau-1}\right)=\frac{\hbar\omega(1-G)}{\exp(\hbar\omega/kT)-1}.$$

We thus identify it with the thermal, or spontaneous emission, noise produced by the atoms. For emphasis, we repeat the point that this noise field and the zero-point field are always additive and Gaussian.

Finally, let us consider the term

$$A(\xi,t)=\langle\psi_a|\exp\{i\xi\alpha^*G^{1/2}e^{i\omega(t-t_1)}a^\dagger\}\times\exp\{i\xi\alpha G^{1/2}e^{-i\omega(t-t_1)}a\}|\psi_a\rangle, \tag{29}$$

where we have set $u(t-t_1)$ of (26a) equal to $G^{1/2}e^{-i\omega(t-t_1)}$ [see (17) and (28)]. This term represents a field which is always a precise amplified or attenuated replica of the input field (excluding the zero-point field). The zero-point field has already been taken into account. To show that it indeed does not affect $A(\xi,t)$ in any way, we note that if the initial state is the vacuum state, i.e., if

$$|\psi_a\rangle=|0_a\rangle$$

then, since

$$\exp(\gamma a)|0_a\rangle=(1+\gamma a+\cdots)|0_a\rangle=|0_a\rangle, \tag{30}$$

where γ may be any complex number, we see that $A(\xi,t)$ reduces immediately to unity. Thus, all moments of the field vanish for all time, and so must the field itself. An interesting and fairly general case for which the input field may be resolved into a signal field plus an additive Gaussian noise field is explicitly worked out in Appendix B. However, to understand the general nature of $A(\xi,t)$, we need only examine its dependence on G. Any moment of the field represented by (29), say the nth, is given by

$$M_n=\left[\frac{\partial^n}{\partial(i\xi)^n}A(\xi,t)\right]_{\xi=0}.$$

By inspection of (29), it is clear, however, that the quantity

$$\left[\frac{\partial^n}{\partial(i\xi G^{1/2})^n}A(\xi,t)\right]_{\xi=0}=G^{-n/2}M_n$$

is independent of G. Thus, at equivalent times in the cycle [i.e., for $\omega(t-t_1)=2\pi l$, where l is an integer], we have

$$M_n(t)=G^{n/2}M_n(t_1).$$

This can occur only if, at the output, this field is an exact amplified replica of the input, with power gain G. And of course if we know the distributions of p and q for one time in a cycle, we know them for all times in the cycle, since the interaction is presumed to have appreciable effects only for times long compared to a cycle.

To sum up this discussion, we see that if, at the input to a linear maser-like amplifier or attenuator the field can be resolved into the sum an "input" field and the Gaussian zero-point field then the output field can be resolved into a sum of three fields; first the Gaussian zero-point field, which remains unchanged, second a Gaussian spontaneous emission or thermal field, and third a precisely amplified or attenuated replica of the input field. The argument can easily be extended to include a sequence of attenuating and/or amplifying processes. The zero-point field simply goes unchanged through the sequence. The other component fields of the output include the precise replica of the input multiplied by the overall gain or loss of the sequence, and the sum of the thermal or spontaneous emission Gaussian noise fields, each such field being of course multiplied by the net gain of all stages following the one in which it is evolved. A particular conclusion that may be drawn from this latter result is that if at the important times in the sequence, i.e., at the input and at the output, the total field is much larger than the zero-point field, then the zero-point field may be neglected everywhere, and quantum-mechanical effects enter the process only in the proper determination of the thermal or spontaneous emission noise power generated by each stage. All of these noise terms are additive and Gaussian everywhere in the sequence.

APPENDIX A

To solve the equations of motion (15) approximately, we take their Laplace transforms. Let $\bar{a}(s)$ be the transform of $a(t)$, $\bar{b}_j^-(s)$ be the transform of $b_j^-(t)$, eliminate $\bar{b}_j^-(s)$ from the coupled equations and we find that

$$\bar{a}(s)=\frac{\left(a-i\sum_j \frac{\kappa_j b_j^-}{s+i\omega_j}\right)}{\left(s+i\omega+\sum_k \frac{\kappa_k^2(\tau_k-1)\Lambda_k}{s+i\omega_k}\right)} \equiv \bar{u}(s)a+\sum \bar{v}_j(s)b_j^-, \quad \text{(A1)}$$

where $\bar{u}(s)$ and $\bar{v}_j(s)$ are the Laplace transforms of $u(t)$ and $v_j(t)$, respectively, while a and b_j^- are the Heisenberg operators at $t=t_1$.

The sum over k in the denominator of (A1) will in general have a real and imaginary part. The imaginary part will make a small correction of order κ^2 to the cavity frequency ω which we shall neglect. The real part which is also small will act as a loss (or gain) term. We approximate the real part of this sum by an integral:

$$\mathrm{Re}\left\{\sum_k \frac{\kappa_k^2(\tau_k-1)\Lambda_k}{s+i\omega_k}\right\} \to \int_{-\infty}^{\infty} \frac{\kappa_k^2(\tau_k-1)\Lambda_k\sigma(\omega_k)d\omega_k}{s+i\omega_k}. \quad \text{(A2)}$$

We assume the numerator is a slowly varying function of ω_k in the neighborhood of $\omega_k=is$. Furthermore, since the pole in the s plane is approximately located at $s=-i\omega$ [provided the integral in (A2) is small], we may write the sum in (A2) as

$$\kappa_\omega^2(\tau_\omega-1)\Lambda_\omega\sigma(\omega)\int_{-\infty}^{\infty}\frac{d\omega_k}{s+i\omega_k} \equiv \pi\kappa^2\sigma(\tau-1)\Lambda. \quad \text{(A3)}$$

We, therefore, define the parameter

$$\mu=2\pi\kappa_\omega^2\sigma(\omega) \quad \text{(A4)}$$

which we require to be small but finite in the limit as $\sigma(\omega)\to\infty$ and $\kappa_\omega^2\to 0$.

With these approximations (A1) becomes

$$\bar{a}(s)=\frac{a}{s+i\omega+\frac{1}{2}\mu(\tau-1)\Lambda} -i\sum_j \frac{\kappa_j b_j^-}{(s+i\omega_j)[s+i\omega+\frac{1}{2}\mu(\tau-1)\Lambda]}. \quad \text{(A5)}$$

Taking the inverse transform we find

$$a(t)=u(t-t_1)a+\textstyle\sum_j v_j(t-t_1)b_j^-, \quad \text{(A6)}$$

where $u(t)$ and $v_j(t)$ are given by (17) of the text.

APPENDIX B

An interesting and rather general case to consider is when the input field is resolvable into a signal field accompanied by an arbitrary amount of Gaussian noise. What is needed for the explicit evaluation of the term $A(\xi,t)$ in the characteristic function is the initial wave function or density matrix for the field. Consider a wave function of the form

$$|\psi_a\rangle=N\exp(\epsilon^* a^\dagger+\delta a^\dagger c^\dagger)|0\rangle, \quad \text{(B1)}$$

where

$$\delta=[\bar{n}_N/(1+\bar{n}_N)]^{1/2},$$
$$\epsilon=[(\bar{n}_S)^{1/2}/(1+\bar{n}_N)]e^{i\varphi},$$
$$N^{-2}=(1+\bar{n}_N)\exp[\bar{n}_N/(1+\bar{n}_N)].$$

In (B1) we have introduced, for mathematical convenience, an auxiliary (fictitious) boson field represented by the promotion operator $c^\dagger$, which is defined to commute with a and $a^\dagger$ and to satisfy the boson commutation relation

$$[c,c^\dagger]=1.$$

Finally the ket $|0\rangle$ represents the vacuum state for both fields, i.e., $|0\rangle\equiv|0_a\rangle|0_c\rangle$. The fictitious field is introduced simply for mathematical convenience, since calculations with the resulting wave function can be accomplished by manipulation of the operators. The density matrix for the physically meaningful field results from taking the trace over the auxiliary field of the density matrix

$$|\psi_a\rangle\langle\psi_a|,$$

yielding

$$\rho_a=N^2\textstyle\sum_{n_c}\exp(\epsilon^* a^\dagger)\langle n_c|\exp(\delta a^\dagger c^\dagger)|0_c\rangle|0_a\rangle \times\langle 0_a|\langle 0_c|\exp(\delta ac)|n_c\rangle\exp(\epsilon a)$$
$$=N^2\textstyle\sum_n(\delta^{2n}/n!)(a^\dagger)^n\exp(\epsilon^* a^\dagger)|0_a\rangle\langle 0_a|(a)^n\exp(\epsilon a).$$

It is clear that the wave function $|\psi_a\rangle$ and the density matrix ρ_a are equivalent for the evaluation of the expectation value of any operator function of a and $a^\dagger$; however, $|\psi_a\rangle$ is the more simple. It will appear that at the initial time $t=t_1$ the physically meaningful field represented by the wave function (B1) is a sum of signal and noise fields, with average signal energy $\bar{n}_S\hbar\omega$ and average noise energy $(\bar{n}_N+\frac{1}{2})\hbar\omega$. $\bar{n}_S$ and $\bar{n}_N$, respectively, have the significance of the average number of signal and noise photons in the "input" field, the $\frac{1}{2}\hbar\omega$ being the zero-point energy.

In working with this wave function we will use two identities. The first is

$$\exp(\delta ac)\exp(\delta a^\dagger c^\dagger)|0\rangle=\frac{1}{1-\delta^2}\exp\left(\frac{\delta a^\dagger c^\dagger}{1-\delta^2}\right)|0\rangle. \quad \text{(B2)}$$

This may be proved directly by expanding the exponentials in power series in their arguments, then by repeated commutation carrying all the a and c factors to the right until they are eliminated, and finally summing the resulting series. The second identity is

$$e^{ya}e^{za^\dagger}=e^{yz}e^{za^\dagger}e^{ya}, \tag{B3}$$

which holds when y and z commute with each other and with a and $a^\dagger$.[22]

With the wave function (B1) let's examine the field represented by the term (29) of the characteristic function. We have, from (29),

$$A(\xi,t)=\langle\psi_a|\exp(i\xi\lambda^*a^\dagger)\exp(i\xi\lambda a)|\psi_a\rangle, \tag{B4}$$

where

$$\lambda\equiv\alpha G^{1/2}\exp[-i\omega(t-t_1)].$$

Using (B3) to commute the central two factors of (B4) and writing out the wave function from (B1), we have

$$A(\xi,t)=N^2\exp(\xi^2|\lambda|^2)\langle 0|\exp(\delta ac)\exp[(\epsilon+i\xi\lambda)a] \times\exp[(\epsilon^*+i\xi\lambda^*)a^\dagger]\exp(\delta a^\dagger c^\dagger)|0\rangle. \tag{B5}$$

Using (B3) and (B2) in turn, we can carry the term $\exp(\delta ac)$ to the right. The result is

$$A(\xi,t)=[N^2/(1-\delta^2)]\exp(\xi^2|\lambda|^2) \times\langle 0|\exp[(\epsilon+i\xi\lambda)a]\exp[(\epsilon^*+i\xi\lambda^*)a^\dagger] \times\exp[\delta(\epsilon^*+i\xi\lambda^*)c]\exp[\delta a^\dagger c^\dagger/(1-\delta^2)]|0\rangle. \tag{B6}$$

Next we use (B3) successively to carry the terms $\exp[\delta(\epsilon^*+i\xi\lambda^*)c]$ and $\exp[(\epsilon+i\xi\lambda)a]$ to the right until they disappear (i.e., become unity) against $|0\rangle$ [see (30)]. The remaining operators, exponential functions of $a^\dagger$, $c^\dagger$, and $a^\dagger c^\dagger$, similarly become unity when applied to the left to $\langle 0|$, and thus we obtain the result

$$A(\xi,t)=\frac{N^2}{1-\delta^2}\exp\left(\xi^2|\lambda|^2+\frac{(\epsilon+i\xi\lambda)(\epsilon^*+i\xi\lambda^*)}{1-\delta^2}\right).$$

Putting in the values of the parameters from (B1) and (B4) we have

$$A(\xi,t)=\exp\{i\xi(G\bar{n}_S)^{1/2}[\alpha e^{-i[\omega(t-t_1)+\varphi]}+\alpha^*e^{i[\omega(t-t_1)+\varphi]}]\} \times\exp(-\xi^2|\alpha|^2G\bar{n}_N). \tag{B7}$$

Following our earlier discussion of the complete characteristic function, we see that the field represented by (B7) may be resolved into the sum of two statistically independent fields. There is first the signal field, represented by the exponential term in A whose argument is linear in ξ. This is a field whose amplitude and phase are precisely defined and whose average energy is

$$G\hbar\omega\bar{n}_S.$$

Second, there is a Gaussian noise field represented by the term

$$\exp(-\xi^2|\alpha|^2G\bar{n}_N).$$

Its average energy is

$$G\hbar\omega\bar{n}_N.$$

We see, therefore, that the wave function (B1) represents an input which is a sum of signal and noise fields (plus the Gaussian zero-point field when the complete characteristic function is considered). Also we see that at times $t>t_1$, after the interaction is turned on, both the signal and noise fields represented in $A(\xi,t)$ simply grow or decay with power gain G, in accord with the discussion of $A(\xi,t)$ in the text.