Elementary Particle Theory of Composite Particles*

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Any nonrelativistic theory may be rewritten by introducing fictitious elementary particles with arbitrary properties. No physical predictions are affected, provided that the interaction part of the Hamiltonian is correspondingly modified. The fictitious elementary particle provides a good representation of a real composite particle if the modified interaction is sufficiently weakened for perturbation theory to work. It corresponds to a truly elementary particle with infinite bare mass, and hence with $Z=0$. We show how the latter condition yields a sum rule for the coupling of a composite particle to its constituents as a function of energy. The sum rule can be used to evaluate such coupling constants as that for the proton-electron-hydrogen vertex. The mathematical method used is that developed by Schmidt for the study of the Fredholm equation, and corresponds to the extraction of a single factor from the full Fredholm determinant.

I. INTRODUCTION

THIS is the first of a series of articles, in which we hope to develop a method for the calculation of strong interaction processes.

In this first paper we show how it is possible to introduce fictitious elementary particles with arbitrary properties ("quasi-particles") into any nonrelativistic theory without changing any physical predictions. In order to accomplish this the interaction among the original, truly elementary, particles must be modified according to well-defined rules.

In the second and third papers we will show that the introduction of quasi-particles in nonrelativistic theories can always be managed in such a way that the modified interaction is weaker than the original one, and in fact weak enough so that perturbation theory works. The quasi-particles must be chosen to correspond to real bound particles, or to resonances, or, more generally, to Regge poles.

In the fourth paper we will extend these ideas to the fully relativistic case.[1] Here we shall see that the quasi-particles can provide the force that makes their introduction a necessity.

In further papers we hope to be able to offer a proof (or at least an argument) that the introduction of quasi-particles in relativistic theories may render the full series of Feynman diagrams convergent. And ultimately, we hope to start a program of numerical calculation.

There are some special problems which are discussed in detail in this paper. A theory modified by the introduction of elementary particles is actually physically equivalent to the original theory only if their bare energy is much larger than any energy explored by experiment. Or to put it another way, the quasi-particles must be introduced with infinite unrenormalized mass.[2] If there is a stable bound state which we wish to represent by introducing a quasi-particle, then (as shown in Sec. IV) its infinite bare mass implies that the fictitious elementary particle must have renormalization factor Z equal to zero. In fact, many authors[3] have proposed $Z=0$ as a way of distinguishing bound from elementary particles. But what we prove here is that this is the only possible way of making this distinction.[4]

This does not quite answer the question, of whether experiment *can* decide what sort of elementary particles exist, since we do not show whether experiment can tell whether a particle has infinite bare mass, or if $Z=0$. But we are able to show (in Sec. V) that the condition $Z=0$ provides a sum rule[5] for the coupling of the particle to its presumed constituents at various energies. This sum rule, for weakly bound systems, determines the coupling constant of the particle. An elementary particle (with $0<Z<1$) would have a smaller coupling constant.[6]

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[1] This has already been discussed in a preliminary way by the author in the *Proceedings of the 1962 High-Energy Conference at CERN* (CERN, Geneva, 1962), p. 683. A similar but perhaps inequivalent approach to the problem of introducing composite particle fields into a Lagrangian has been developed by A. Salam, Nuovo Cimento **25**, 224 (1962).

[2] This point has also been made by M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961). However, they work in the dispersion formalism, so in their case the equivalence of composite and elementary particle theories is built in. In their work the bare mass has to be defined in a manner very different from our Eq. (3); also, the infinity of the bare mass is for them a definition and not a theorem.

[3] J. C. Howard and B. Jouvet, Nuovo Cimento **18**, 466 (1960); M. J. Vaughan, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961); R. Acharya (to be published); A. Salam, reference 1. Doubtless there are many other references for $Z=0$ of which the author is unaware.

[4] A similar equivalence theorem has been proven by Vaughan, Aaron, and Amado, reference 3, for the special case where V is a separable potential, and apparently also in a more general case. We present the proof here in our own language for reasons of clarity and completeness, and also to facilitate the extension of these methods to resonances and to multiparticle processes. We want to stress that our primary interest throughout lies not in questions of principle about the definition of "elementary," but in exploiting the fact that the introduction of fictitious elementary particles into a theory will change the interaction of the theory and make perturbation theory work.

[5] This sum rule is equivalent to Eq. (20) of Vaughan, Aaron, and Amado, reference 3, and also to a nonrelativistic version of a formula of Acharya, reference 3. We have attempted to express it here in a more useful form.

[6] This upper limit on coupling constants is of the same sort as that discovered by M. Ruderman and S. Gasiorowicz, Nuovo Cimento **8**, 861 (1958). See also M. Ruderman, Phys. Rev. **127**, 312 (1962).

The general mathematical method employed in introducing quasi-particles is based on one of the classic techniques for solving linear integral equations, the Schmidt method.[7] We provide in the Appendix a general treatment of the Schmidt method, and we explain its relation to the Fredholm approach.

II. SYNTHETIC ELEMENTARY PARTICLES

Suppose there are no elementary particles in a particular channel described by a given Hamiltonian H. We shall show how to construct a new Hamiltonian $\mathbf{H}$ which does involve an elementary particle, but which yields precisely the same physical predictions as does the original H.

The original H is split as usual into an unperturbed part H_0 and an interaction V. The continuum eigenstates of H_0 are labeled by their energy E, and perhaps some discrete quantum numbers n, so that

$$H_0|E,n\rangle=E|E,n\rangle, \quad (E>0) \tag{1}$$

$$\langle E',n'|E,n\rangle=\delta_{n'n}\delta(E'-E). \tag{2}$$

The assumption that there are no elementary particles in H just tells us that H_0 has no discrete eigenstates, so that the $|E,n\rangle$ form a complete set. [There is no need to be too explicit as to the constitution of these states; a two body system with definite J may be kept in mind as a typical case.]

The reconstructed Hamiltonian $\mathbf{H}$ will also be split into a new unperturbed part $\mathbf{H}_0$ and a new interaction $\mathbf{V}$. The unperturbed part will again have a spectrum of orthonormal continuum eigenstates,

$$\mathbf{H}_0|E,n\rangle=E|E,n\rangle,$$

but it will also have a discrete elementary particle eigenstate $|0\rangle$,

$$\mathbf{H}_0|0\rangle=E_0|0\rangle, \tag{3}$$

$$\langle En|0\rangle=0, \tag{4}$$

$$\langle 0|0\rangle=1. \tag{5}$$

Together, $|0\rangle$ and the $|E,n\rangle$ form a complete set spanning a new Hilbert space. (If our notation were impeccable, we should have to distinguish the continuum eigenstates of $\mathbf{H}_0$ by boldface type, since the Hilbert space on which $\mathbf{H}$ acts is larger than the original one. But no confusion should arise.)

In order that $\mathbf{H}$ should be physically equivalent to H, the new interaction $\mathbf{V}$ must be specified in terms of V according to some rules. But the specification is not unique; it depends on the choice of "bare vertices" $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$, which can be any linear combinations of the continuum states $|E,n\rangle$ and $\langle E,n|$. The rules for constructing the matrix elements of the new interaction $\mathbf{V}$ are:

$$\langle E'n'|\mathbf{V}|En\rangle=\langle E'n'|V|En\rangle-U_{n'}(E')\bar{U}_n(E), \tag{6}$$

$$\langle E'n'|\mathbf{V}|0\rangle=(-NE_0)^{1/2}U_{n'}(E'), \tag{7}$$

$$\langle 0|\mathbf{V}|En\rangle=(-NE_0)^{1/2}\bar{U}_n(E), \tag{8}$$

$$\langle 0|\mathbf{V}|0\rangle=-E_0(1-N), \tag{9}$$

where

$$U_n(E)=\langle En|V|\Gamma\rangle, \tag{10}$$

$$\bar{U}_n(E)=\langle\bar{\Gamma}|V|En\rangle, \tag{11}$$

and

$$N=1-\langle\bar{\Gamma}|V|\Gamma\rangle. \tag{12}$$

This prescription for $\mathbf{V}$ may be written more concisely if we define a "reduced" interaction (acting in the original Hilbert space):

$$V_1\equiv V-V|\Gamma\rangle\langle\bar{\Gamma}|V. \tag{13}$$

Then (6)–(9) may be written

$$\langle E'n'|\mathbf{V}|En\rangle=\langle E'n'|V_1|En\rangle, \tag{14}$$

$$\langle E'n'|\mathbf{V}|0\rangle=(-E_0/N)^{1/2}\langle E'n'|V_1|\Gamma\rangle, \tag{15}$$

$$\langle 0|\mathbf{V}|En\rangle=(-E_0/N)^{1/2}\langle\bar{\Gamma}|V_1|En\rangle, \tag{16}$$

$$\langle 0|\mathbf{V}|0\rangle=(-E_0/N)\langle\bar{\Gamma}|V_1|\Gamma\rangle. \tag{17}$$

But it would be a mistake to summarize our prescription by saying simply that

$$\mathbf{V}=V_1,$$

$$|0\rangle=(-E_0/N)^{1/2}|\Gamma\rangle, \quad \langle 0|=(-E_0/N)^{1/2}\langle\bar{\Gamma}|,$$

for $\mathbf{V}$ and V_1 act in different Hilbert spaces. In particular, $|0\rangle$ is orthogonal to all continuum states $|En\rangle$, whereas $|\Gamma\rangle$ certainly is not.

With (6)–(9), or equivalently with (14)–(17), we can now prove that $\mathbf{H}$ yields precisely the same predictions as does H for all processes occurring at an energy such that

$$|W|\ll|E_0|.$$

The two theories become entirely equivalent[2] only in the limit as $|E_0|\to\infty$. (In Sec. IV we show that in this limit the renormalization factor Z of a stable elementary particle approaches zero.[3]) The condition that $|E_0|$ be infinite is the sole memory retained by the new Hamiltonian that there really is not any elementary particle in the channel. Physical consequences of this condition will be discussed in Sec. V.

Before closing this section, we pause to make a remark which will become important in future articles of this series. There is no compulsion to choose the "incoming bare vertex" $\langle\bar{\Gamma}|$ as the adjoint of $|\Gamma\rangle$, nor to choose either or both as constants. (In fact, in our next paper we shall show that the "ideal" choice of $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$ is such that they are not adjoints of each other, and such that they both depend on the energy.) It follows then that the new interaction $\mathbf{V}$ may be energy dependent, and may also be non-Hermitian. But physical predictions are still the same in the two theories.

[7] See, e.g., R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), 1st English ed., Vol. I, p. 155. The original reference quoted by Courant and Hilbert is: E. Schmidt, Math. Ann. **64**, 161 (1907).

III. EQUIVALENCE THEOREM[4]

All physical predictions of the original Hamiltonian H can be derived from an operator $T(W)$, defined for all complex energies W by the Lippmann-Schwinger integral equation

$$T(W)=V+V[W-H_0]^{-1}T(W). \tag{18}$$

For example, the S matrix is

$$S_{n'n}(E)=\delta_{n'n}-2\pi i\langle En'|\,T(E+i\epsilon)\,|En\rangle \tag{19}$$

and the stable one-particle states have energies at the poles of matrix elements of $T(W)$ for negative real W.

Likewise, the physical predictions of the reconstructed Hamiltonian $\mathbf{H}$ can be derived in the same way from an operator $\mathbf{T}(W)$, defined by

$$\mathbf{T}(W)=\mathbf{V}+\mathbf{V}[W-\mathbf{H}_0]^{-1}\mathbf{T}(W). \tag{20}$$

It is our task now to show that if $\mathbf{V}$ is chosen according to the prescriptions (14)–(17), then in the limit as $E_0\to\pm\infty$

$$\langle E'n'|\,\mathbf{T}(W)\,|En\rangle\to\langle E'n'|\,T(W)\,|En\rangle.$$

To do this we shall have to rewrite the formulas for both $T(W)$ and $\mathbf{T}(W)$.

A. Original Theory

Suppose that the "reduced" interaction V_1 defined by Eq. (13) were the whole interaction. Then, the physical T operator would not be $T(W)$, but $T_1(W)$, defined by

$$T_1(W)=V_1+V_1[W-H_0]^{-1}T_1(W), \tag{21}$$

or less concisely

$$T_1(W)=V_1+\sum_n\int dE\;V_1|E,n\rangle(W-E)^{-1}\langle E,n|\,T_1(W). \tag{22}$$

It is the key point of the classic Schmidt method[7] of solving integral equations that the solution of an equation like (18) can be obtained immediately if we know the solution of the reduced equation (21). A little algebra [see Appendix, (A14) and (A15)] shows that the answer is

$$T(W)=T_1(W)+N^{-2}T_1(W)\,|\Gamma\rangle\Delta(W)\langle\bar{\Gamma}|\,T_1(W), \tag{23}$$

where the "propagator" is

$$\Delta(W)=[1-J(W)]^{-1}, \tag{24}$$

$$J(W)=N^{-2}\langle\bar{\Gamma}|\,V_1[W-H_0]^{-1}T_1(W)\,|\Gamma\rangle \tag{25}$$

$$=1-N^{-1}+N^{-2}\langle\bar{\Gamma}|\,T_1(W)\,|\Gamma\rangle. \tag{26}$$

[The general Schmidt method is described in detail in the Appendix, where we also discuss its connection with the more familiar Fredholm (or N/D) approach. We just remark in passing here that if $D(W)$ and $D_1(W)$ are the Fredholm determinants for the integral equations for $T(W)$ [Eq. (18)] and $T_1(W)$ [Eq. (21)], respectively, then

$$D(W)=D_1(W)\Delta^{-1}(W). \tag{27}$$

So the Schmidt method extracts one factor from the full Fredholm determinant. In our next paper we shall show that the usual Fredholm method just corresponds to a particular choice of bare vertices.]

B. New Theory

We can expand Eq. (20) in intermediate states [as we did to get (22)], but the sum will here include the discrete elementary particle state $|0\rangle$, as well as the continuum states $|E,n\rangle$. In order to isolate the effects of the elementary particle, let us define a "proper" $\mathbf{T}$-operator $\mathbf{T}_1(W)$ as what $\mathbf{T}(W)$ would be if the elementary particle were omitted in sums over intermediate states. That is,

$$\mathbf{T}_1(W)=\mathbf{V}+\sum_n\int dE\;\mathbf{V}|E,n\rangle(W-E)^{-1}\langle E,n|\,\mathbf{T}_1(W). \tag{28}$$

It is well known[8] that $\mathbf{T}(W)$ can be expressed in terms of $\mathbf{T}_1(W)$ in a simple way:

$$\mathbf{T}(W)=\mathbf{T}_1(W)+\mathbf{T}_1(W)\,|0\rangle\boldsymbol{\Delta}(W)\langle 0|\,\mathbf{T}_1(W), \tag{29}$$

$$\boldsymbol{\Delta}(W)=[W-E_0-\boldsymbol{\Pi}(W)]^{-1}, \tag{30}$$

$$\boldsymbol{\Pi}(W)=\langle 0|\,\mathbf{T}_1(W)\,|0\rangle. \tag{31}$$

The physical significance of these equations is apparent. In Eq. (29), $\langle 0|\,\mathbf{T}_1$ is the complete vertex that converts the incoming particles into a virtual elementary particle; $\boldsymbol{\Delta}$ is the complete elementary particle propagator; $\mathbf{T}_1|0\rangle$ is the complete final vertex; and $\mathbf{T}_1$ is the sum of graphs that do not arise from one-elementary-particle exchange. Also, $\boldsymbol{\Pi}$ is the proper self-energy insertion, and (31) may be recognized as one of Dyson's equations.

C. Comparison

According to the prescriptions (14)–(17), all matrix elements of $\mathbf{V}$ are equal to corresponding matrix elements of V_1, if we perform the substitutions

$$|0\rangle\to(-E_0/N)^{1/2}|\Gamma\rangle,$$
$$\langle 0|\to(-E_0/N)^{1/2}\langle\bar{\Gamma}|.$$

By taking matrix elements of (22) and (28), we see that the same equalities must hold between corresponding matrix elements of $\mathbf{T}_1(W)$ and $T_1(W)$:

$$\langle E'n'|\,\mathbf{T}_1(W)\,|En\rangle=\langle E'n'|\,T_1(W)\,|En\rangle, \tag{32}$$

[8] See, e.g., B. Zumino, in *Lectures on Field Theory*, edited by E. R. Caianiello (Academic Press Inc., New York, 1961), p. 40; G. C. Wick, Rev. Mod. Phys. **27**, 339 (1955). Wick treats the case where there actually is a stable one-particle state, and Zumino discusses unstable particles. The algebra is the same in both cases, and actually has nothing to do with the existence of real particles.

$$\langle 0|\mathbf{T}_1(W)|En\rangle=(-E_0/N)^{1/2}\langle\bar{\Gamma}|T_1(W)|En\rangle, \quad (33)$$

$$\langle En|\mathbf{T}_1(W)|0\rangle=(-E_0/N)^{1/2}\langle En|T_1(W)|\Gamma\rangle, \quad (34)$$

$$\langle 0|\mathbf{T}_1(W)|0\rangle=(-E_0/N)\langle\bar{\Gamma}|T_1(W)|\Gamma\rangle. \quad (35)$$

It follows from (29) and (32), (33), (34) that the continuum matrix elements of the operator $\mathbf{T}(W)$ are

$$\langle E'n'|\mathbf{T}(W)|En\rangle=\langle E'n'|T_1(W)|En\rangle - (E_0/N)\mathbf{\Delta}(W)\langle E'n'|T_1(W)|\Gamma\rangle\langle\bar{\Gamma}|T_1(W)|En\rangle, \quad (36)$$

and from (31), (35), and (26) we see that the self-energy part is

$$\mathbf{\Pi}(W)=-E_0\{1-N+NJ(W)\} \quad (37)$$

so that (30) gives the propagator as

$$\mathbf{\Delta}(W)=-E_0^{-1}N^{-1}[1-J(W)-(W/E_0N)]^{-1}. \quad (38)$$

Combining (36) and (38), we get

$$\langle E'n'|\mathbf{T}(W)|En\rangle=\langle E'n'|T_1(W)|En\rangle +N^{-2}[1-J(W)-(W/E_0N)]^{-1}\langle E'n'|T_1(W)|\Gamma\rangle \times\langle\bar{\Gamma}|T_1(W)|En\rangle. \quad (39)$$

This is to be compared with the continuum matrix elements of $T(W)$ in the original theory, which are given by (23) and (24) as

$$\langle E'n'|T(W)|En\rangle=\langle E'n'|T_1(W)|En\rangle +N^{-2}[1-J(W)]^{-1}\langle E'n'|T_1(W)|\Gamma\rangle \times\langle\bar{\Gamma}|T_1(W)|En\rangle. \quad (40)$$

Clearly, (39) and (40) are equal for energies W such that $|W|$ is sufficiently small in comparison with $|E_0|$.

Hence, the original theory and the theory modified by the introduction of the quasi-particle can only be distinguished by experiments at high energy. Only in the limit $|E_0|\to\infty$ are the two theories entirely equivalent.

IV. REAL PARTICLES

Now we know how to put an elementary particle into a theory in which it originally did not appear. Indeed, we can put in as many as we like, with arbitrary bare vertices Γ, without affecting any physical predictions. But so far, we have not connected this purely mathematical trickery with the existence of real bound states and resonances. We now consider how we should choose these synthetic elementary particles to best represent the real ones.

Suppose that there is a physical bound state with energy $-B<0$. It must correspond to a pole of $T(W)$ [defined by (18)] at $W=-B$. Let us think of how such a pole could arise. If V were sufficiently weak, the integral equation for $T(W)$ could be solved by perturbation theory, which gives the Neumann series (i.e., the Born series)

$$T(W)=V+V[W-H_0]^{-1}V+\cdots. \quad (41)$$

But no term in this series has any poles. Hence the bound state can only exist if V is too strong for (41) to converge, at least for W in some neighborhood of $-B$.

It was just to handle such integral equations that the Schmidt method[7] was developed. The trick is to try to approximate the interaction V by the separable interaction

$$V_S=V|\Gamma\rangle\langle\bar{\Gamma}|V. \quad (42)$$

If V_S is a good enough approximant to V, then the reduced interaction V_1, given by (13) as $V-V_S$, will be weak enough so that the corresponding T-operator $T_1(W)$ will not have any poles.

The method for accomplishing this will be discussed in the next paper of this series. For the present, we will just assume that it has been done, and that the reduced T-operator $T_1(W)$ does not have a pole at $-B$. But then the only place that the pole can arise in $T(W)$ is in the propagator $\Delta(W)$ in Eq. (23). So the binding energy is to be found as the root of the equation

$$J(-B)=1. \quad (43)$$

This appears paradoxically as if the physical binding energy depended upon the arbitrary separable potential V_S. In fact, V_S is not entirely arbitrary, since it must approximate V well enough so that V_1 is too weak to give T_1 a pole. But this still allows a range of possible choices of V_S. We are forced to the surprising conclusion that, however, we choose V_S within this range, the value of B obtained from (43) will be the same!

The Schmidt method allows us to compute not only the binding energy but also the wave function of any bound state. To see this, let us consider how $T(W)$ behaves as $W\to-B$. If the propagator $\Delta(W)$ has a pole at $W=-B$ with residue $\mathfrak{z}$, then from (23) we see that as $W\to-B$

$$T(W)\to\frac{\mathfrak{z}N^{-2}T_1(-B)|\Gamma\rangle\langle\bar{\Gamma}|T_1(-B)}{W+B}. \quad (44)$$

But we can write a general formal solution of (18) for $T(W)$:

$$T(W)=[W-H_0][W-H]^{-1}V. \quad (45)$$

If the bound state $|\mathfrak{B}\rangle$ is defined by

$$H|\mathfrak{B}\rangle=-B|\mathfrak{B}\rangle, \quad (46)$$

$$\langle\mathfrak{B}|\mathfrak{B}\rangle=1, \quad (47)$$

then (45) shows that as $W\to-B$

$$T(W)\to-\frac{[H_0+B]|\mathfrak{B}\rangle\langle\mathfrak{B}|V}{W+B} =\frac{[H_0+B]|\mathfrak{B}\rangle\langle\mathfrak{B}|[H_0+B]}{W+B}. \quad (48)$$

It follows upon comparison of (44) with (48) that

$$|\mathfrak{B}\rangle = -\mathfrak{z}^{1/2}N^{-1}[H_0+B]^{-1}T_1(-B)|\Gamma\rangle, \quad (49)$$

$$\langle\mathfrak{B}| = -\mathfrak{z}^{1/2}N^{-1}\langle\bar{\Gamma}|T_1(-B)[H_0+B]^{-1}. \quad (50)$$

(We are making a convenient choice of phase for $|\mathfrak{B}\rangle$. If $\langle\bar{\Gamma}|$ is chosen as the adjoint, or minus the adjoint, of $|\Gamma\rangle$, then $\mathfrak{z}^{1/2}$ must be, respectively, real or pure imaginary, and so the propagator has, respectively, positive or negative residue.) A more explicit formula for the wave function is

$$\langle En|\mathfrak{B}\rangle = -\frac{\mathfrak{z}^{1/2}N^{-1}}{E+B}\langle En|T_1(-B)|\Gamma\rangle. \quad (51)$$

For completeness, we will give some corresponding results for the case where the stable particle $\mathfrak{B}$ is really elementary. Here, instead of reducing the potential $\mathbf{V}$, we assume from the beginning that it is too weak to give a bound state pole in the "proper" operator $\mathbf{T}_1(W)$. Then we see from (29) that the pole can only arise in the propagator $\mathbf{\Delta}(W)$, and so

$$B+E_0+\mathbf{\Pi}(-B)=0. \quad (52)$$

This does not allow us to calculate the energy $-B$, but rather provides a relation between it and the unrenormalized energy E_0.

The physical particle wave function can be calculated here by the same method as in the nonelementary case. We obtain

$$|\mathfrak{B}\rangle = -\frac{Z^{1/2}}{\mathbf{H}_0+B}\mathbf{T}_1(-B)|0\rangle, \quad (53)$$

$$\langle\mathfrak{B}| = -\langle 0|\mathbf{T}_1(-B)\frac{Z^{1/2}}{\mathbf{H}_0+B}, \quad (54)$$

where Z is the residue of the pole of $\mathbf{\Delta}(W)$ at $W=-B$. [It is apparent from (53) and (54) that $Z^{1/2}$ is real, so that $Z>0$.] The continuum part of the wave function is then

$$\langle En|\mathfrak{B}\rangle = \frac{-Z^{1/2}}{E+B}\langle En|\mathbf{T}_1(-B)|0\rangle. \quad (55)$$

However, there is now also a component along the discrete bare elementary particle state

$$\langle 0|\mathfrak{B}\rangle = \frac{-Z^{1/2}}{E_0+B}\langle 0|\mathbf{T}_1(-B)|0\rangle. \quad (56)$$

Using (31) and (52), we get

$$\langle 0|\mathfrak{B}\rangle = Z^{1/2} \quad (57)$$

which verifies that Z [defined here as the residue of $\mathbf{\Delta}(W)$ at $W=-B$] is, in fact, just the conventional renormalization constant.

If the interaction $\mathbf{V}$ in the elementary particle theory is chosen according to our rules, then (37) tells us that the condition (52) determining the energy $-B$ of the particle may be written

$$J(-B)=1+(B/E_0N). \quad (58)$$

Also, the wave function (55) is

$$\langle En|\mathfrak{B}\rangle = -\frac{(-E_0Z/N)^{1/2}}{E+B}\langle En|T_1(-B)|\Gamma\rangle. \quad (59)$$

Furthermore, (38) gives the residue of $\mathbf{\Delta}(W)$ as

$$Z=-(\mathfrak{z}/E_0N)[1-(\mathfrak{z}/E_0N)]^{-1}. \quad (60)$$

To get (60), we used the definition of $\mathfrak{z}$ as the residue of $\Delta(W)$, which gives

$$-\mathfrak{z}^{-1}=\left(\frac{d}{dW}J(W)\right)_{W=-B}. \quad (61)$$

Combining (59) and (60), we see that the wave function of the physical elementary particle is

$$\langle En|\mathfrak{B}\rangle = -\frac{\mathfrak{z}^{1/2}N^{-1}}{(E+B)[1-(\mathfrak{z}/E_0N)]^{1/2}} \times\langle En|T_1(-B)|\Gamma\rangle. \quad (62)$$

As $|E_0|\to\infty$, Eq. (58) for B becomes the same as Eq. (43) in the nonelementary case, and Eq. (63) for the wave function becomes the same as Eq. (51). This is only to be expected from the equivalence theorem proved in Sec. III; since the energy and wave function of a stable one-particle state are observables, they are equal in the original theory and in the theory modified by the introduction of a quasi-particle with infinite bare mass.

A more significant result is obtained from (60): In the limit as $|E_0|\to\infty$, the renormalization constant $Z\to 0$. (It should be recalled that N and $\mathfrak{z}$ are defined by the original Hamiltonian H, and by our choice of $|\Gamma\rangle$ and $\langle\bar{\Gamma}|$; they do not depend upon E_0.) More will be said about the condition $Z=0$ in the next section.

Other types of real particles, such as Regge poles and resonances, will be treated in our next paper.

V. SIGNIFICANCE OF $Z=0$

We have seen that $Z^{1/2}$ is the matrix element between a physical elementary particle state and the corresponding bare state. It follows then that

$$0<Z<1. \quad (63)$$

It has been often remarked that this inequality sets an upper limit[6] on the coupling constant of the particle to any set of constituents, and that this limit is attained when $Z=0$, i.e., when the particle is actually not elementary. We offer here a very simple derivation of this result and give a convenient expression for the maximum coupling constant.

The physical one-particle state $|\mathfrak{B}\rangle$ obeys the

Schrödinger equation (46), which may be written

$$|\mathfrak{B}\rangle = -[H_0+B]^{-1}V|\mathfrak{B}\rangle. \tag{64}$$

Therefore, the continuum part of its wave function is

$$\langle E,n|\mathfrak{B}\rangle = -[1/(E+B)]\langle E,n|V|\mathfrak{B}\rangle. \tag{65}$$

But since $|\mathfrak{B}\rangle$ is normalized, we must have

$$1=Z+\int dE \sum_n |\langle E,n|\mathfrak{B}\rangle|^2, \tag{66}$$

where Z is a sum over bare "elementary particle" states $|\beta\rangle$

$$Z=\sum_\beta |\langle\beta|\mathfrak{B}\rangle|^2. \tag{67}$$

In particular, if there is one elementary particle state $|0\rangle$ then Z is given by (57), and if there are no elementary particles in H then $Z=0$. Combining (65) and (66) gives

$$1-Z=\int dE \sum_n \frac{|\langle E,n|V|\mathfrak{B}\rangle|^2}{E+B}. \tag{68}$$

Imagine for a moment that the state $|\mathfrak{B}\rangle$ was not stable, but instead was the physical "in" state corresponding to a bare state of energy $E>0$. Then if there were a resonance at E, the decay matrix element of the resonance into $|E,n\rangle$ would be

$$T_n(E)=\langle E,n|V|\mathfrak{B}\rangle,$$

and, therefore, the total decay rate of the $\mathfrak{B}$ particle would be

$$\omega(E)=2\pi\sum_n |\langle E,n|V|\mathfrak{B}\rangle|^2. \tag{69}$$

(The density-of-states factor is included in the normalization of $|E,n\rangle$, which has the dimensions of $E^{-1/2}$.) We see then that (68) just tells us that[5]

$$1-Z=\frac{1}{2\pi}\int_0^\infty dE \frac{\omega(E)}{(E+B)^2}. \tag{70}$$

Clearly $\omega(E)$ is proportional to some effective coupling parameter, and so (70) sets an upper bound on this parameter[6]; the maximum is attained when $Z=0$.

In the limit as $E\to 0$, $\omega(E)$ will always have the behavior

$$\omega(E)\cong A\sqrt{E} \tag{71}$$

provided that there is a two-body S-wave state into which a very low energy $\mathfrak{B}$ particle could decay. If the binding energy B of the physical bound state is sufficiently small, than (71) can be used over the whole range of integration in (70), and we get

$$1-Z=\tfrac{1}{4}AB^{-1/2}. \tag{72}$$

In other words, for particles which are only weakly bound, the decay rate the particle would have if it had energy $E>0$ (instead of $-B<0$) is, for small E,

$$\omega(E)\cong 4(1-Z)(EB)^{1/2}. \tag{73}$$

For example, if the deuteron were slightly heavier it could decay by the mode

$$d\to p+n.$$

The effective p-n-d interaction Lagrangian may be written

$$\mathfrak{L}_{\rm eff}=G_{pnd}\bar{\psi}_p\gamma_5\gamma_\mu\psi_n{}^c\varphi_d{}^\mu. \tag{74}$$

(Here $\psi_n{}^c$ is the charge conjugate of the neutron field, and has opposite parity to ψ_p.) Then, if the deuteron mass were m_p+m_n+E, with E sufficiently small, the deuteron decay rate would be

$$\omega(E)=\frac{|G_{pnd}|^2}{\pi}\frac{(m_pm_n)^{3/2}}{(m_p+m_n)^{5/2}}(2E)^{1/2}. \tag{75}$$

Comparing with (73), we see that[9]

$$\frac{|G_{pnd}|^2}{4\pi}=(1-Z_d)\frac{(m_p+m_n)^{5/2}}{(m_pm_n)^{3/2}}\left(\frac{B_d}{2}\right)^{1/2} \tag{76}$$

$$=0.19(1-Z_d). \tag{77}$$

If $|G_{pnd}|^2/4\pi$ were less than 0.19 we would conclude that the deuteron is an elementary particle, while if $|G_{pnd}|^2/4\pi$ were greater than 0.19, we should have to call it a ghost.

Similarly, the effective interaction among the proton, electron, and 1s hydrogen atom may be written for $F=1$ as in (74), or for $F=0$:

$$\mathfrak{L}_{\rm eff}=G_{pe\rm H}\bar{\psi}_p\gamma_5\psi_e{}^c\varphi_{\rm H}. \tag{78}$$

The "decay" rate is given in either hyperfine state by replacing m_p, m_n, and G_{pnd} in (75) by m_p, m_e, and $G_{pe\rm H}$. Using (73) gives, then, (since $m_e\ll m_p$)

$$\frac{|G_{pe\rm H}|^2}{4\pi}=(1-Z_{\rm H})\frac{m_p}{m_e{}^{3/2}}\left(\frac{B_{\rm H}}{2}\right)^{1/2} \tag{79}$$

$$=6.6(1-Z_{\rm H}). \tag{80}$$

It can easily be shown that 0.19 is just the value of $|G|^2/4\pi$ needed to get the right scattering length in triplet n-p scattering. Aside from this, the presence of long-range forces in the hydrogen atom, and of anomalous thresholds in both examples, makes it doubtful whether these formulas for $|G|^2/4\pi$ have any physical utility. If they do, then the place to measure one-deuteron-exchange or one-hydrogen atom-exchange would be in backward antineutron-proton or positron-proton scattering. The residue of the pole in the u channel is proportional to $|G|^2/4\pi$.

[9] This type of formula for coupling constants can be derived in a more familiar manner by noting that the binding energy determines the behavior of the exponential tail of a bound-state wave function. For very small binding energy, the normalization integral is dominated by the exponential tail, so its coefficient (which is the coupling constant) is determined by the binding energy. A calculation of the Σ-Λ-π coupling has been carried out on these lines by Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **6**, 377 (1961). We have applied (73) to $\Sigma\to\Lambda+\pi$ and very easily get their coupling, with a needed factor of 2 correctly supplied.

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APPENDIX: THE SCHMIDT METHOD AND THE FREDHOLM METHOD

Any linear integral equation with kernel K can be solved if we know the resolvent F, defined by

$$F=K+KF. \tag{A1}$$

Let us consider instead of (A1) another integral equation

$$F_1=K_1+K_1F_1 \tag{A2}$$

whose kernel K_1 differs from K by a sum of a finite number of separable (or "degenerate") kernels:

$$K_1=K-\sum_s|s\rangle\langle\bar{s}|. \tag{A3}$$

The vectors $|s\rangle$, $|\bar{s}\rangle$ need not be orthonormal, and certainly do not form a complete set. Also, since we have not said any thing about the Hermiticity properties of K, the vectors $\langle\bar{s}|$ may or may not be related to the adjoints of the $|s\rangle$.

We want to show first how to solve (A1) in terms of (A2). Using (A3), Eq. (A1) may be written

$$F=K_1+\sum_s|s\rangle\langle\bar{s}|(1+F)+K_1F. \tag{A4}$$

If we regard the first two terms on the right-hand side as known, then this is a linear integral equation for F with kernel K_1, and can therefore be solved using F_1:

$$F=F_1+\sum_s(1+F_1)|s\rangle\langle\bar{s}|(1+F). \tag{A5}$$

In order to eliminate the unknown $\langle\bar{s}|(1+F)$ we take the matrix element of (A5) with $\langle\bar{s}|$:

$$\sum_t[\delta_{st}-\langle\bar{s}|(1+F_1)|t\rangle]\langle\bar{t}|(1+F)=\langle\bar{s}|(1+F_1). \tag{A6}$$

If we define a matrix Δ_{st} by

$$(\Delta^{-1})_{st}=\delta_{st}-\langle\bar{s}|(1+F_1)|t\rangle, \tag{A7}$$

then (A5) reads

$$F=F_1+\sum_{st}(1+F_1)|s\rangle\Delta_{st}\langle\bar{t}|(1+F_1). \tag{A8}$$

Equations (A7) and (A8) solve the problem of obtaining F if we know F_1.

In particular, we can write the Lippmann-Schwinger equation (18) in the form (A1) if we define

$$F(W)\equiv T(W)[W-H_0]^{-1}, \tag{A9}$$

$$K(W)\equiv V[W-H_0]^{-1}. \tag{A10}$$

Also, the reduced L–S equation (21) may be written in the form (A2) if we define

$$F_1(W)\equiv T_1(W)[W-H_0]^{-1}, \tag{A11}$$

$$K_1(W)\equiv V_1[W-H_0]^{-1}$$
$$=K(W)-V|\Gamma\rangle\langle\bar{\Gamma}|V[W-H_0]^{-1} \tag{A12}$$
$$=K(W)-N^{-2}V_1|\Gamma\rangle\langle\bar{\Gamma}|K_1(W). \tag{A13}$$

Then (A8) tells us that

$$T(W)=T_1(W)+N^{-2}[1+F_1(W)]V_1|\Gamma\rangle\Delta(W)$$
$$\times\langle\bar{\Gamma}|K_1(W)[1+F_1(W)][W-H_0]$$
$$=T_1(W)+N^{-2}T_1(W)|\Gamma\rangle\Delta(W)\langle\bar{\Gamma}|T_1(W), \tag{A14}$$

where (A7) gives the propagator as

$$\Delta(W)=[1-J(W)]^{-1}, \tag{A15}$$

and

$$J(W)=N^{-2}\langle\bar{\Gamma}|K_1(W)[1+F_1(W)]V_1|\Gamma\rangle$$
$$=N^{-2}\langle\bar{\Gamma}|V_1[W-H_0]^{-1}T_1(W)|\Gamma\rangle$$
$$=N^{-2}\langle\bar{\Gamma}|\{T_1(W)-V_1\}|\Gamma\rangle$$
$$=1-N^{-1}+N^{-2}\langle\bar{\Gamma}|T_1(W)|\Gamma\rangle.$$

To get the last line, we use the fact that

$$\langle\bar{\Gamma}|V_1|\Gamma\rangle=N\langle\bar{\Gamma}|V|\Gamma\rangle=N(1-N).$$

This verifies the formulas (23)–(26).

We end by describing the relation between the Fredholm and Schmidt methods. The Fredholm determinants for any kernels K or K_1 are given by

$$D=\exp\left\{-\int_0^1\mathrm{Tr}F[\lambda]d\lambda\right\}, \tag{A16}$$

$$D_1=\exp\left\{-\int_0^1\mathrm{Tr}F_1[\lambda]d\lambda\right\}, \tag{A17}$$

where we define $F[\lambda]$ and $F_1[\lambda]$ by inserting a λ into (A1) and (A2):

$$F[\lambda]=K+\lambda KF[\lambda], \tag{A18}$$

$$F_1[\lambda]=K_1+\lambda K_1F_1[\lambda]. \tag{A19}$$

The relation between $F[\lambda]$ and $F_1[\lambda]$ is given, according to (A8), by

$$F[\lambda]=F_1[\lambda]$$
$$+\sum_{st}(1+\lambda F_1[\lambda])|s\rangle\Delta_{st}[\lambda]\langle\bar{t}|(1+\lambda F_1[\lambda]), \tag{A20}$$

$$(\Delta^{-1}[\lambda])_{st}=\delta_{st}-\lambda\langle\bar{s}|(1+\lambda F_1[\lambda])|t\rangle. \tag{A21}$$

Then the traces are related by

$$\mathrm{Tr}F[\lambda]=\mathrm{Tr}F_1[\lambda]$$
$$+\sum_{st}\Delta_{st}[\lambda]\langle\bar{t}|(1+\lambda F_1[\lambda])^2|s\rangle. \tag{A22}$$

To evaluate the second term, let us differentiate (A19) with respect to λ. We get

$$F_1'[\lambda]=\lambda K_1 F_1'[\lambda]+K_1 F_1[\lambda].$$

The solution of this integral equation for F_1' is

$$F_1'[\lambda]=F_1^2[\lambda].$$

Using this in (A21) gives

$$\left(\frac{d}{d\lambda}\Delta^{-1}[\lambda]\right)_{st}=-\langle\bar{s}|\,(1+\lambda F_1[\lambda])^2\,|t\rangle,$$

and so (A22) may be written

$$\mathrm{Tr}F[\lambda]=\mathrm{Tr}F_1[\lambda]+\frac{d}{d\lambda}\ln\,\mathrm{Det}\Delta[\lambda]. \qquad \text{(A23)}$$

Using (A23) with (A16) and (A17), we have finally

$$D=D_1/\mathrm{Det}\Delta. \qquad \text{(A24)}$$

Equation (27) is a special case of this general relation.

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Isotopic Spin in $K \to 3\pi$*

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Because recent data on $K_2^0 \to \pi^+\pi^-\pi^0$ are at variance with the $\Delta T=1/2$ rule while the data on $K^+ \to 3\pi$ are not, the charge space kinematics of $K \to 3\pi$ are re-examined. Matrix elements are assumed to be at most linearly dependent on the usual variables s_i, and it follows that only four of the seven possible 3π states can contribute to the decay. Of these states, two have $T=1$, the third has $T=2$ and the fourth $T=3$. The possible values of ΔT are $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, and accordingly, the most general interaction Hamiltonian is written as the sum of four parts $H_{n/2}$, each corresponding to $\Delta T=n/2$ ($n=1, 3, 5, 7$). It is then possible to express the matrix elements, rates and spectra of all the modes of $K \to 3\pi$ in terms of the reduced matrix elements of $H_{n/2}$ between the four 3π states and the K meson. The analysis reveals that, provided the branching ratio of $K_2^0 \to 3\pi^0$ to $K_2^0 \to \pi^+\pi^-\pi^0$ is $\frac{3}{2}$, the present data are consistent with an interaction Hamiltonian containing only $\Delta T=\frac{1}{2}$ and $\frac{3}{2}$, and a 3π final state of isotopic spin one.

INTRODUCTION

RECENT experiments on $K_2^0 \to \pi^+\pi^-\pi^0$ indicate that while the slope[1] of the π^0 spectrum may be consistent with the $\Delta T=\frac{1}{2}$ rule,[2] the rate of decay[3] is not.[4] In the case of K^+ decay, however, the rates[3] and spectra[5,6] of the τ and τ' decay modes all seem to be consistent with the predictions of $\Delta T=\frac{1}{2}$.[2,4] Because of this discrepancy, it seems appropriate to give a systematic restatement of the charge space kinematics of $K \to 3\pi$.

Dalitz[4] has shown that the τ to τ' branching ratio depends not on ΔT being $\frac{1}{2}$, but rather on the isotopic spin of the final state being equal to one; and that if the interaction Hamiltonian contains both $\Delta T=\frac{1}{2}$ and $\Delta T=\frac{3}{2}$, the admixture of $\Delta T=\frac{3}{2}$ affects only the relative rates for $K^+ \to 3\pi$ and $K_2^0 \to 3\pi$. Similarly, Weinberg's relation[2] between the spectra of τ and τ' is, as we shall show below, a consequence only of the final state having $T=1$; and further, as regards the slopes, an admixture of $\Delta T=\frac{3}{2}$ will show up only in the slope of the $K_2^0 \to \pi^+\pi^-\pi^0$ spectrum. Hence, even if the $\Delta T=\frac{1}{2}$ rule has to be abandoned, it may still be true that the final state of $K \to 3\pi$ has isotopic spin equal to one. Our analysis shows that such a conclusion is, in fact, consistent with the present data, provided the branching ratio of $K_2^0 \to \pi^0\pi^0\pi^0$ to $K_2^0 \to \pi^+\pi^-\pi^0$ is assumed to be $\frac{3}{2}$.

THE LINEAR APPROXIMATION

We use the linear approximation, which appears to be in good agreement with the τ and τ' experimental data, and write the matrix element for

$$K^\rho \to \pi_1{}^\alpha+\pi_2{}^\beta+\pi_3{}^\gamma,$$

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