Theoretical Nucleon-Nucleon Potential*

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The two-pion exchange nucleon-nucleon potential is calculated on the basis of the Mandelstam representation of the field-theoretic scattering amplitude. The results contain parameters referring to low-energy pion-nucleon scattering and the renormalized coupling constant g only. A comparison of these results with phenomenological potentials is made.

I. INTRODUCTION

THERE is no field-theoretic reason why the interaction of two nucleons should be exactly describable in terms of a potential that, apart from the energy dependence implied by the inclusion of such terms as spin orbit and tensor forces, is independent of the scattering energy. Indeed, it is impossible for such a potential to describe nucleon-nucleon scattering at high energies, above the threshold for meson production. However, the idea of regarding the interaction of two nucleons as being approximately equivalent to the interaction through such an energy-independent potential has played an important role both in the theory of nuclear structure and in the theory of nucleon-nucleon scattering.

Some field-theoretic justification for these ideas came from the form of the one-pion-exchange contribution to the field theory scattering amplitude. It was shown [see, for example, Iwadare, Otsuki, and Tamagaki[1]] that this one-pion-exchange contribution was equivalent, nonrelativistically, to the first-order Born approximation to the scattering amplitude in a certain potential, the one-pion-exchange potential (OPEP), and that this potential should be the dominant interaction at large distances. These theoretical arguments were substantiated experimentally by the work of Cziffra, MacGregor, Moravcsik, and Stapp[2] who showed the significance of the one-pion-exchange term for giving an understanding of the nucleon-nucleon partial-wave phase shifts with large angular momenta. From this analysis it can be concluded that OPEP does, in fact, give a good description of the nucleon-nucleon interaction at large distances, distances greater than 1.2 inverse pion masses.[3]

We here look at the low-energy nucleon-nucleon interaction problem by means of the Mandelstam representation for the elastic scattering amplitude.[4] These relations again demonstrate the notion that the one-pion-exchange term does give the form of the interaction at very large distances, and also, for the many-pion exchange terms in general, the more pions that are exchanged the shorter is the equivalent interaction range. More than that, the Mandelstam representation along with the unitarity condition gives a method of calculating the many-pion contributions to the elastic nucleon-nucleon scattering amplitude once the amplitudes for nucleon-antinucleon annihilation processes into these many-pion states are known.[4] Amati, Leader, and Vitale[5] have performed the calculation of the two-pion-exchange contribution to the elastic scattering amplitude using an approximate knowledge of the nucleon-antinucleon annihilation amplitude into two pions or, more precisely, an analytic continuation of the low-energy pion-nucleon scattering amplitude. We here show, by an extension of the method of Charap and Fubini,[6,7] that this two-pion contribution is equivalent, in a certain approximation, to an energy-independent potential contribution. We claim that this potential should give a reasonable description of the way in which two nucleons interact for separation distances somewhat smaller than those at which the OPEP alone is dominant.

We attempt to justify this claim by comparing at distances greater than 0.5 inverse pion masses the central, spin-spin, spin-orbit, and tensor parts of this potential in both isotopic spin states with potentials

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[1] J. Iwadare, S. Otsuki, and R. Tamagaki, Suppl. Progr. Theoret. Phys. (Kyoto) No. 3 (1956).

[2] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959); M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, University of California Radiation Laboratory Report UCRL-5566 (unpublished).

[3] G. Breit and M. H. Hull, Nucl. Phys. **15**, 216 (1960).

[4] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[5] D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960); **18**, 409 (1960), hereafter referred to as I and II, respectively.

[6] This method was given by Charap and Fubini (reference 7) for a spinless nucleon, the realistic case involving the nucleon spin and isospin was treated by Charap and Tausner (reference 16), however this last paper leans very heavily on perturbation theory (in fact, their results were limited to the consideration of the fourth-order Feynman diagram). Their results are not very useful from a practical point of view. Nevertheless, they were able, from their analysis, to provide an unambiguous definition of a nucleon-nucleon local potential.

[7] J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959); **15**, 73 (1960).

constructed phenomenologically to fit low-energy nucleon-nucleon scattering data.

To make this paper self-contained, we will include in Sec. II the definitions and results of reference 5 which were extensively used during our work. In Sec. III, we will give a constructive definition of the potential in terms of the pion-nucleon scattering parameters. The "potential approximation" itself will also be discussed. Section IV is devoted to the explicit expressions for the potential. Finally, in Sec. V the calculated potentials will be compared to the phenomenological potential obtained recently by Breit *et al.*[8] (the Yale potential); the eventual role of the three-pion-exchange contributions, especially the 3-pion $T=0$ $J=1$ resonance (ω meson) will also be discussed.

II. THE FIELD THEORY SCATTERING AMPLITUDE

The notation used in this paper will be the same as in the papers of Amati, Leader, and Vitale,[5] hereafter referred to as I and II. For completeness, we include in this section those definitions and results relevant to the work of the later sections.

We are to discuss the elastic scattering of two nucleons from a state with 4-momenta, n_1, p_1, to a state with 4-momenta n_2, p_2.

Apart from spin and isotopic spin this process is characterized by three independent 4-vectors, chosen here as

$$\begin{aligned} N&=\tfrac{1}{2}(n_1+n_2),\\ P&=\tfrac{1}{2}(p_1+p_2),\\ \Delta&=\tfrac{1}{2}(n_1-n_2)=\tfrac{1}{2}(p_2-p_1). \end{aligned} \tag{2.1}$$

Only two independent scalars can be constructed from these vectors. We here define the three scalars

$$\begin{aligned} w&=-(n_1+p_1)^2,\\ t&=-(n_1-n_2)^2,\\ \bar{t}&=-(p_1-n_2)^2. \end{aligned} \tag{2.2}$$

These are related by the equation

$$w+t+\bar{t}=4m^2, \tag{2.3}$$

where m is the nucleon mass (taken to be the same for both neutron and proton).

In the center-of-mass system of the two nucleons [(2.1) and (2.2)] reduce to

$$N=(-\mathbf{K},E),\quad P=(\mathbf{K},E),\quad \Delta=(\boldsymbol{\Delta},0),$$

with

$$\mathbf{K}=\tfrac{1}{2}(\mathbf{p}_1+\mathbf{p}_2),\quad \boldsymbol{\Delta}=\tfrac{1}{2}(\mathbf{p}_2-\mathbf{p}_1),\quad |\mathbf{p}_1|=|\mathbf{p}_2|=k,$$

and

$$\begin{aligned} w&=4(k^2+m^2)=4E^2,\\ t&=-2k^2(1-\cos\theta)=-4\Delta^2,\\ \bar{t}&=-2k^2(1+\cos\theta), \end{aligned} \tag{2.4}$$

[8] K. E. Lassila, M. H. Hull, H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

where E and k are, respectively, the energy and momentum of each nucleon and θ is the angle of scattering in the c.m. system.

We will take the particles labeled n (initial and final 4-momenta n_1, n_2) and p (initial and final 4-momenta p_1 and p_2) to be distinguishable. The Pauli exclusion principle will be taken account of explicitly in the definition of the S matrix.

In order to make the spin dependence of the scattering amplitude explicit, one must define a set of five spinor invariants, Amati *et al.*[5] take the five [see I (2.19)]

$$\begin{aligned} &P_1=1^n1^p,\quad P_2=i(\gamma^n\cdot P1^p+1^n\gamma^p\cdot N),\\ &P_3=-\gamma^n\cdot P\gamma^p\cdot N,\quad P_4=\gamma^n\cdot\gamma^p,\quad P_5=\gamma_5{}^n\gamma_5{}^p. \end{aligned} \tag{2.5}$$

The spinor operators $1^{n(p)}$, $\gamma^{n(p)}$, and $\gamma_5{}^{n(p)}$ are the usual matrices operating on the 4-component spinors $u(n_i)u(p_i)$ ($i=1,2$) representing the spin of the particles labeled n and p. Similarly one defines two invariant isotopic spin operators 1^n1^p and $\tau^n\cdot\tau^p$ operating on the two-component isospinors $\chi_{ni}\chi_{pi}$ which carry the isotopic spin of the particles.

The field theory S matrix for the scattering of two similar nucleons can then be written as

$$\begin{aligned} S=1+i\delta^4(n_1+p_1-n_2-p_2)(m/2\pi E)^2&\\ \times\bar{u}(p_2)\chi_{p_2}\bar{u}(n_2)\chi_{n_2}Mu(n_1)\chi_{n_1}u(p_1)\chi_{p_1}&\\ -\text{antisymmetrized term}.& \end{aligned} \tag{2.6}$$

M is the causal matrix, which can be written in the general form

$$M=\textstyle\sum_i\{3p_i{}^+(w,t,\bar{t})+2p_i{}^-(w,t,\bar{t})\tau^n\cdot\tau^p\}P_i, \tag{2.7}$$

where $p_i{}^+(w,t,\bar{t})$ and $p_i{}^-(w,t,\bar{t})$ are scalar functions of w, t, and $\bar{t}$ only. The antisymmetrized term is obtained from the direct term by interchanging all of the quantum numbers (momentum, spin, and isospin) of the incoming nucleons. The Pauli principle makes this term necessary.

The functions $p_i{}^\pm(w,t,\bar{t})$ have been examined in I and II. It is shown in these papers that, apart from the presence of the deuteron pole, the Mandelstam representations for the functions $p_i{}^\pm(w,t,\bar{t})$ are of the form

$$\begin{aligned} p_i{}^\pm(w,t,\bar{t})=&\begin{Bmatrix} 0\\ -\tfrac{1}{2}g^2\dfrac{\delta_{i5}}{\mu^2-t} \end{Bmatrix}\\ &+\frac{1}{\pi}\int_{4\mu^2}^\infty\frac{\rho_i{}^\pm(\omega,t')\mp(-1)^i\rho_i{}^\pm(\bar{t},t')}{t'-t}dt'\\ &+\frac{1}{\pi^2}\int_{9\mu^2}^\infty\frac{dt'}{t'-t}\int_{4m^2}^\infty\chi_i{}^\pm(x,t')\left[\frac{1}{x-w}\mp\frac{1}{x-\bar{t}}\right]dx. \end{aligned} \tag{2.8}$$

g is the rationalized renormalized pseudoscalar coupling constant $g^2\simeq4\pi\times14.4$; μ is the π-meson mass.

The functions $\rho_i^{\pm}(wt')$ are analytic functions of w except for a cut along the real axis from $w=4m^2$ to ∞ and are real for $w<4m^2$. The functions $\chi_i^{\pm}(x,t')$ are real Mandelstam spectral functions.

The first term in this expression for $p_i^{\pm}(w,t,\bar{t})$ is the one-pion-exchange pole. The term

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho_i^{\pm}(w,t')\mp(-1)^i\rho_i^{\pm}(\bar{t},t')}{t'-t}dt'$$

has been separated from the remaining contributions of the spectral functions $\chi_i(x,t')$ by defining them to give the contribution of the two-pion-exchange processes. This separation can be made unambiguously by considering the unitarity relation in the nucleon-antinucleon elastic scattering channel. In this channel, t is the energy, w and $\bar{t}$ are the two momentum transfers.

In terms of the causal matrix M the unitarity relation is [see I (3.9)]

$$i\langle N\bar{N}|M^{\dagger}-M|N\bar{N}\rangle = [1/(2\pi)^2t]\sum\langle N\bar{N}|\tau^{\dagger}|2\pi\rangle\langle 2\pi|\tau|N\bar{N}\rangle+J_n, \quad (2.9)$$

where the sum is taken over all two-pion intermediate states that conserve four-momenta. J_n gives the contribution of the intermediate states which contain more than two pions. We define the functions $\rho_i^{\pm}(w,t)$ such that a substitution of (2.8) into (2.9) gives

$$\sum_i[\rho_i^{\pm}(w,t)\mp(-1)^i\rho_i^{\pm}(\bar{t},t)]P_i=[1/2(2\pi)^2t]\mathfrak{M}, \quad (2.10)$$

$$\mathfrak{M}=\sum\langle\bar{p}_1p_2|\tau^{\dagger}|2\pi\rangle\langle 2\pi|\tau|n_1\bar{n}_2\rangle. \quad (2.11)$$

The spectral functions $\chi_i^{\pm}(xt)$ come from the three and more pion contributions contained in J_n and so are zero for $t<9\mu^2$ as explicitly given in (2.8). $\mathfrak{M}$ corresponds to the graph of Fig. 1 in which the intermediate pions are on the mass shell. n_1 and $-n_2$ are the 4-momenta of the incoming nucleon and antinucleon, respectively; p_2 and $-p_1$ those of the outgoing nucleon and antinucleon. $\langle 2\pi|\tau|n_1\bar{n}_2\rangle$ is the annihilation amplitude for the process $N+\bar{N}\rightarrow 2\pi$ (Fig. 2) and is an analytic continuation of the pion-nucleon scattering amplitude. This amplitude can be written in the form

$$\langle\pi_\beta\pi_\alpha|\tau|n_1\bar{n}_2\rangle = \bar{V}(-n_2)[-A+\tfrac{1}{2}i\gamma\cdot(q_2-q_1)B]u(n_1), \quad (2.12)$$

with

$$A_{\beta\alpha}=\delta_{\beta\alpha}A^++\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^-,\qquad B_{\beta\alpha}=\delta_{\beta\alpha}B^++\tfrac{1}{2}[\tau_\beta,\tau_\alpha]B^-, \quad (2.13)$$

where α and β are the isotopic spin indices of the two pions, q_1 and q_2 are their momenta. $A^{\pm}$ and $B^{\pm}$ are scalar functions of the variables s, $\bar{s}$, and t where

$$\begin{aligned}t&=-(n_1-n_2)^2=-(q_1+q_2)^2,\\ s&=-(n_1-q_1)^2=-(n^2+q^2-2nq\cos\phi_n),\\ \bar{s}&=-(n_1-q_2)^2=-(n^2+q^2+2nq\cos\phi_n),\\ t+s+\bar{s}&=2m^2+2\mu^2,\end{aligned} \quad (2.14)$$

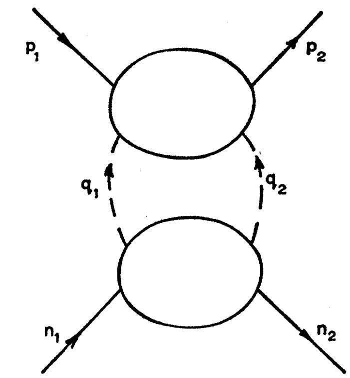

FIG. 1. The two-pion-exchange graph.

where n and q are the magnitudes of the nucleon and pion 3 momenta and ϕ_n is the angle between them, in the $N\bar{N}$ c.m. system. The t defined here coincides with the t of (2.2). s and $\bar{s}$ depend on the angle ϕ_n.

The amplitude $\langle\pi_\beta\pi_\alpha|\tau|\bar{p}_1p_2\rangle^*=\langle p_2\bar{p}_1|\tau^{\dagger}|\pi_\alpha\pi_\beta\rangle$ and has a similar form in terms of the spin and isotopic spin matrices γ^p and τ^p, and the same scalar functions $A^{\pm}$ and $B^{\pm}$ of t and the new variable ϕ_p the angle between the pion and the outgoing nucleon in the $N\bar{N}$ c.m. system.

Inserting these expressions for $\langle 2\pi|\tau|\bar{p}_1p_2\rangle$ and $\langle 2\pi|\tau|n_1\bar{n}_2\rangle$ into (2.11), we obtain, taking care with the appropriate phase-space factors

$$\mathfrak{M}=3\mathfrak{M}^++2\tau^n\cdot\tau^p\mathfrak{M}^-, \quad (2.15)$$

where

$$\mathfrak{M}^{\pm}=\tfrac{1}{16}[t(t-4\mu^2)]^{1/2}\int[-A^{\pm *}+i\gamma^p\cdot(q_2-q_1)B^{\pm *}]\times[-A^{\pm}+i\gamma^n\cdot(q_2-q_1)B^{\pm}], \quad (2.16)$$

where the integration is over the angular direction of the three vector $\mathbf{q}_2-\mathbf{q}_1$. Because of energy conservation in the intermediate states we have that the magnitude of this vector is fixed as

$$(\mathbf{q}_2-\mathbf{q}_1)^2=t-4\mu^2. \quad (2.17)$$

Provided we know $A^{\pm}$ and $B^{\pm}$ in the appropriate regions for the integration of (2.16), $\mathfrak{M}$ and, therefore, from (2.10), the functions $\rho_i^{\pm}(w,t)$ can be calculated. Since no complete solution of the pion-nucleon scattering problem is known, the calculation cannot be done exactly. However, in this paper we are only interested in nucleon-nucleon scattering near threshold ($w\simeq 4m^2$) and also, as will be explained later, most directly concerned with the values of t and from (2.3) $\bar{t}$, not too large. The calculation of $\rho_i^{\pm}(wt)$, $\rho_i^{\pm}(\bar{t}t)$ in this region ($w\simeq 4m^2$, t, $\bar{t}$ small) involves a knowledge of $A^{\pm}$ and $B^{\pm}$ near to the physical region for low-energy pion-nucleon scattering, a region in which $A^{\pm}$ and $B^{\pm}$ are known experimentally. As in I and II we here extrapolate out of the physical region by using the

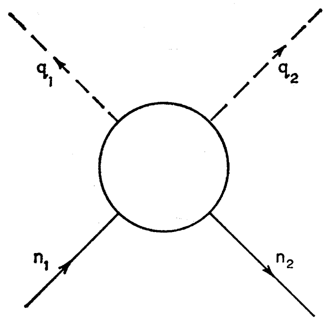

FIG. 2. The nucleon-antinucleon two-pion annihilation graph.

amplitudes $A^{\pm}$, $B^{\pm}$ constructed by Bowcock, Cottingham, and Lurié.[9]

These authors construct an approximate scattering amplitude based on the Cini-Fubini representation[10] for low-energy scattering amplitudes. This approximation to the scattering amplitude includes the one-nucleon pole term, and the $(\frac{3}{2},\frac{3}{2})$ resonance contribution to the πN rescattering corrections, itself approximated by a pole term. Terms are also included which have cuts in the t variable, these are there to take account of pion-pion interactions in S and P states. Since we are interested in these amplitudes for small t, interactions in higher partial waves than P have not been considered since these will probably only be significant for larger t values. The B.C.L. amplitudes, for the process of Fig. 2 with the definitions (2.14), are

$$A^{\pm}=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}\left[\frac{1}{s_r-s}\pm\frac{1}{s_r-\bar{s}}\right]G_A+\begin{pmatrix}1\\s-\bar{s}\end{pmatrix}\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\alpha^{\pm}(t')dt'}{t'-t}+\begin{pmatrix}1\\0\end{pmatrix}C_A{}^{+},$$

$$B^{\pm}=g^2\left(\frac{1}{m^2-s}\mp\frac{1}{m^2-\bar{s}}\right)-\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}\left[\frac{1}{s_r-s}\mp\frac{1}{s_r-\bar{s}}\right]G_B+\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\sigma^{-}(t')dt'}{t'-t}+\begin{pmatrix}0\\1\end{pmatrix}C_B{}^{-},\qquad(2.18)$$

$s_r=(\frac{3}{2}\,\frac{3}{2}$ total resonance energy$)^2$,

$$G_A=\left[\frac{1}{3}\frac{s_r^{1/2}-m}{E_r-m}+\frac{s_r^{1/2}+m}{E_r+m}\left(1+\frac{t}{2q_r^2}\right)\right]\frac{4s_r^{1/2}q_r^2g^2}{3m^2},$$

$$G_B=\left[\frac{1}{3}\frac{1}{E_r-m}-\frac{1}{E_r+m}\left(1+\frac{t}{2q_r^2}\right)\right]\frac{4s_r^{1/2}q_r^2g^2}{3m^2},$$

E_r is the nucleon energy at the $(\frac{3}{2},\frac{3}{2})$ resonance,

$$q_r=[E_r^2-m^2]^{1/2}.$$

We have normalized G_A and G_B so that the integral over our expression for $\sin^2\delta_{33}(s)$ is equal to the area under the curve obtained from the usual effective range formula.[11] We will call the contributions to $A^{\pm}$ and $B^{\pm}$ due to the nucleon pole and the $(\frac{3}{2},\frac{3}{2})$ resonance the C.G.L.N. contributions.[12] The functions $\alpha^{\pm}(t)$ and $\sigma^{-}(t)$ represent the effect of S- and P-wave $\pi\pi$ interactions on the πN scattering amplitude. The effects of such interactions will be discussed separately.

The evaluation of $\mathfrak{M}^{\pm}$ using the B.C.L. amplitudes in the integral expression (2.16) has been carried out in I and II.

The crossing symmetric form of $\mathfrak{M}^{\pm}$ is obtained explicitly since the πN scattering amplitude is itself crossing symmetric. Also, since the five invariant amplitudes P_i are linearly independent, the functions $\rho_i{}^{\pm}(w,t)$ and $\rho_i{}^{\pm}(\bar{t},t)$ can be individually calculated as has been done in I and II.

III. DEFINITION OF THE POTENTIAL

We wish to define a potential $V(\mathbf{x})$ which, when inserted into a Schrödinger equation, gives a T matrix, or the related scattering matrix

$$S=1-2\pi i\delta(E_f-E_i)T\qquad(3.1)$$

such that it will reproduce, in an energy range sufficiently below the meson production threshold, the same scattering amplitude as field theory. We, therefore, require:

$$\langle\mathbf{p}_2|T|\mathbf{p}_1\rangle=-\frac{1}{(2\pi)^3}\bar{u}(p_2)\chi_{p_2}{}^{\dagger}\bar{u}(n_2)\times\chi_{n_2}{}^{\dagger}Mu(n_1)\chi_{n_1}u(p_1)\chi_{p_1},\qquad(3.2)$$

where M is the field-theoretic causal matrix defined by (2.6).[13]

The spin and isotopic spin dependence of M implies that the potential must itself be spin and isotopic spin dependent. In fact, corresponding to the set of five independent spinor invariants needed to construct M we have to employ five types of spin-dependent potential. It is convenient to use the standard set as first given by Okubo and Marshak,[14] and consider a potential

$$V(\mathbf{x})=\sum\nolimits_{\alpha}[3V_{\alpha}{}^{+}(r)+2V_{\alpha}{}^{-}(r)\tau^{n}\cdot\tau^{p}]\Omega_{\alpha},\qquad(3.3)$$

with

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[9] J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo Cimento **16**, 918 (1960); **19**, 142 (1961); hereafter referred to as B.C.L.

[10] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **3**, 352 (1960).

[11] J. Ashkin, J. P. Blaser, F. Feiner, and M. O. Stern, Phys. Rev. **105**, 724 (1955).

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957); hereafter referred to as C.G.L.N.

[13] The potential thus defined is only the direct potential between two distinguishable nucleons. The exchange character of nuclear forces can be taken account of by antisymmetrizing the scattering amplitude resulting from this potential. See, for example, G. Breit, Ann. Phys. (N. Y.) **16**, 346 (1961).

[14] S. Okubo and R. E. Marshak, Ann. Phys. (N. Y.) 4, 166 (1958).

$$\begin{aligned}
&\alpha=(C,SO,T,SO2,SS),\\
&\Omega_C=1,\\
&\Omega_{SO}=\tfrac{1}{2}(\boldsymbol{\sigma}^n+\boldsymbol{\sigma}^p)\cdot\mathbf{L},\\
&\Omega_T=(1/r^2)(3\boldsymbol{\sigma}^n\cdot\mathbf{x}\boldsymbol{\sigma}^p\cdot\mathbf{x})-\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p,\\
&\Omega_{SO2}=\tfrac{1}{2}[\boldsymbol{\sigma}^n\cdot\mathbf{L}\boldsymbol{\sigma}^p\cdot\mathbf{L}+\boldsymbol{\sigma}^p\cdot\mathbf{L}\boldsymbol{\sigma}^n\cdot\mathbf{L}],\\
&\Omega_{SS}=\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p,
\end{aligned}\tag{3.4}$$

with

$$\mathbf{x}=\mathbf{x}_n-\mathbf{x}_p,\qquad \mathbf{L}=\mathbf{x}\times\boldsymbol{\Delta},$$
$$\boldsymbol{\Delta}=\tfrac{1}{2}(\mathbf{p}_2-\mathbf{p}_1),\qquad r=|\mathbf{x}|.$$

$\boldsymbol{\sigma}^n$ and $\boldsymbol{\sigma}^p$ are the Pauli spin operators for the nucleons labeled n and p.

The functions $V_\alpha^\pm(r)$ are independent of the nucleons' spin and isospin, and the combinations

$$\begin{aligned}
V_\alpha^0&=3V_\alpha^+(r)-6V_\alpha^-(r),\\
V_\alpha^1&=3V_\alpha^+(r)+2V_\alpha^-(r)
\end{aligned}\tag{3.5}$$

are the central, spin-orbit, tensor, quadratic spin-orbit, and spin-spin potentials in the isotopic spin states 0 and 1, respectively.

In momentum space the potential can be written as

$$\tilde{V}(\boldsymbol{\Delta})=\textstyle\sum_\alpha[3\tilde{V}_\alpha^+(t)+2\tilde{V}_\alpha^-(t)\boldsymbol{\tau}^n\cdot\boldsymbol{\tau}^p]\tilde{\Omega}_\alpha,\tag{3.6}$$

with

$$\begin{aligned}
&\tilde{\Omega}_C=1,\\
&\tilde{\Omega}_{SO}=-\tfrac{1}{2}i(\boldsymbol{\sigma}^n+\boldsymbol{\sigma}^p)\cdot(\mathbf{p}_1\times\mathbf{p}_2),\\
&\tilde{\Omega}_T=4[\Delta^2\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p-3\boldsymbol{\sigma}^n\cdot\boldsymbol{\Delta}\boldsymbol{\sigma}^p\cdot\boldsymbol{\Delta}],\\
&\tilde{\Omega}_{SO2}=\boldsymbol{\sigma}^n\cdot(\mathbf{p}_1\times\mathbf{p}_2)\boldsymbol{\sigma}^p\cdot(\mathbf{p}_1\times\mathbf{p}_2),\\
&\tilde{\Omega}_{SS}=\boldsymbol{\sigma}^n\cdot\boldsymbol{\sigma}^p.
\end{aligned}\tag{3.7}$$

The T matrix associated with V can also be written in terms of these five new invariants:

$$T=\textstyle\sum_\alpha[3T_\alpha^+(k^2,t)+2T_\alpha^-(k^2,t)\boldsymbol{\tau}^n\cdot\boldsymbol{\tau}^p]\tilde{\Omega}_\alpha.\tag{3.8}$$

It is also convenient to express the field-theory amplitude M in terms of these invariants $\tilde{\Omega}_\alpha$.[15] The transformation coefficients from the Dirac spin operators P_i to the Pauli spin operators can be obtained by direct calculation and

$$P_i=\textstyle\sum_\alpha X_{i\alpha}\tilde{\Omega}_\alpha.\tag{3.9}$$

Throughout this paper, except for the analysis referred to in reference 16, we will make the adiabatic approximation. That is, we will neglect all terms of order of magnitude k^2/m^2 compared to unity. In this approximation the transformation matrix X is independent of energy and is

$$\begin{array}{lccccc}
\alpha= & C & SO & T & SO2 & SS\\
 & 1, & -1/2m^2, & 0, & -1/16m^4, & 0\\
 & -2m, & -1/m, & 0, & -3/8m^3, & 0\\
X_{i\alpha}= & m^2, & 3/2, & 0, & -9/16m^2, & 0\\
 & 1, & 3/2m^2, & -1/12m^2, & -1/16m^4, & t/6m^2\\
 & 0, & 0, & 1/12m^2, & 0, & t/12m^2
\end{array}.\tag{3.10}$$

[15] Care has to be taken in obtaining this result since the adiabatic approximation cannot be used directly. See reference 16.

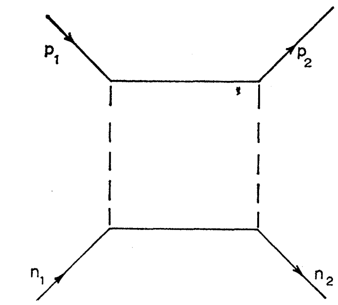

FIG. 3. The second-order Born approximation graph for the $N-N$ scattering amplitude.

In general, it is not possible to exactly satisfy Eq. (3.2) by a potential in which the $V_\alpha^\pm(r)$ [or $\tilde{V}_\alpha^\pm(t)$] are energy independent. However, at least for the one-pion and two-pion exchange contributions to the field-theoretic matrix M, we will see that an energy-independent potential can be constructed which approximately satisfies (3.2) for energies below the inelastic threshold.

First, the one-pion contribution to the potential is

$$\begin{aligned}
V_{C,\mathrm{OPEP}}^\pm=V_{SO,\mathrm{OPEP}}^\pm&=V_{SO2,\mathrm{OPEP}}^\pm\\
&=V_{T,\mathrm{OPEP}}^+=V_{SS,\mathrm{OPEP}}^+=0,\qquad(3.11)\\
V_{T,\mathrm{OPEP}}^-&=(g^2\mu^2e^{-\mu r}/96\pi m^2r)(1+3/\mu r+3/\mu^2r^2),\\
V_{SS,\mathrm{OPEP}}^-&=g^2\mu^2e^{-\mu r}/96\pi m^2r,
\end{aligned}$$

and is so defined because in the Born approximation it gives exactly the same contribution to T as the one-pion pole does to M. The ability of an energy-independent potential V_{OPEP} to reproduce the field-theoretic one-pion pole is because this one-pion exchange contribution is an energy-independent one (it is a function of t only, not of w).

The two-pion exchange contribution to M as given by the $\rho_i^\pm(w,t)$ functions [Eqs. (2.7), (2.8)] are not energy independent. In fact, in the Mandelstam representation, and from this direct calculation, they have the form, apart from subtraction terms

$$\rho_i^\pm(w,t)=\frac{1}{\pi}\int_{4m^2}^\infty\frac{y_i^\pm(w',t)dw'}{w'-w}.\tag{3.12}$$

As functions of w they have cuts beginning at the elastic threshold $w=4m^2$ and, therefore, are strongly dependent on w. However, the only contribution to the $\rho_i^\pm(w,t)$ which has this cut in the low-energy region is given by the 2-nucleon intermediate state in the $N+N\to N+N$ channel (Fig. 3). If we call this contribution $\rho_{N,i}^\pm(w,t)$ and let

$$\rho_i^\pm(w,t)=\rho_{N,i}^\pm(w,t)+\rho_{R,i}^\pm(w,t),\tag{3.13}$$

then

$$\frac{1}{\pi}\sum_i P_i\int_{4\mu^2}^\infty\frac{3\rho_{N,i}^+(wt')+2\rho_{N,i}^-(w,t')\boldsymbol{\tau}^n\cdot\boldsymbol{\tau}^p}{t'-t}dt'\tag{3.14}$$

is just the contribution of the fourth-order graph to the scattering amplitude M. It can be shown, as was done by Charap and Tausner,[16] that (3.14) is equal to

$$-(2\pi)^3 T_{\text{OPEP},2}+\frac{1}{\pi}\sum_{\alpha}\tilde{\Omega}_\alpha\int_{4\mu^2}^{\infty}\frac{dt'}{t'-t}\times[3\rho_\alpha'^{+}(w,t')+2\rho_\alpha'^{-}(w,t')\tau^n\cdot\tau^p], \qquad (3.15)$$

where $T_{\text{OPEP},2}$ is the second-order iterated OPEP contribution to the potential scattering amplitude $\langle \mathbf{p}_2|T|\mathbf{p}_1\rangle$, i.e.,

$$T_{\text{OPEP},2}(k^2,t)=\int d\mathbf{q}\frac{\langle\mathbf{p}_2|V_{\text{OPEP}}|\mathbf{q}\rangle\langle\mathbf{q}|V_{\text{OPEP}}|\mathbf{p}_1\rangle m}{q^2-k^2-i\epsilon}, \qquad (3.16)$$

and the functions $\rho_\alpha'^{\pm}(wt)$ have now a weak energy dependence in the sense that the cut in the w variable begins above the meson production threshold.

The functions $\rho_{R,i}^{\pm}(wt)$ which contain contributions from the $(\frac{3}{2},\frac{3}{2})$ resonance (see Fig. 4) also have only cuts in the w plane beginning above the meson production threshold. In fact, in the approximation of taking the resonance to be narrow, these cuts start above the resonance production threshold and again the functions $\rho_{R,i}^{\pm}(w,t)$ are weakly energy dependent.

The contributions from the crossed terms which contain the crossed graphs (see Fig. 5) are given, by (2.3), as

$$\rho_i^{\pm}(\bar{t},t')=\rho_i^{\pm}(-t-4k^2,\,t'). \qquad (3.17)$$

From (3.12) apart from subtraction terms

$$\rho_i^{\pm}(\bar{t},t')=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{y_i^{\pm}(w',t')}{w'+t'+4k^2}dw'. \qquad (3.18)$$

The cut in the energy variable is here very distant from the low-energy physical region and the energy dependence of such terms is very small.

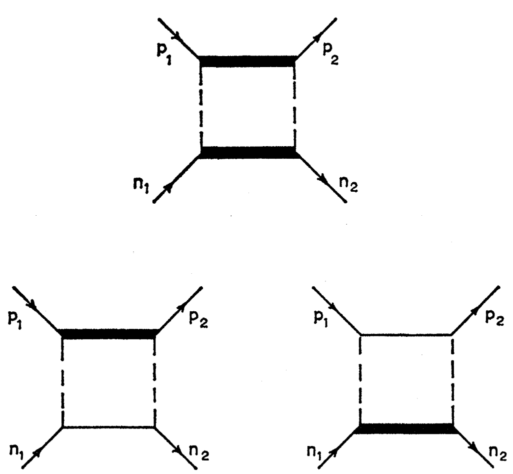

FIG. 4. The uncrossed graphs containing contributions from the $\frac{3}{2}$, $\frac{3}{2}$ resonance.

[16] J. M. Charap and M. J. Tausner, Nuovo Cimento 18, 316 (1960).

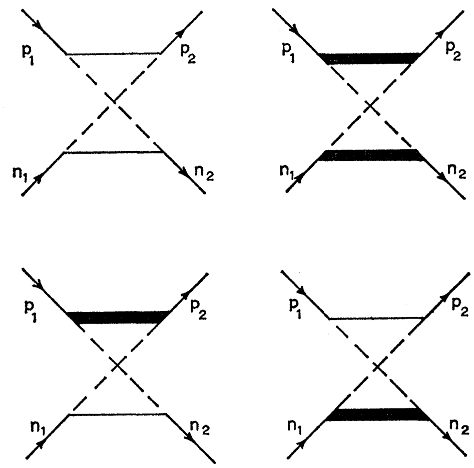

FIG. 5. The crossed graphs.

The $\pi\pi$ interactions, treated by the B.C.L. method only influence the subtraction terms in (3.12) and (3.18). When only S- and P-wave $\pi\pi$ interactions are included, these subtraction terms are at the most first order polynomials in $w\pm\bar{t}$ and so they too have only slight energy dependence over the physical energy region below the inelastic threshold.

An energy-independent two-pion-exchange potential can be defined in the approximation that we neglect the energy dependence of $\rho_\alpha'^{\pm}(wt')$, $\rho_{R,i}^{\pm}(wt')$, and $\rho_i^{\pm}(\bar{t},t')$. That is we make the approximation[17]

$$\begin{aligned}\rho_\alpha'^{\pm}(w,t')&=\rho_\alpha'^{\pm}(4m^2,t'),\\ \rho_{R,i}^{\pm}(wt')&=\rho_{R,i}^{\pm}(4m^2,t'),\\ \rho_i^{\pm}(\bar{t}t')&=\rho_i^{\pm}(0,t').\end{aligned} \qquad (3.19)$$

The field-theory scattering amplitude can then be written in the form:

$$\begin{aligned}M(k^2,t)=&-(2\pi)^3(\tilde{V}_{\text{OPEP}}+T_{\text{OPEP},2})\\ &+\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\left\{\int_{4\mu^2}\frac{dt'}{t'-t}[3\eta_\alpha^{+}(t')+2\eta_\alpha^{-}(t')\tau^n\cdot\tau^p]\right\}\\ &+\left(\frac{1}{\pi}\right)^2\sum_\alpha\tilde{\Omega}_\alpha\left\{\int_{9\mu^2}^{\infty}dt'\int_0^{\infty}dk'^2\right.\\ &\left.\times\frac{3\chi_\alpha^{+}(k'^2t')+2\chi_\alpha^{-}(k'^2t')\tau^n\cdot\tau^p}{(t'-t)(k'^2-k^2)}\right\}\\ &+\left(\frac{1}{\pi}\right)^2\sum_\alpha\tilde{\Omega}_\alpha\left\{\int_{9\mu^2}^{\infty}dt'\int_0^{\infty}dk'^2\right.\\ &\left.\times\frac{3\chi_\alpha'^{+}(k'^2t')+2\chi_\alpha'^{-}(k'^2t')\tau^n\cdot\tau^p}{(t'-t)(k'^2+m^2+t/4+k^2)}\right\}. \end{aligned} \qquad (3.20)$$

[17] This is essentially the Cini-Fubini approximation (reference 10).

The functions $\eta_\alpha{}^{\pm}(t')$ are the contributions of the three terms (3.19) and are independent of w, namely,

$$\eta_\alpha{}^{\pm}(t)=\rho_\alpha{}'^{\pm}(4m^2,t)+\sum_i X_{i\alpha}[\rho_{R,i}{}^{\pm}(4m^2,t)\mp(-1)^i\rho_i{}^{\pm}(0,t)]. \quad (3.20^1)$$

Explicit expressions for these functions will be given in the next section. The last terms of Eq. (3.20) come from the exchange of more than two pions.

Consider now a potential defined by

$$\tilde{V}(\mathbf{\Delta})=\tilde{V}_{\text{OPEP}}-\left(\frac{1}{2\pi}\right)^3\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\times\left\{\int_{4\mu^2}^\infty dt'\frac{3\eta_\alpha{}^+(t')+2\eta_\alpha{}^-(t')\tau^n\cdot\tau^p}{t'-t}\right\}, \quad (3.21)$$

that is, the OPEP and a continuous superposition of Yukawa potentials. Although the Mandelstam representation has not been proved for such a general potential, we will assume it to be true. Then one has[18]

$$\langle\mathbf{p}_2|T|\mathbf{p}_1\rangle=\tilde{V}_{\text{OPEP}}+T_{\text{OPEP},2}-\left(\frac{1}{2\pi}\right)^3\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\left\{\int_{4\mu^2}^\infty dt'\frac{3\eta_\alpha{}^+(t')+2\eta_\alpha{}^-(t')\tau^n\cdot\tau^p}{t'-t}\right\}-\left(\frac{1}{2\pi}\right)^3\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\left\{\int_{9\mu^2}^\infty\frac{dt'}{t'-t}\int_0^\infty dk'^2\times\frac{3\xi_\alpha{}^+(k'^2t')+2\xi_\alpha{}^-(k'^2t')\tau^n\cdot\tau^p}{k'^2-k^2}\right\} \quad (3.22)$$

we now have that, apart from the contributions of the double spectral functions, Eq. (3.2) is indeed satisfied with definition (3.21) of the energy-independent potential.

The last double spectral function of (3.20) has very little energy dependence, the t dependence is such that it has the form of a 3π exchange potential. The double spectral functions of both M and T which have the cuts in the energy variable beginning in the low-energy elastic region are responsible for making both the scattering amplitudes M and T unitary. Since the unitarity condition for the field-theory amplitude is different from the potential amplitude for energies above the inelastic threshold, these two contributions will not be the same. However, the dependence of both of these terms on the t variable should again not be as strong as the dependence of the one- and two-pion-exchange contributions, since the cut in the t variable is more distant from the physical region. The difference between M and T should be small for the partial waves with large angular momentum. We might expect, therefore, that the potential defined by (3.21) and given in configuration space by the OPEP, (4.2), and (4.3) will give an understanding of the way in which two nucleons interact at distances away from the core region.

IV. EXPLICIT FORMULAS FOR THE POTENTIALS

From (3.22) the two-pion-exchange potential (TPEP) to be added to the OPEP is[19]

$$\tilde{V}_{\text{TPEP}}(\mathbf{\Delta})=-\left(\frac{1}{2\pi}\right)^3\frac{1}{\pi}\sum_\alpha\tilde{\Omega}_\alpha\times\left\{\int_{4\mu^2}^\infty dt'\frac{3\eta_\alpha{}^+(t')+2\eta_\alpha{}^-(t')\tau^n\cdot\tau^p}{t'-t}\right\}. \quad (4.1)$$

In configuration space

$$V_{\text{TPEP}}(\mathbf{x})=\sum_\alpha\Omega_\alpha\{3U_\alpha{}^+(r)+2U_\alpha{}^-(r)\tau^n\cdot\tau^p\}, \quad (4.2)$$

where $U_\alpha{}^{\pm}(r)$ can be obtained by expressing $V_{\text{TPEP}}(\mathbf{x})$ as the inverse Fourier transform of $\tilde{V}_{\text{TPEP}}(\mathbf{\Delta})$. Taking care of the operator character of the $\tilde{\Omega}_\alpha$'s one finds

$$\begin{aligned}
U_C{}^{\pm}(r)&=-\left(\frac{1}{2\pi}\right)^2\int_4^\infty\eta_C{}^{\pm}(t)\frac{e^{-rt^{1/2}}}{r}dt,\\
U_{SO}{}^{\pm}(r)&=+\left(\frac{1}{2\pi}\right)^2\int_4^\infty\eta_{SO}{}^{\pm}(t)\frac{e^{-rt^{1/2}}}{r^2}t^{1/2}\left(1+\frac{1}{rt^{1/2}}\right)dt,\\
U_T{}^{\pm}(r)&=-\left(\frac{1}{2\pi}\right)^2\int_4^\infty\eta_T{}^{\pm}(t)\frac{e^{-rt^{1/2}}}{r}t\left(1+\frac{3}{rt^{1/2}}+\frac{3}{r^2t}\right)dt,\\
U_{SO2}{}^{\pm}(r)&=-\left(\frac{1}{2\pi}\right)^2\int_4^\infty\eta_{SO2}{}^{\pm}(t)\frac{e^{-rt^{1/2}}}{r^3}t\left(1+\frac{2}{rt^{1/2}}\right)dt,\\
U_{SS}{}^{\pm}(r)&=-\left(\frac{1}{2\pi}\right)^2\int_4^\infty\eta_{SS}{}^{\pm}(t)\frac{e^{-rt^{1/2}}}{r}dt.
\end{aligned} \quad (4.3)$$

We will separate the different contributions to the functions $\eta_\alpha{}^{\pm}(t)$ into three parts

$$\eta_\alpha{}^{\pm}(t)=\eta_{A,\alpha}{}^{\pm}(t)+\eta_{S,\alpha}{}^{\pm}(t)+\eta_{P,\alpha}{}^{\pm}(t). \quad (4.4)$$

The functions $\eta_{A,\alpha}{}^{\pm}(t)$ contain terms coming from the iteration of the "C.G.L.N. contributions" to $\rho_i{}^{\pm}(wt)$ and $\rho_i{}^{\pm}(\bar{t}t)$ minus the iterated OPEP contributions. The function $\eta_{S,\alpha}{}^+(t)$ contains the contributions due to $\pi\pi$ interactions in S states [the function $\alpha^+(t)$ of

[18] J. Bowcock and A. Martin, Nuovo Cimento 14, 516 (1959); R. Blankenbecler, M. L. Goldberger, N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) 10, 62 (1960).

[19] The results of this section are given in the system of units. $\hbar=\mu=c=1$.

(2.18)] and also the constant $C_A{}^+$. The function $\eta_{P,\alpha}{}^-(t)$ contains the contributions from $\pi\pi$ P-wave interactions [the functions $\alpha^-(t)$ and $\sigma^-(t)$ of (2.18)] and also the constant $C_B{}^-$.

Our method, given in the Appendix, of getting relatively simple analytic formulas for $\eta_{A,C}{}^\pm(t)$, $\eta_{A,SO}{}^\pm(t)$, $\eta_{A,T}{}^\pm(t)$, and $\eta_{A,SS}{}^\pm(t)$ cannot be applied to the function $\eta_{A,SO2}{}^\pm(t)$. For $\eta_{A,SO2}{}^\pm(t)$ the formulas are formally much longer. Although there is no difficulty in principle to calculate them, they are an order of magnitude μ^4/m^4 smaller than the central potential and μ^2/m^2 smaller than the other potentials: For the time being we will not consider them further.

With the definitions

$$\begin{aligned} q&=(\tfrac{1}{4}t-1)^{1/2},\\ N&=(1/32\pi^2)q/t^{1/2},\\ x_1&=\tfrac{1}{2}t-1,\\ x_2&=s_r-m^2-1+\tfrac{1}{2}t,\\ G_A'&=2mG_A/x_2,\\ \phi_1&=\tan^{-1}(2mq/x_1),\\ \phi_2&=\tan^{-1}(2mq/x_2),\\ \Theta&=\tan^{-1}(1/2mq),\\ \eta&=-2+8mq(\tfrac{1}{2}\pi-\phi_1)/t+\frac{4m}{qt}\tan^{-1}\left(\frac{4mq^3}{4m^2q^2+x_1^2}\right). \end{aligned} \tag{4.5}$$

We have

$$\begin{aligned} \eta_{A,C}{}^+(t)=\frac{\pi N}{m^2}\bigg\{&2(G_B-g^2)^2+\frac{2g^2}{x_2^2-x_1^2}[x_2^2(G_A'-g^2)-x_1^2(G_B-g^2)]\frac{x_1\phi_1}{mq}+(G_B-G_A')\left[\frac{2x_2^2g^2}{x_2^2-x_1^2}-\tfrac{3}{2}G_B-\tfrac{1}{2}G_A'\right]\frac{x_2\phi_2}{mq}\\ &+g^4\left[-\frac{\Theta}{2mq}+\frac{3\pi x_1}{4mq}-\frac{\pi}{4mq}+\frac{x_1^2}{x_1^2+4m^2q^2}\right]+(G_B-G_A')^2\frac{x_2^2}{x_2^2+4m^2q^2}\bigg\}, \end{aligned} \tag{4.6a}$$

$$\begin{aligned} \eta_{A,C}{}^-(t)=\frac{\pi N}{m^2}\bigg\{&g^2(G_B-G_A')\frac{x_1^2x_2\phi_1}{(x_2^2-x_1^2)mq}-(G_B-G_A')\left[\frac{x_1x_2g^2}{(x_2^2-x_1^2)}+(\tfrac{1}{8})(G_B-G_A')\right]\frac{x_2\phi_2}{mq}\\ &+g^4\left[\frac{x_1^2}{x_1^2+4m^2q^2}+\frac{\Theta}{2mq}-\frac{3\pi x_1}{4mq}+\frac{\pi}{4mq}\right]+(G_B-G_A')^2\frac{x_2^2}{4(x_2^2+4m^2q^2)}\bigg\}, \end{aligned} \tag{4.6b}$$

$$\begin{aligned} \eta_{A,SO}{}^+(t)=\frac{\pi N}{2m^4}\bigg\{&-g^4\left(1+\eta+\frac{4m^2q^2}{x_1^2+4m^2q^2}\right)-G_B^2-G_BG_A'+\frac{4g^2G_A'x_2^2}{x_2^2-x_1^2}-\frac{g^2x_1\phi_1}{mq}\left[\frac{g^2}{2}+\frac{8m^2q^2(x_2^2+x_1^2)}{(x_2^2-x_1^2)^2}\left(G_B+\frac{x_2G_A}{2mq^2}\right)\right]\\ &+\frac{2g^2x_2^2x_1\phi_1}{mq}\frac{8m^2q^2(G_B-G_A')}{(x_2^2-x_1^2)^2}+(G_B-G_A')\left(G_B+\frac{x_2G_A}{2mq^2}\right)\frac{2mq\phi_2}{x_2}\\ &-\frac{8g^2mqx_2\phi_2}{(x_2^2-x_1^2)^2}\left[(x_2^2+x_1^2)(G_B-G_A')-2x_1^2\left(G_B+\frac{x_2G_A}{2mq^2}\right)\right]+\frac{x_2^2(G_B^2-G_A'^2)}{x_2^2+4m^2q^2}\bigg\}, \end{aligned} \tag{4.6c}$$

$$\begin{aligned} \eta_{A,SO}{}^-(t)=\frac{\pi N}{2m^4}\bigg\{&+g^4\left(\eta+\frac{x_1^2}{x_1^2+4m^2q^2}\right)-\frac{G_B}{4}(G_B-G_A')-4g^2G_A\frac{x_1m}{x_2^2-x_1^2}+\frac{g^4x_1\phi_1}{2mq}-(G_B-G_A')\left(G_B+\frac{x_2G_A}{2mq^2}\right)\frac{mq\phi_2}{2x_2}\\ &+\frac{4mqx_2\phi_1}{(x_2^2-x_1^2)^2}g^2\left[2x_1^2\left(G_B+\frac{x_2G_A}{2mq^2}\right)-(x_2^2+x_1^2)(G_B-G_A')\right]\\ &+\frac{4mqx_1\phi_2}{(x_2^2-x_1^2)^2}g^2\left[2x_2^2(G_B-G_A')-(x_2^2+x_1^2)\left(G_B+\frac{x_2G_A}{2mq^2}\right)\right]+\frac{x_2^2(G_B^2-G_A'^2)}{4(x_2^2+4m^2q^2)}\bigg\}, \end{aligned} \tag{4.6d}$$

$$\begin{aligned} \eta_{A,T}{}^+(t)=\frac{\pi N}{m^4}\bigg\{&\frac{G_B^2}{8}\left(1-\frac{x_2\phi_2}{2mq}-\frac{2mq}{3x_2}\phi_2\right)-\frac{g^2G_B}{6}\left[1-\frac{2mqx_2\phi_2}{x_2^2-x_1^2}\left(1+\frac{x_2^2}{4m^2q^2}\right)\right]\\ &-\frac{g^2G_B}{6}\frac{2mqx_1\phi_1}{x_2^2-x_1^2}\left(1+\frac{x_1^2}{4m^2q^2}\right)+\frac{g^4}{6}\left[\frac{1}{2}-\frac{x_1\phi_1}{4mq}+\frac{mx_1}{qt}\left(\frac{\pi}{2}-\phi_1\right)-\frac{m\Theta}{qt}\right]\bigg\}, \end{aligned} \tag{4.6e}$$

$$\begin{aligned} \eta_{A,T}{}^-(t)=\frac{\pi N}{m^4}\bigg\{&\frac{G_B^2}{96}\left(1-\frac{x_2\phi_2}{2mq}+\frac{2mq}{x_2}\phi_2\right)+\frac{g^2G_B}{6}\frac{mq}{x_2^2-x_1^2}\left[x_2\phi_1\left(1+\frac{x_1^2}{4m^2q^2}\right)-x_1\phi_2\left(1+\frac{x_2^2}{4m^2q^2}\right)\right]\\ &+\frac{g^4}{12}\left[1-\frac{x_1\phi_1}{2mq}-\frac{2mx_1}{qt}\left(\frac{\pi}{2}-\phi_1\right)+\frac{2m}{qt}\Theta\right]\bigg\}, \end{aligned} \tag{4.6f}$$

$$\eta_{A,SS}{}^\pm(t)=-2t\eta_{A,T}{}^\pm(t) \tag{4.6g}$$

the parameters involved in these contributions are the well determined g^2, G_A, and G_B.[20]

We now consider the effects of $\pi\pi$ interactions, firstly the well-established $J=T=1$ $\pi\pi$ resonance (the ρ meson).[21] We here treat the resonance by the same method as B.C.L.[9] and take it to be sufficiently narrow to be considered, for low nucleon-nucleon scattering energies, as a single-particle exchange pole. In this approximation to (2.18)

$$\begin{aligned}\alpha^-(t)&=6\pi^2C_2\delta(t_r-t),\\ \sigma^-(t)&=-12\pi^2(C_1+2mC_2)\delta(t_r-t).\end{aligned}\tag{4.7}$$

For a narrow resonance the interference term between it and the "C.G.L.N. contributions" will be small and

$$\begin{aligned}\eta_{\rho,C}^-(t)&=-DC_1^2\delta(t_r-t),\\ \eta_{\rho,SO}^-(t)&=-(D/m^2)[\tfrac{3}{2}C_1^2+4mC_1C_2]\delta(t_r-t),\\ \eta_{\rho,T}^-(t)&=-(1/2t_r)\eta_{\rho,SS}^-(t)\\ &=D[C_1+2mC_2]^2\delta(t_r-t)/12m^2,\\ D&=(3\pi^2/8\Gamma)[(t_r-4)^3/t_r]^{1/2}.\end{aligned}\tag{4.8}$$

We here take the resonance parameters of B.C.L. which are[22]

$$\begin{aligned}C_2/C_1&=g_V/m,\\ C_1&=-0.6[t_r/(\tfrac{1}{4}t_r-1)^3]^{1/2}\Gamma=-1.0,\\ t_r^{1/2}&=4.7,\ t_r^{-1/2}\Gamma=0.73,\end{aligned}\tag{4.9}$$

where $t_r^{1/2}$ is the resonance energy, $t_r^{-1/2}\Gamma$ is the full width at half-maximum, and g_V is the nucleon isovector gyromagnetic ratio (experimentally $g_V=1.83$).

With these parameters, and transforming to configuration space we have

$$\begin{aligned}U_{\rho,C}^-(r)&=3\mathfrak{N}^2e^{-rt_r^{1/2}}/4r,\\ U_{\rho,SO}^-(r)&=-\frac{3\mathfrak{N}^2(\frac{3}{2}+4g_V)t_r^{1/2}e^{-rt_r^{1/2}}}{4m^2r^2}\left(1+\frac{1}{rt_r^{1/2}}\right),\\ U_{\rho,T}^-(r)&=-\frac{3\mathfrak{N}^2(1+2g_V)^2t_re^{-rt_r^{1/2}}}{48m^2r}\left(1+\frac{3}{rt_r^{1/2}}+\frac{3}{r^2t_r}\right),\\ U_{\rho,SS}^-(r)&=3\mathfrak{N}^2t_r(1+2g_V)^2e^{-rt_r^{1/2}}/24m^2r,\end{aligned}\tag{4.10}$$

$$\mathfrak{N}^2=C_1^2(t_r-4)^{3/2}/8t_r^{1/2}\Gamma=0.6,\tag{4.11}$$

with regard to the constant C_B^-. This also contributes to the $N+\bar{N}\rightarrow2\pi$ P-wave amplitude. However, the value of this parameter from B.C.L. is so small as to produce a negligible effect on the potential. Here, we have set $C_B^-=0$; therefore,

$$U_{P,\alpha}^-(r)=U_{\rho,\alpha}^-.\tag{4.12}$$

Turning now to the $\pi\pi$ S-wave interactions, we believe that at the present time there is no firm experimental evidence for them being strong in this state.[23] For the moment we have not included any $\pi\pi$ S-wave interaction terms in this potential calculation [we have set $\alpha^+(t)=0$]. The effects of such interactions will be discussed in Sec. V. The constant C_A^+ does contribute significantly to the $N\bar{N}\rightarrow2\pi$ S-wave amplitude. It is also related to the pion-nucleon S-wave scattering lengths and has been determined from the experimental values of these parameters.

The determination has been done in such a way as to ensure that the model formula (2.18) for the πN scattering amplitude predicts these πN S-wave scattering lengths (taken here to be $a_1=0.17$, $a_3=-0.089$.[24] With the B.C.L. parameters and $\alpha^+(t)=0$ this value is

$$C_A^+=-0.9(4\pi).\tag{4.13}$$

The contribution to the potential is

$$\eta_{S,C}^+(t)=2\pi NC_A^+\left[C_A^++\frac{2(G_B-g^2)}{m}-\frac{x_2(G_B-G_A')\phi_2}{m^2q}+\frac{g^2x_1\phi_1}{m^2q}\right],\tag{4.14}$$

$$\eta_{S,SO}^+(t)=-(1/2m^2)\eta_{S,C}^+(t).\tag{4.15}$$

C_A^+ and the $N\bar{N}\rightarrow2\pi$ S-wave amplitude in general, has no effect on the tensor and spin-spin potentials.

Our way of normalizing the S- and P-wave $N\bar{N}\rightarrow2\pi$ amplitudes by adding these constants C_A^+ and C_B^-, the values of which are directly related to experimental πN scattering lengths, is in some way equivalent to the Ball and Wong normalization procedure.[25] The importance of such a normalization has been already emphasized by Moravcsik and Noyes[26] in connection with the "pair damping" assumption in earlier nucleon-nucleon calculations.

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[20] An interesting feature of the "C.G.L.N. contributions" is that in a μ/m expansion, $\eta_{A,C}^{\pm}(t)$ is formally an order of magnitude m^2/μ^2 larger than the other potentials, but that for $\eta_{A,C}^{\pm}(t)$ large cancellations take place in the leading terms (since $g^2\cong G_B\cong G_A'$ for t small). These canceling terms are contained implicitly in the model formula used for the scattering amplitude. Explicitly they are the parts of the general πN scattering amplitude that refer to S-wave scattering. We have used here a pion-nucleon scattering formula that takes account of the single nucleon term and the $(\frac{3}{2},\frac{3}{2})$ resonance rescattering correction. It was first shown by Chew, Goldberger, Low, and Nambu (reference 12), that the leading terms in a μ/m expansion of this formula give only πN S-wave scattering, and although the individual contributions are large, they are approximately equal in magnitude and of opposite sign.

[21] E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. Letters **7**, 192 (1961). D. Duane Carmony and Remy T. Van de Walle, *ibid.* **8**, 73 (1962); see also reference 29.

[22] The B.C.L. parameters now need some revision, for example, the resonance energy is now established by direct experiment to be $\simeq 5.4\mu$ (see reference 21). However, we think that the use of more realistic parameters than B.C.L. will not change any of the principal features of these contributions.

[23] However, on the theoretical side, the analysis of J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (N. Y) **17**, 1 (1962); and J. Hamilton (private communication), indicate that these interactions are probably not negligible.

[24] W. S. Woolcock, in *Proceedings of the Aix-en-Provence Conference, 1961* (Centre d'Etudes Nucleaires de Saclay, Seine et Oise, 1961).

[25] J. Ball and D. Wong, Phys. Rev. Letters **6**, 29 (1961).

[26] M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. **11**, 95 (1961). We are grateful to Professor H. Noyes for having called our attention to this question.

V. DISCUSSION OF THE RESULTS

In this paper we have attempted to calculate a nucleon-nucleon potential which will give an understanding of the gross features of the mechanism of the nucleon-nucleon interaction (its tensor, spin-orbit characteristics, etc.). Detailed phase-shift analyses have been made on the low-energy nucleon-nucleon scattering data, and it has been shown that these data can be approximately fitted by means of phenomenological local potentials.[8] Regarding the detailed phase-shift analyses, Amati, Leader, and Vitale are in the process of calculating nucleon-nucleon phase parameters directly from the one- and two-pion exchange as given in I and II. Our analysis is intended to be complementary to their approach. Instead of solving the Schrödinger equation with our potential[27] we compare it directly to two phenomenological potentials constructed to reproduce the experimental results. The phenomenological potentials we have used for comparison are those of Breit *et al.*,[8] which we call here the Yale potential, and also that of Hamada.[28]

These two potentials are very similar in form, they are both constructed using the five forms discussed in Sec. IV. They agree generally in sign and order of magnitude especially in the outer parts, and we believe that at least these features of the phenomenological potentials are correct. The comparison of our potential with the Yale potential for $r>0.5$ (units $\hbar=\mu=c=1$) is given in Figs. 6 to 13. In the figures the superscripts 0 and 1 refer to the isotopic spin states 0 and 1 [see Eq. (3.5)]. The curves labeled A are the complete potentials obtained from this calculation including the $\pi\pi$ P-wave resonance (ρ meson) with the parameters given in Sec. IV. The curves B are the one- and two-pion-exchange potentials calculated with no $\pi\pi$ interaction. They are given to show the extent to which the resonance is itself responsible for the calculated potential, and also to facilitate the modification of the complete potential A when more reliable resonance parameters are used than (4.9). The curves C are obtained by neglecting the C.G.L.N. contributions to the $N\bar{N}\to\pi\pi$ P-wave amplitudes. That is, for curves C, we take the $N\bar{N}\to\pi\pi$ P-wave amplitude to contain only the resonance contribution. We wish to emphasize that the potential has been calculated from a knowledge of the pion-nucleon scattering amplitude and the coupling constant g^2 only, with no arbitrary parameters. The calculation does, however, depend on the B.C.L. model formulas that were used for the $N\bar{N}\to\pi\pi$ amplitude. The calculation is, therefore, reliable only to the extent that this model is reliable.

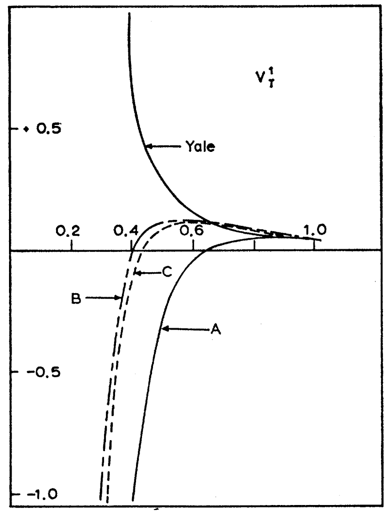

FIG. 7. The isotopic spin one, tensor potential.

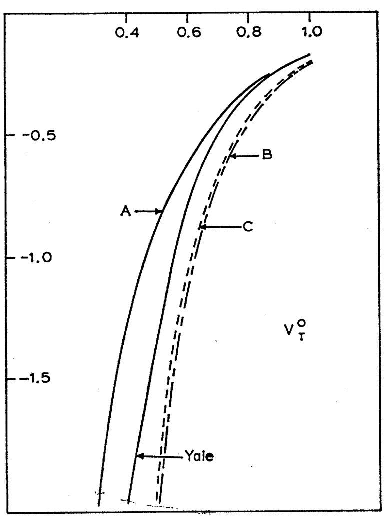

FIG. 6. The tensor potential in the singlet isotopic spin state. Curves A are the "complete" potentials containing all the contributions discussed in IV. The units are $\hbar=\mu=c=1$.

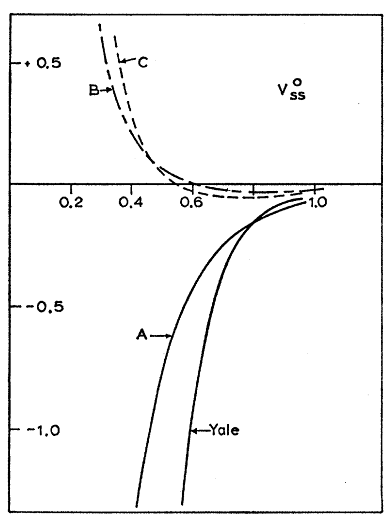

FIG. 8. The isotopic spin zero, spin-spin potential.

[27] The solutions of the Schrödinger equation can be obtained directly from the weight functions $\eta_\alpha^\pm(t)$ by using a rather simple iteration method given by R. Vinh Mau and A. Martin, Nuovo Cimento **20**, 390 (1961).

[28] T. Hamada, Progr. Theoret. Phys. (Kyoto) **24**, 1033 (1960); **25**, 247 (1961). See also T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).

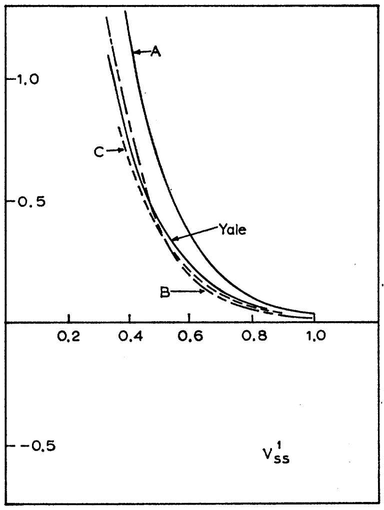

FIG. 9. The isotopic spin one, spin-spin potential.

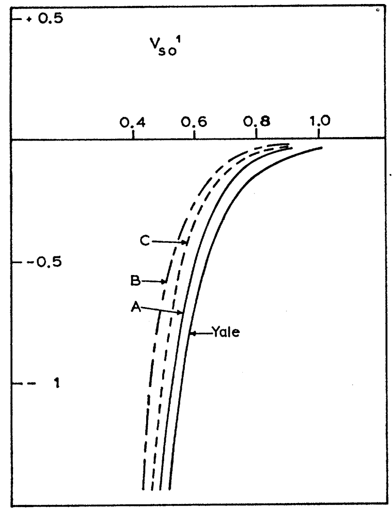

FIG. 11. The isotopic spin one, spin-orbit potential.

A casual glance at the figures will show that for $r > 0.5$ most of the features of the Yale potential are reproduced by the calculated potential A. Also, although the influence of the ρ resonance does not dominate the calculated potential it is important for an understanding of some of the phenomenological features.

For an examination of the individual potentials we will first discuss the tensor and spin-spin potentials ($V_T^{0,1}$ and $V_{SS}^{0,1}$) since these are independent of $\pi\pi$ interactions in S waves. The signs and strengths of all four of these potentials are in good agreement with Yale, especially at large distances. Concerning the tensor potentials, the outer parts are, in fact, dominated by the well-established one-pion-exchange contributions. Proceeding towards the inner regions of V_T^0 and V_T^1 the Yale and Hamada potentials deviate from OPEP by terms of the same sign, although for V_T^1 the deviation of Hamada from OPEP is somewhat bigger than that of Yale. The calculated V_T^0 and V_T^1 deviate from OPEP in the same way as phenomenology. For V_T^1 we favor the Hamada potential (which has our sign change) rather than Yale. Regarding V_{SS}^0 and V_{SS}^1 the one-pion-exchange contribution is small, at least in the region shown in the graphs, most of the calculated potential comes from the two-pion-exchange term. For V_{SS}^1 Yale and Hamada are almost the same and are in reasonable agreement with this calculation. For V_{SS}^0 the Hamada potential has the same sign as Yale but is not as strong. The calculated V_{SS}^0 falls in between these two. It should also be noted that for this potential in particular the influence of the $\pi\pi$ resonance is strong and helps in giving agreement with phenomenology.

For the spin-spin and tensor potentials, in both isotopic spin states we can say that the one- and two-pion contributions alone are in good qualitative agreement with phenomenology at least for $r > 0.5$.

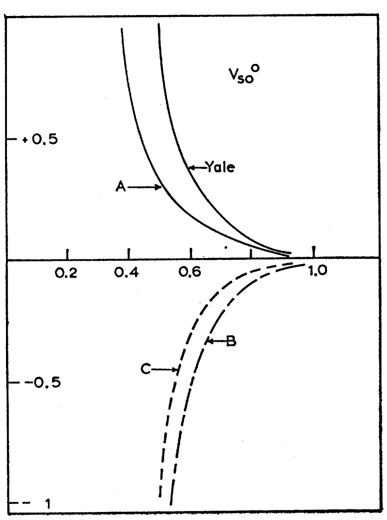

FIG. 10. The isotopic spin zero, spin-orbit potential.

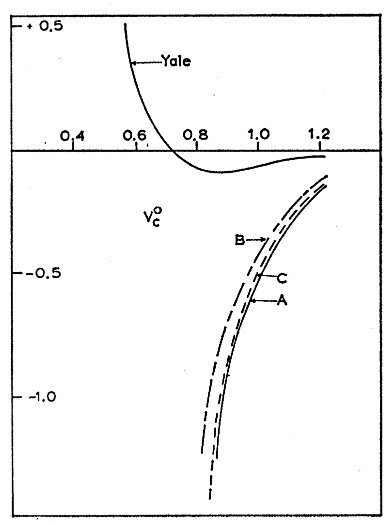

FIG. 12. The isotopic spin zero, central potential.

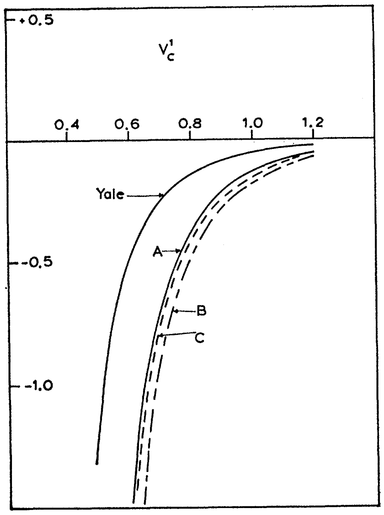

FIG. 13. The isotopic spin one, central potential.

The potentials $V_C^{0,1}$ and $V_{SO}^{0,1}$ do depend on what is assumed concerning $\pi\pi$ S-wave interactions, and this is especially true of the central potential. The calculation of these potentials is correspondingly of more doubtful validity than is the calculation of $V_T^{0,1}$ and $V_{SS}^{0,1}$. Also, these potentials have no contributions from OPEP. In this calculation they are determined only from the two-pion-exchange contributions.

Considering the spin-orbit potentials, the V_{SO}^1 of Yale and Hamada are in very good agreement. The calculated two-pion contribution to V_{SO}^1 can be seen to be in good agreement with Yale. Although this agreement might be to some extent spurious we think the two-pion contribution as calculated here to be the largest individual contributing part. As for V_{SO}^0 it does not seem to be as well established phenomenologically as V_{SO}^1. However, both Yale and Hamada agree that it has opposite sign to V_{SO}^1 (see Figs. 10 and 11). A feature given by our calculation, however, as with V_{SS}^0 the ρ resonance gives a large contribution to the calculated V_{SO}^0. Also the sign of this potential is in fact sensitive to small changes in the ρ resonance parameters, and so cannot be regarded as reliably established by this calculation. However, although the resonance parameters that we have used are less reliable than the other parameters that occur in this calculation, as far as a comparison can be made the two-pion contribution alone can give an understanding of the size and sign of this potential.

As regards the central potentials, our results are in much worse agreement with Yale than are the spin-orbit, tensor, and spin-spin potentials. This is especially true of V_C^0. Although the signs of our central potentials are in agreement with Yale at very large distance, the Yale potential V_C^0 quickly becomes repulsive, as is the Hamada V_C^0, as you move closer to the core, while the calculated potential remains strongly attractive.

The curve B for V_C^0 shows that the calculated potential without taking account of the $\pi\pi$ P-wave resonance is attractive, and the inclusion of the resonance makes the potential more attractive. The inclusion of this resonance with more reliable parameters cannot therefore help matters very much. The only other possibility within the framework of this calculation is to include $\pi\pi$ S-wave interaction. However, the effect of the exchange of two pions in a relative S state is of unambiguous sign and gives a central attraction.

It happens that even if the total $\pi\pi$ S-wave exchange contribution to the central potential is subtracted from the calculated potential, the remaining part is still attractive. Therefore, we cannot obtain a repulsive V_C^0 within the framework of this calculation by the inclusion of $\pi\pi$ S-wave interactions either repulsive or attractive. It would seem, therefore, that this feature of the nucleon-nucleon potential (repulsive V_C^0 at quite large distances), if it is to be taken seriously, comes from three and more pion exchange effects.

One part of the three-pion-exchange contribution that can be considered without too much difficulty is the exchange of the ω resonance.[29] There will, of course, be other 3π exchange contributions; however, since at the present time the ω is the most well established 3π effect, it is of interest to discuss it here.

The ω resonance has $T=0$ [29,30] and, like the ρ, is a narrow resonance. We will here discuss it, as we have done for the ρ by considering it as the exchange of a single particle. The strength of the coupling of this particle to the nucleon has not yet been reliably determined. If we assume however, as current evidence suggests, that the ω has the same quantum numbers as the photon[30] and is the resonance responsible for much of the isoscalar nucleon electromagnetic form factors,[31] then we can include the ω exchange in the same way as we include the ρ exchange. In analogy with (4.10) the contribution of the ω to the NN potential will then be

$$
\begin{aligned}
U_{\omega,C}^{+}(r)&=3\mathfrak{N}'^2e^{-rt_r'^{1/2}}/4r,\\
U_{\omega,SO}^{+}(r)&=-\frac{3\mathfrak{N}'^2t_r'^{1/2}e^{-rt_r'^{1/2}}}{4m^2r^2}\left(1+\frac{1}{rt_r'^{1/2}}\right)(\tfrac{3}{2}+4g_S),\\
U_{\omega,T}^{+}(r)&=-\frac{3\mathfrak{N}'^2(1+2g_S)^2t_r'e^{-rt_r'^{1/2}}}{48m^2r}\left(1+\frac{3}{rt_r'^{1/2}}+\frac{3}{r^2t_r'}\right),\\
U_{\omega,SS}^{+}(r)&=3\mathfrak{N}'^2(1+2g_S)^2t_r'e^{-rt_r'^{1/2}}/24m^2r,
\end{aligned}
\tag{5.1}
$$

g_S = isoscalar gyromagnetic ratio ($g_S=-0.06$), ω resonance energy $t_r'^{1/2}=5.7$, $\mathfrak{N}'^2$ is a constant which is

[29] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961).
[30] N. H. Xuong and G. R. Lynch, Phys. Rev. Letters **7**, 327 (1961).
[31] S. Bergia, A. Stanghellini, S. Fubini, and C. Villi, Phys. Rev. Letters **6**, 367 (1961).

given by the strength of the coupling of the ω meson to the nucleon [analogous to $\mathfrak{N}^2$ of (4.10)].

The approximate equality of the 2π and 3π resonance energies ($t_r \simeq t_r'$) implies that the functional forms of the ρ and ω contributions are approximately the same.

The other features of these two contributions are, however, different. The first difference is that the ω, since it has zero isotopic spin, gives, in the nucleon-nucleon channel, the same contribution to both isotopic spin states.

Also, from a comparison of (5.1) and (4.10) it can be seen that the ρ and the ω give relatively different contributions to the different potential forms. In particular, the smallness of the nucleon isoscalar anomalous magnetic moment compared to the isovector moment implies that apart from the over all constant $\mathfrak{N}'^2$ and the different isospin factor the ω contribution to the spin spin and tensor potentials is an order of magnitude $[(1+2g_S)/(1+2g_V)]^2 \simeq 1/30$ smaller than the ρ contribution. Therefore, unless $\mathfrak{N}'^2$ is very much larger than $\mathfrak{N}^2$ the ω contributions to the spin spin and tensor potentials will be very small.

Similar considerations also apply to the spin-orbit potentials. In this case, however, the ω contributes a term of order of magnitude $(1.5+4g_S)/(1.5+4g_V) \simeq 1/7$ that of the ρ. The ω contribution to the spin-orbit potential need not, therefore, be so very small. The contribution of the ω to the central potential is, however, apart from the coupling constant $\mathfrak{N}'$, of the same size as the ρ contribution. Also, it is repulsive in both isotopic spin states.

We have no reliable knowledge of this coupling constant $\mathfrak{N}'$, however, the frequency of production of the ω resonance relative to the ρ in proton antiproton annihilation processes suggests that $\mathfrak{N}'$ could be at least as large as $\mathfrak{N}$. Provided $\mathfrak{N}'$ is not very much greater than $\mathfrak{N}$, the influence of the ω on $V_{SS}(r)$, $V_T(r)$, and to a lesser extent $V_{SO}(r)$ will not be large.

However, the influence on the central potential could be large and is certainly repulsive. The two-pion contributions to the central potential as calculated here, appear to be too attractive. If more refined calculations, including some three-pion-exchange effects are to improve the agreement of this calculation with experiment, the additional contributions must be repulsive. The ω-meson contribution has this desirable feature; however, the inclusion of the ω alone, to this calculation, will still not give good agreement with Yale in both isotopic spin states.

VI. CONCLUSIONS

The work of Breit *et al.* and Hamada shows that a reasonable fit can be obtained to the wealth of nucleon-nucleon scattering data below 310-MeV lab energy, in terms of the scattering amplitude calculated in a potential. A potential that, apart from the energy dependence implied by the five potential forms used, is energy independent (see Sec. II). Such work does not prove, and would not be expected to prove, that such a potential gives an exact understanding of the nucleon-nucleon interaction. It does, however, give some experimental support to the claim that the field-theory scattering amplitude is approximately the same as the scattering amplitude in such a potential.

We have calculated the one- and two-pion-exchange contributions to these equivalent potentials. It must be emphasized that there are no arbitrary parameters involved in this calculation, only parameters coming from other well established branches of meson physics. Some improvement of this calculation is certainly possible especially if more sophisticated $N\bar{N} \to \pi\pi$ amplitudes are used than those of the B.C.L. model. However, presuming that such a more reliable analysis does not change the qualitative features of these results, it can be said that the one- and two-pion contributions alone do give an understanding of many of the features of the phenomenological potentials at distances greater than $0.6\mu^{-1}$.

This success gives much support to the general method of probing into the form of the nucleon-nucleon interactions by calculating first the one pion, then the two pion and so on exchange contributions to the scattering amplitude and hoping that a good approximation can be obtained by keeping only the first few terms.

It is premature to place very great faith in the exact forms of the potentials as calculated here, especially at the smallest distances shown ($\sim 0.5\mu^{-1}$). This is particularly true of the central potentials since the phenomenological repulsion of V_C^0 is not given by the one- and two-pion-exchange contributions alone. We believe, however, that the inclusion, when it is possible, of the three-pion-exchange terms, could much improve the situation. Also, that a reliable treatment of these effects along with a more realistic calculation of the two-pion-exchange terms, including, for example, $\pi\pi$ S-wave scattering corrections, could give a quantitatively reliable form of low-energy nucleon-nucleon interactions at much smaller distances than those at which OPEP alone is dominant.

ACKNOWLEDGMENTS

This work was started while both of us were at the Theoretical Study Division of CERN. We are deeply grateful to Professor D. Amati for many discussions and comments.

APPENDIX

The purpose of this Appendix is to give the method of obtaining the functions $\eta_{A,C}^{\pm}(t)$, $\eta_{A,SO}^{\pm}(t)$, $\eta_{A,T}(t)$, and $\eta_{A,SS}^{\pm}(t)$ given in Sec. IV. For simplicity we will write throughout this Appendix $\eta_\alpha^{\pm}(t)$, $\rho_\alpha^{\pm}(t)$, $\rho_i^{\pm}(t)$, instead of $\eta_{A,\alpha}^{\pm}(t)$, etc.

From (3.20′) the functions $\eta_\alpha^\pm(t)$ are defined as

$$\eta_\alpha^\pm(t)=\sum_i X_{\alpha,i}[\rho_i^\pm(4m^2t)\mp(-1)^i\rho_i^\pm(0,t)], \quad (A1)$$

where

$$\rho_i^\pm(4m^2,t)=\rho_i'^\pm(4m^2,t)+\rho_{R,i}^\pm(4m^2,t).$$

The first term of (A1) is the contribution from the "uncrossed" terms minus the second-order iterated OPEP, and the second is the "crossed" terms contribution.

Using a similar notation to I and II let us define the functions

$$q=(\tfrac{1}{4}t-1)^{1/2},$$
$$\kappa=(m^2-\tfrac{1}{4}t)^{1/2},$$
$$\xi(x',x'')=x''(x'-x'')-2wq^2, \quad (A2)$$
$$\beta(x',x'')=(x'-x'')^2+4wq^2,$$
$$\zeta(x',x'')=x''^2+4q^2\kappa^2,$$

$$I^0(x',x'')=\frac{4\pi}{[\zeta\beta-\xi^2]^{1/2}}\tan^{-1}\left[\frac{(\zeta\beta-\xi^2)^{1/2}}{\xi+\zeta}\right], \quad (A3)$$

and also

$$I(x)=\frac{2\pi}{q\kappa}\tan^{-1}\left(\frac{2q\kappa}{x}\right), \quad (A4)$$

$$R(x',x'')=(1/2\bar{t})[I(x')+I(x'')-(x'+x'')I^0(x',x'')],$$
$$S(x',x'')=(1/2w)[I(x')-I(x'')+(x'-x'')I^0(x',x'')], \quad (A5)$$

and from these, the functions

$$\theta_1^+(w,x',x'')=\frac{m^2Ng^4}{w\bar{t}}\Bigg\{(w+\bar{t})\left[\frac{\pi}{\kappa^2}-\frac{1}{8\kappa^2}[x'I(x')+x''I(x'')]\right]-(2w-\bar{t})\left(\frac{x'+x''}{2}\right)R+(w-2\bar{t})\left(\frac{x'-x''}{2}\right)S+q^2(w-\bar{t})I^0(x',x'')\Bigg\},$$

$$\theta_2^+(w,x',x'')=\frac{m^2Ng^4}{w\bar{t}}\Bigg\{(w-\bar{t})\left[\frac{\pi}{\kappa^2}-\frac{1}{8\kappa^2}[x'I(x')+x''I(x'')]\right]-(2w+\bar{t})\left(\frac{x'+x''}{2}\right)R+(w+2\bar{t})\left(\frac{x'-x''}{2}\right)S+q^2(w+\bar{t})I^0(x',x'')\Bigg\}, \quad (A6)$$

$$\theta_3^+(w,x',x'')=(1/m^2)\theta_1^+(w,x',x''),$$

$$\theta_4^+(w,x',x'')=Ng^4\left[-\left(\frac{x'+x''}{2}\right)R+\left(\frac{x'-x''}{2}\right)S+q^2I^0(x',x'')\right],$$

$$\theta_5^+(w,x',x'')=0.$$

It is convenient to consider separately the contributions of the three terms to the $\rho_i^\pm(w,t)$ functions: (1) the iterated $(\frac{3}{2},\frac{3}{2})$ resonance terms, (2) the "mixed" terms, and (3) the fourth order terms (see Figs. 3–4).

The results of I and II on the contributions of these different parts can be written as follows:

(1) The iterated $(\frac{3}{2},\frac{3}{2})$ resonance:

$$\rho_1^+(w,t)=NG_A^2I^0(x_2,x_2)+2mNG_AG_BR(x_2x_2)+(G_B^2/g^4)\theta_1^+(w,x_2,x_2),$$
$$\rho_2^+(w,t)=NG_AG_BR(x_2x_2)+(G_B^2/g^4)\theta_2^+(w,x_2,x_2),$$
$$\rho_3^+(w,t)=(G_B^2/g^4)\theta_3^+(w,x_2,x_2), \quad (A7)$$
$$\rho_4^+(w,t)=(G_B^2/g^4)\theta_4^+(w,x_2,x_2),$$
$$\rho_5^+(w,t)=0,$$
$$\rho_i^-(w,t)=-\tfrac{1}{4}\rho_i^+(w,t).$$

(2) The "mixed" terms:

$$\rho_1^+(w,t)=2mNg^2G_A[-R(x_1,x_2)+S(x_1,x_2)]-(2G_B/g^2)\theta_1^+(w,x_1,x_2),$$
$$\rho_2^+(w,t)=-Ng^2G_A[R(x_1,x_2)+S(x_1,x_2)]-(2G_B/g^2)\theta_2^+(w,x_1,x_2),$$
$$\rho_3^+(w,t)=-2(G_B/g^2)\theta_3^+(w,x_1,x_2), \quad (A8)$$
$$\rho_4^+(w,t)=-2(G_B/g^2)\theta_4^+(w,x_1,x_2),$$
$$\rho_5^+(w,t)=0,$$
$$\rho_i^-(w,t)=+\tfrac{1}{2}\rho_i^+(w,t).$$

The fourth-order contributions: Apart from the subtraction of the second-order iterated OPEP these are

$$\rho_i^+(w,t)=\theta_i^+(w,x_1,x_1),$$
$$\rho_i^-(w,t)=-\theta_i^+(w,x_1,x_1). \quad (A9)$$

The forms (A6) for the functions $\theta_1^+(w,x_1,x_2)$ and $\theta_2^+(w,x_1,x_2)$ are not convenient for making the adiabatic approximation both for the crossed and uncrossed terms because of the denominator $w\bar{t}$ in these expressions. However, if the transformation to the "potential" representation is made before the adiabatic limits are taken the expressions become much more simple and we have from the transformation matrix X Eq. (3.10), for the "uncrossed" terms

$$\theta_C^+(4m^2,x',x'') =\frac{Ng^4}{2}\left[-(x'-x'')S(x',x'')+\frac{2\pi}{m^2} -\frac{1}{4m^2}[x'I(x')+x''I(x'')]\right],$$

$$\theta_{SO}^+(4m^2,x',x'') =\frac{Ng^4}{4m^2}\left[4q^2I^0(x'x'')-2(x'+x'')R(x',x'') +(x'-x'')S(x',x'')+\frac{2\pi}{m^2} -\frac{1}{4m^2}[x'I(x')+x''I(x'')]\right], \quad \text{(A10)}$$

$$\theta_T^+(4m^2,x',x'')=-(1/12m^2)\theta_4^+(4m^2,x',x''),$$

$$\theta_{SS}^+(4m^2,x',x'')=(t/6m^2)\theta_4^+(4m^2,x',x''),$$

and for the crossed terms

$$\theta_C^+(0,x',x'') =\frac{Ng^4}{2}\left[-(x'+x'')R(x',x'')+\frac{2\pi}{m^2} -\frac{1}{4m^2}[x'I(x')+x''I(x'')]\right],$$

$$\theta_{SO}^+(0,x',x'') =\frac{Ng^4}{4m^2}\left[-4q^2I^0(x',x'')+(x'+x'')R(x',x'') -2(x'-x'')S(x',x'')+\frac{2\pi}{m^2} -\frac{1}{4m^2}[x'I(x')+x''I(x'')]\right], \quad \text{(A11)}$$

$$\theta_T^+(0,x',x'')=(1/12m^2)\theta_4^+(0,x',x''),$$

$$\theta_{SS}^+(0,x',x'')=-(t/6m^2)\theta_4^+(0,x',x'').$$

From these expressions [and (A7) and (A8)] it is not difficult to obtain the iterated $(\frac{3}{2},\frac{3}{2})$ resonance and mixed contributions to the functions $\eta_\alpha^\pm(t)$ [Eq. (A1)]. These are, respectively, the terms in Eq. (4.6) which have no explicit g^2 dependence and those which depend only linearly on g^2. The "uncrossed" contributions to the fourth order terms (terms of order g^4) are more involved because of the subtraction of the iterated OPEP. The reader is referred to the work of Charap and Tausner for a discussion of this point.