

## Discussion of Electron Optics Limit of Optically Pumped Alkali Vapor\*

R. R. LEWIS

Department of Physics, University of Michigan, Ann Arbor, Michigan

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Recent experiments on the behavior of thermal electrons in a dilute, optically pumped sodium vapor can be appropriately described in terms of the "electron optics" limit. The formalism developed for "neutron optics" is rederived for this purpose. This description gives a simple interpretation to the existence of an electron polarization, and of a shifted (but not broadened) electron spin resonance.

IN a recent experiment of Hobart and Franken,<sup>1</sup> the spin resonance for free electrons in an optically pumped sodium vapor was observed. It was found to exhibit a frequency shift dependent on temperature and light intensity. Both the polarization of the free electrons, and the shift of the spin resonance were interpreted as consequences of the spin-dependent scattering of thermal electrons from polarized sodium atoms.

The purpose of this note is simply to point out the utility of a description of these phenomena in the same formalism as that developed for thermal neutron scattering,<sup>2</sup> especially the use of the *pseudopotential*.<sup>3</sup> This formalism is adequate for the description of the scattering of a particle of wavelength  $\lambda$ , in a medium of scattering centers of radius  $a$ , separated by a mean distance  $d$ , if

$$a \ll \lambda \ll d.$$

For thermal neutrons in a solid,

$$a \simeq 10^{-13} \text{ cm}, \quad \lambda \simeq 10^{-9} \text{ cm}, \quad d \simeq 10^{-8} \text{ cm};$$

while for thermal electrons in sodium vapor,

$$a \simeq 10^{-8} \text{ cm}, \quad \lambda \simeq 10^{-7} \text{ cm}, \quad d \simeq 10^{-6} \text{ cm}.$$

The simplifications which arise in this limit are twofold. (a) Only the lowest angular momentum states interact appreciably (*s* waves, *p* waves...). (b) Only the asymptotic form of the scattered wave is required. Together, these imply that the exact asymptotic form of the scattered wave from a single scattering center,

$$\psi \underset{r \rightarrow \infty}{\simeq} e^{ikz} + f \frac{e^{ikr}}{r}, \quad f = \frac{e^{2id} - 1}{2ik},$$

can be derived by computing the scattering *in Born approximation* (perturbation theory) with the pseudopotential,

$$V(\mathbf{r}) = -\frac{2\pi\hbar^2}{m} f \delta(\mathbf{r}).$$

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<sup>1</sup> J. Hobart, thesis, University of Michigan, 1962 (unpublished).

<sup>2</sup> D. J. Hughes, *Neutron Optics* (Interscience Publishers, Inc., New York, 1954).

<sup>3</sup> This device is due originally to E. Fermi, *Ricerca Sci.* **7**, 13 (1936). It is rederived in Blatt and Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 74, and further generalized in K. Huang and C. N. Yang, *Phys. Rev.* **105**, 767 (1957).

The subsequent discussion of the behavior of a single electron in such a medium is greatly facilitated by the use of this pseudopotential.<sup>4</sup>

Both for the case of thermal neutrons and of thermal electrons in an alkali vapor, the method can be generalized to include the effects of spin by introducing different scattering amplitudes  $f_1$ ,  $f_3$  for the  $^1S$  and  $^3S$  states, neglecting all other states. This corresponds to choosing

$$V(\mathbf{r}) = -\frac{2\pi\hbar^2}{m} \delta(\mathbf{r}) \left[ f_1 \left( \frac{1 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4} \right) + f_3 \left( \frac{3 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4} \right) \right],$$

where  $\boldsymbol{\sigma}_1$ ,  $\boldsymbol{\sigma}_2$  are the spin operators for electron and sodium, respectively. We neglect the effects of nuclear spin and hyperfine structure, and the scattering from the buffer gas.

### A. SINGLE SCATTERING OF ELECTRONS

It is an elementary calculation to express the electron scattering cross section and polarization in terms of  $f_1$ ,  $f_3$ . The results show that the scattering from an *unpolarized* sodium atom cannot produce electron polarization. But if the sodium atom has polarization  $\mathbf{P}$ , then the single scattering of an unpolarized electron will produce an electron polarization proportional to  $\mathbf{P}$ .

These results can be verified with a density matrix formalism as follows: Let  $M$  be the  $4 \times 4$  scattering matrix:

$$M = f_1 \left[ \frac{1 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4} \right] + f_3 \left[ \frac{3 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2}{4} \right].$$

It is also convenient to write this as

$$M = a + (1/\sqrt{3})b\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,$$

where  $a = (f_1 + 3f_3)/4$  is called the coherent (or non-spin-flip) amplitude, and  $b = \sqrt{3}(f_3 - f_1)/4$  is called the incoherent (or spin-flip) amplitude. The physical origin of the difference between  $f_3$ ,  $f_1$  lies in the effective spin dependence of the interaction of two identical electrons (exchange effects). This formalism includes both direct and exchange amplitudes. Then the  $4 \times 4$  density matrices describing the incident and scattered beams are related by

$$\rho_f = M \rho_i M^\dagger.$$

<sup>4</sup> We consider only the rather trivial problem of the motion of a single electron in the gas of many sodium atoms. The pseudopotential has also been extended to the more difficult problem of many bodies (i.e., the motion of a sodium atom in the gas).

The differential cross section is

$$d\sigma/d\Omega = \text{tr}\rho_f/\text{tr}\rho_i,$$

and the polarization of either electrons or sodium, in initial or final state, is

$$\mathbf{P} = \text{tr}(\boldsymbol{\sigma}_1, \rho)/\text{tr}(\rho).$$

For unpolarized electrons and sodium we set  $\rho_i = \frac{1}{4}\mathbf{I}$  and then by an easy calculation<sup>5</sup> find the final polarization to be zero and the cross section to be

$$d\sigma/d\Omega = \frac{1}{4}[|f_1|^2 + 3|f_3|^2] = |a|^2 + |b|^2,$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \pi[|f_1|^2 + 3|f_3|^2] = 4\pi[|a|^2 + |b|^2].$$

For reference we note the relation between total scattering and imaginary parts of these amplitudes (optical theorem):

$$\sigma = \frac{4\pi}{k} \left[ \frac{\text{Im}f_1 + 3\text{Im}f_3}{4} \right] = \frac{4\pi}{k} \text{Im}a.$$

If instead, the sodium atoms and electrons are polarized with polarization  $\mathbf{P}(\text{Na})$ ,  $\mathbf{P}(e)$ , respectively, we set

$$\rho_i = \frac{1}{4}[1 + \boldsymbol{\sigma}_1 \cdot \mathbf{P}(e)][1 + \boldsymbol{\sigma}_2 \cdot \mathbf{P}(\text{Na})],$$

and repeat the calculation. The cross section is now found to depend on the polarization:

$$d\sigma/d\Omega = \frac{1}{4}[|f_1|^2 + 3|f_3|^2] + \mathbf{P}(e) \cdot \mathbf{P}(\text{Na}) \frac{1}{4}[|f_3|^2 - |f_1|^2],$$

$$= [|a|^2 + |b|^2] - \mathbf{P}(e) \cdot \mathbf{P}(\text{Na})$$

$$\times \left[ \frac{2}{3}|b|^2 - \frac{1}{\sqrt{3}}(ab^* + a^*b) \right].$$

This result can again be expressed in terms of the imaginary parts of the scattering amplitudes through the optical theorem:

$$d\sigma/d\Omega = \frac{1}{k} \text{Im}a + \mathbf{P}(e) \cdot \mathbf{P}(\text{Na}) \frac{1}{\sqrt{3}k} \text{Im}b.$$

## B. MULTIPLE SCATTERING OF ELECTRONS

The motion of a single electron in a gas of sodium atoms can be found by evaluating the dispersion law; that is, by finding the propagation law for plane wave solutions of various wavelengths and polarizations. We consider the general solution for arbitrary boundary conditions, as a superposition of these plane wave solutions. Of course, the solutions will be plane waves only for a large, homogeneous sample of the gas.

<sup>5</sup> The labor can be reduced by use of the properties of projection operators  $P_1 = (1 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)/4$ ,  $P_3 = (3 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)/4$ ,  $P_1^2 = P_1$ ,  $P_3^2 = P_3$ ,  $P_1 P_3 = 0$ .

The dispersion law is easily calculated in the pseudo-potential approximation by evaluating the energy of a given plane wave state in perturbation theory. For the electron state we choose  $\phi = ue^{ik \cdot \mathbf{r}}/L^{3/2}$ , and denote the state of the  $N$  gas atoms simply by  $\Psi$ . Then, calling the energy  $E = \hbar^2 k_0^2/2m$ , we obtain

$$\frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2 k^2}{2m} + \langle V \rangle,$$

with

$$V = -\frac{2\pi\hbar^2}{m} \sum_{m=1}^N \delta(\mathbf{r} - \mathbf{r}_i)$$

$$\times \left[ f_1 \left( \frac{1 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2(i)}{4} \right) + f_3 \left( \frac{3 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2(i)}{4} \right) \right].$$

Evaluating the spatial part of this matrix element, we have

$$\langle V \rangle = -\frac{2\pi\hbar^2}{m} \frac{N}{L^3} \left( u \left| f_1 \left( \frac{1 - \boldsymbol{\sigma}_1 \cdot \mathbf{P}}{4} \right) + f_3 \left( \frac{3 + \boldsymbol{\sigma}_1 \cdot \mathbf{P}}{4} \right) \right| u \right),$$

where  $\mathbf{P} = (1/N) \sum_i \langle \Psi | \boldsymbol{\sigma}(i) | \Psi \rangle$  is the mean polarization of the sodium atoms. For electrons with spin parallel or antiparallel to  $\mathbf{P}$ , we find a dispersion law

$$\frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2 k^2}{2m} - \frac{2\pi\hbar^2}{m} \mathfrak{U} \left[ a \pm P \frac{b}{\sqrt{3}} \right],$$

where  $\mathfrak{U} = N/L^3$  is the number density of sodium atoms; the upper sign refers to spin parallel to  $\mathbf{P}$ , the lower sign to spin antiparallel. Defining in the usual way an index of refraction<sup>6</sup>  $\nu = k/k_0$ , we obtain

$$\nu^2 = 1 + \frac{4\pi\mathfrak{U}}{k_0^2} \left[ a \pm P \frac{b}{\sqrt{3}} \right].$$

If there is, in addition, a uniform magnetic field  $\mathbf{H}$  present, parallel to  $\mathbf{P}$ , then the term  $\pm\mu_0 H$  should be added to the right-hand side,<sup>7</sup> due to the interaction of the electron moment with the field.

The existence of such a dispersion law implies that we are in the "electron optics" limit; the further calculation of solutions of the wave equation will be in strong analogy with the familiar theory of geometrical optics.

<sup>6</sup> This result gives the complex index of refraction. If we take real and imaginary parts of both sides, the (real) index of refraction and the absorption coefficient can be obtained. These are defined as  $\nu = n + i\kappa/2k_0$ , so that the form of the plane wave solution in the medium becomes  $e^{i\mathbf{n}k_0\mathbf{r}} e^{-\kappa z/2}$  and hence the intensity of the wave decreases by  $e^{-1}$  in distance  $\kappa^{-1}$ . It is assumed throughout that  $n-1 \ll 1$ ;  $\kappa/k_0 \ll 1$ .

<sup>7</sup> This term constitutes the "magnetic scattering", and has an interesting history in the scattering of neutrons in ferromagnetic materials; see D. J. Hughes, *Neutron Optics* (Interscience Publishers, Inc., New York, 1954), Chap. 5. For our application, the difference between using  $\mathbf{B}$  and  $\mathbf{H}$  is negligible.

The dependence of the index on spin orientation means the medium is doubly refracting, the analog of an optically active medium (circular dichroism). For an unpolarized medium ( $P=0$ ) we find

$$n^2 = 1 + \frac{4\pi\mathfrak{N}}{k_0^2} \operatorname{Re}(a),$$

$$\kappa = \frac{4\pi\mathfrak{N}}{k_0} \operatorname{Im}(a) = \mathfrak{N}\sigma.$$

The plane wave solutions propagate with phase  $e^{in_k x}$  and amplitude  $e^{-\kappa x/2}$ , and hence intensity

$$e^{-\kappa x} = \exp(-\mathfrak{N}\sigma x).$$

This establishes the expected relation between total scattering cross section and attenuation of the beam.

For a polarized medium (and magnetic field) we obtain for the two indices of refraction

$$n(P) = n \left[ 1 \pm \frac{4\pi\mathfrak{N}P}{\sqrt{3}k_0^2} \operatorname{Re}(b) \pm \mu_0 H \right],$$

$$\kappa(P) = \frac{4\pi\mathfrak{N}}{k_0} \operatorname{Im}(a) \pm \frac{4\pi\mathfrak{N}}{\sqrt{3}k_0} P \operatorname{Im}(b).$$

The medium now has different phase change and absorption for electrons with spins up and down, and hence the attenuation will polarize the electrons. An elementary derivation shows that the electron polarization on passage through a plane parallel slab (perpendicular to  $P, H$ ) of thickness  $x$ , is

$$P(e) = -\tanh \left[ \frac{4\pi\mathfrak{N}P}{\sqrt{3}k_0} \operatorname{Im}(b) x \right].$$

Such considerations underly the explanation of the magnitude of the electron polarization in the bulb.

### C. ELECTRON SPIN RESONANCE

The reader has no doubt already realized that the dispersion law derived in the previous section implies a shift in the electron spin resonance frequency. The energy difference of electrons with spins up and down (and *same* linear momentum) is, taking real parts,

$$\Delta E = +2\mu_0 H - \frac{4\pi\hbar^2}{\sqrt{3}m} \mathfrak{N}P \operatorname{Re}(b),$$

with the additional term proportional to sodium density and polarization, and to the difference between triplet and singlet amplitudes. From the imaginary part we

derive the energy width

$$\Gamma = \frac{\pi\hbar^2}{m} \mathfrak{N} \left[ \operatorname{Im}(a) \pm P \operatorname{Im}\left(\frac{b}{\sqrt{3}}\right) \right].$$

Since we have neglected the other collisions with walls and buffer gas, this is not the only source of line width. It shows that there is, however, an additional contribution to the width which is also proportional to sodium density and polarization. However, since the observed width of the electron spin resonance is the sum of the widths of the two spin states, the resonance should show no broadening proportional to  $\mathfrak{N}P$ .

### D. DISCUSSION

(1) The existence of the electron polarization, and of the frequency shift are shown to be directly related to the existence of a spin-dependent dispersion law, and of the electron optics limit. The corresponding limit in neutron optics, including spin effects in polarized media, has been extensively studied. It is interesting to note, however, that the analogy is not complete; for room temperature samples, the nuclear polarization is negligibly small, and so the term  $\sim \mathfrak{N}Pb$  cannot easily be studied. On the other hand, the sample can be ferromagnetic, and the contribution of the electronic polarization to the "magnetic scattering" can be observed.

(2) The measurement of the absolute magnitude of the electron polarization and the frequency shift, seem to offer a unique opportunity to measure the incoherent amplitude  $b$  directly. Since the scattering phases of alkali atoms at thermal energies are amenable to direct calculation, and are of general interest for many plasma properties, this appears to be a fruitful line of research.

(3) This derivation provides ample reason for expecting a nonlinear relationship between electron polarization and sodium polarization. The linearity of the relationship between frequency shift and sodium polarization seems to be quite general, however.

(4) The extension can be made to electron scattering from systems with spin  $S$ ; this enables one to consider hyperfine structure effects in alkali atoms, and to discuss other atoms such as  $2^3S$  helium, etc. Without proof, we will state the result that for polarized systems with spin  $S$ , one need only substitute into all the above results

$$a = \frac{(S+1)f_+ + Sf_-}{2S+1}, \quad b = \frac{[S(S+1)]^{1/2}}{2S+1} (f_+ - f_-),$$

where  $f_{\pm}$  are the scattering amplitudes in the states with total angular momentum  $S \pm \frac{1}{2}$ .

(5) A gas containing atoms of spin  $S \geq 1$  can also be aligned, rather than polarized. However, in the  $S$ -wave limit, it can be shown that the resulting dispersion law is independent of spin. Hence, the medium would not demonstrate these effects.