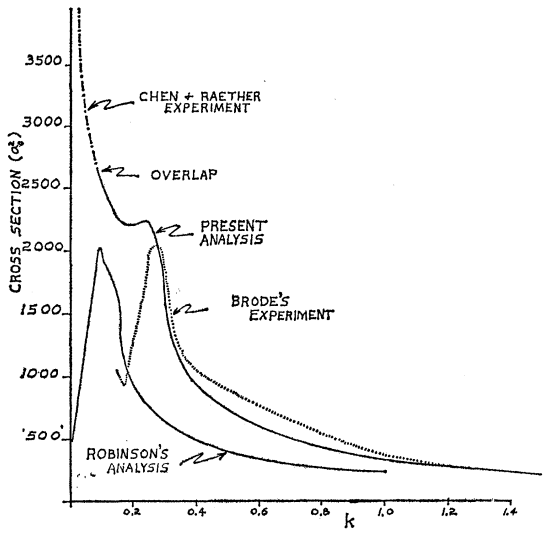

FIG. 3. Comparison of present analysis with that of Robinson and with the experimental results of Brode and of Chen and Raether. (Atomic units, $k=\sqrt{\epsilon}$.)

(<0.1% of the incident energy). The total cross sections were then obtained using the expression

$$\sigma=(4\pi/k^2)\sum_l(2l+1)\sin^2\delta_l. \quad (4)$$

IV. RESULTS

In Fig. 1 the Hartree potential function is plotted and compared with the total potential including polarization. From this figure we see that the polarization contribution is very much larger than the Hartree potential in the region $r>3$, and hence the statement that small errors in $V_H(r)$ should be negligible as compared with V_p is justified. The change in the potential function due to a small change in the parameter fr_0 is also shown. It was found that in the low energy region the cross section varied over a wide range of values with small changes in the cutoff parameter. This indicates a strong dependence on the polarization contribution to the potential for values of r comparable to the atomic radius. In Fig. 2 the cross section for various values of α and fr_0 are plotted. We note that in the energy region $\epsilon>0.4$ ($E>11$ eV) little change is produced in the total cross section by small variations in the cutoff parameter. In Fig. 3 the theoretical cross section which best fits the experimental values given by Brode[9] as well as more recent values given by Chen and Raether[10] is given. We see that a good fit to Brode's values is achieved in all but a very small region in the low energy range, and an almost perfect fit to the values of Chen and Raether in the thermal region is obtained ($E\sim0.06$–0.075 eV).

V. CONCLUSIONS

From the results obtained it appears that the model discussed describes the collision process. Clearly, if the method used for selecting the value of fr_0 is valid for different atoms, then a simple model may be used to describe low energy electron scattering as only the experimental polarizability is required.

[9] R. B. Brode, Phys. Rev. **34**, 673 (1929).
[10] C. L. Chen and M. Raether, Phys. Rev. **128**, 2679 (1962).

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Low-Energy Spectrum of Electrons Scattered on Bound Electrons*

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The spectrum of electrons at low energies and large scattering angles resulting from the scattering of high-energy electrons on bound atomic electrons is calculated to lowest order in αZ and highest order in the incident energy. Relativistic Coulomb field effects are included by the use of wave functions correct to two orders in αZ. Inclusion of these relativistic Coulomb effects leads to a cross section significantly different from that obtained previously by the use of the plane wave approximation. The results show that the low-energy spectrum of electrons scattered on bound atomic electrons completely dominates the peak predicted by Parzen and co-workers in the low-energy spectrum of bremsstrahlung-producing electrons.

I. INTRODUCTION

IT has been pointed out by Parzen and co-workers[1] that the measurement of the energy spectrum of the electrons which have lost energy in bremsstrahlung production would complement the direct measurement of the bremsstrahlung energy spectrum and thus provide a check on the Bethe-Heitler formula. Of particular interest is the form of the bremsstrahlung spectrum near the high-energy limit. This portion of the bremsstrahlung spectrum corresponds to the low-energy spectrum of the scattered electrons. Using the Bethe-Heitler formula, Parzen *et al.* have shown that there is a peak in the spectrum of electrons for very low energies and large scattering angles.

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[1] D. G. Keiffer and G. Parzen, Phys. Rev. **101**, 1244 (1956); P. T. McCormick, D. G. Keiffer, and G. Parzen, *ibid.* **103**, 29 (1956).

As pointed out by Ford and Mullin,[2] there are two other processes which may mask the Parzen peak. First, there is the process of a single Møller scattering followed by a nuclear Coulomb scattering into large angles. Since this is a multiple scattering, it depends on the thickness of the scattering foil. Second, there is the single scattering of an electron on a bound atomic electron. Since there is the nuclear recoil to conserve momentum, this process can give rise to low-energy electrons at large scattering angles. Ford and Mullin evaluated this cross section by using a Dirac plane wave description of the incident and outgoing electrons and a "nonrelativistic" description of the bound electron. However, for large angle scattering it is important that large momentum be transferred to the nucleus. Since it is the high-momentum components of the bound state which produce this momentum transfer, a more accurate description of this state is necessary. Further, since the collisions which produce large-angle scattering must occur very near to the nucleus, the low-energy outgoing electron is strongly influenced by the retarding field of the nucleus. Consequently, it is to be expected that relativistic Coulomb effects must be included in the description of the low-energy outgoing electron and the bound target electron.

We have re-evaluated the spectrum of low-energy electrons which have been scattered through large angles on bound atomic electrons. The motion of the electrons has been described by using the first Born approximation for the scattering wave functions. In this way, the first-order relativistic Coulomb effects on the motion of the incident and scattered electrons have been included. First-order relativistic Coulomb corrections have been included in the description of the bound target electron. For high energies, we find that the incident electron and the high-energy outgoing electron are accurately described by plane waves. However, relativistic Coulomb corrections to the wave functions of the slow outgoing electron and the bound target electron make contributions to the scattering matrix element which are of the same order as the contributions from the plane wave term of the wave function of the slow outgoing electron and the "nonrelativistic" term of the wave function of the bound electron. We have evaluated the cross section to lowest order in αZ for high-energy incident electrons and have found that the inclusion of the Coulomb effects gives a cross section that is of order αZ lower from that obtained by Ford and Mullin. However, the low-energy spectrum of the electrons scattered through large angles on bound atomic electrons remains dominant over the corresponding spectrum of bremsstrahlung producing electrons. Thus, the peak predicted by Parzen *et al.* in the low-energy spectrum of bremsstrahlung electrons will not be detectable even if the scattering foil thickness is sufficiently reduced so that multiple scattering processes are neglectible.

[2] G. W. Ford and C. J. Mullin, Phys. Rev. **110**, 520 (1958).

II. SCATTERING OF ELECTRONS ON BOUND ELECTRONS

The matrix element for electron-electron scattering is[3]

$$M_{fi}=\alpha\int d\mathbf{r}d\mathbf{r}'\,|\mathbf{r}-\mathbf{r}'|^{-1}\exp[i(W_1-W_{1'})|\mathbf{r}-\mathbf{r}'|]$$
$$\times\bar{\psi}_{1'}(\mathbf{r})\gamma_\mu\psi_1(\mathbf{r})\bar{\psi}_{2'}(\mathbf{r}')\gamma_\mu\psi_2(\mathbf{r}')-(1'\rightleftarrows 2'). \quad (1)$$

Here the initial and final states of the two electrons are denoted, respectively, by the unprimed and primed subscripts 1 and 2. The initial state ψ_2 describes the bound electron. The bracket $(1'\rightleftarrows 2')$ represents the exchange term obtained by interchange of $1'$ and $2'$. The following discussion will be restricted to the case of large-angle scattering. In particular, we take electron $2'$ to emerge from the scattering center at an angle relative to the incident direction exceeding $\pi/2$. The differential cross section for scattering into state $2'$ is

$$d\sigma_B=(2\pi)^{-5}(W_1/p_1)d\Omega_{2'}dW_{2'}W_{2'}p_{2'}\int d\mathbf{p}_{1'}\tfrac{1}{4}\sum|M_{fi}|^2$$
$$\times\delta(W_1+W_2-W_{1'}-W_{2'}), \quad (2)$$

where $\frac{1}{4}\sum$ represents the average over initial and sum over final spin states.

In order to obtain the matrix element correct to the lowest nonvanishing order in αZ, we use the first Born approximation for the continuum states 1, $1'$, and $2'$.

$$\psi(\mathbf{r})=\left\{e^{i\mathbf{p}\cdot\mathbf{r}}-\frac{1}{4\pi}[H_0(\mathbf{r})+W]\right.$$
$$\left.\times\int\frac{e^{\pm ip|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(r')e^{i\mathbf{p}\cdot\mathbf{r}'}d\mathbf{r}'\right\}U(\mathbf{p}), \quad (3)$$

where the sign of the exponent in the integrand is determined by the boundary conditions. For a Coulomb potential, $V(r)=-(\alpha Z/r)e^{-\mu r}$, where μ is a positive parameter which is allowed to vanish at the end of the calculation.

For the bound state, the wave function of a hydrogen-like atom is used. The greatest contribution to the cross section will come from the K-shell electrons since large momentum can then be transferred to the nucleus. The hydrogen-like ground state is exactly

$$\psi_2(\mathbf{r})=Ne^{-\lambda r}r^{\gamma-1}\left[1+\frac{i\lambda}{m(1+\gamma)}\boldsymbol{\alpha}\cdot\hat{r}\right]U(\mathbf{p}_2) \quad (4)$$

with

$$\lambda=\alpha Zm,$$
$$W_2=m\gamma=m(1-\alpha^2Z^2)^{1/2},$$

[3] See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 146.

and

$$|N|^2=\frac{(1+\gamma)(2\lambda)^{2\gamma+1}}{8\pi\Gamma(2\gamma+1)}.$$

This wave function has been expressed in terms of the plane-wave spinor $U(\mathbf{p}_2)$ to facilitate the evaluation of the spin summations. In all results the momentum $\mathbf{p}_2$ is set equal to zero. Neglecting second-order terms, γ can be approximated by unity and $\psi_2(\mathbf{r})$ can be written:

$$\psi_2(\mathbf{r})=Ne^{-\lambda r}\left[1+\frac{i\lambda}{2m}\boldsymbol{\alpha}\cdot\hat{r}\right]U(\mathbf{p}_2). \tag{5}$$

Note that the bound state energy, W_2, can be described accurately to two orders in αZ by the rest mass energy, m.

An expansion of $\psi_2(\mathbf{r})$ in powers of αZ can be made in the form

$$\psi_2(\mathbf{r})=\lim_{\epsilon\to 0}Ne^{-\epsilon r}(1-\alpha Zmr+\tfrac{1}{2}i\alpha Z\boldsymbol{\alpha}\cdot\hat{r})U(\mathbf{p}_2),$$

where the factor $e^{-\epsilon r}$ is included to insure the convergence of the resulting integrals. Indeed, McVoy and Fano[4] have shown that the first two terms in the expression of the matrix element in powers of αZ are properly obtained by expanding the wave function in this form. However, although this form does lead to an expression for the matrix element which is formally correct to order αZ, it is not applicable for our purposes because it gives rise to a factor of the form

$$\{[(\mathbf{p}_1-\mathbf{p}_{1'})^2-p_{2'}{}^2]^2+4p_{2'}{}^2\epsilon^2\}^{-1}$$

in the square of the matrix element. Since $(\mathbf{p}_1-\mathbf{p}_1')^2-p_{2'}{}^2$ has a zero for $2\mathbf{p}_1\cdot\mathbf{p}_1'=p_1{}^2+p_{1'}{}^2-p_{2'}{}^2$, this factor leads to a singularity in the subsequent angular integration in the limit that ϵ vanishes. If the factor $e^{-\lambda r}$ is not expanded in the bound-state wave function, ϵ is replaced by λ and the divergence in the angular integration is replaced by a finite term of order $1/\lambda$. Thus, we obtain an unexpected lowering in the order of the differential cross section.

The zeroth-order matrix element results from (1) with the wave functions as given in (3) and (5) evaluated in the limit that αZ and λ approach zero.[5] In this limit

$$M_{fi}=2(2\pi)^4\alpha N(P_1-P_{1'})^{-2}\times\delta(\mathbf{p}_1-\mathbf{p}_{1'}-\mathbf{p}_{2'})-(1'\rightleftarrows 2'), \tag{6}$$

where

$$(P_1-P_{1'})^2=(\mathbf{p}_1-\mathbf{p}_{1'})^2-(W_1-W_{1'})^2.$$

This term vanishes identically for the case of large-angle scattering since the argument of the δ function is nonzero. The matrix element is thus of first order in αZ.

The matrix element to first order is rather complicated. Since, however, we are interested in the case for which

$$W_1,\ W_{1'}\gg W_{2'},\ m$$

the matrix element can be considerably simplified by retaining in it only those terms which contribute the highest power of W_1/m in the cross section. With the aid of the energy relation

$$W_1+m=W_{1'}+W_{2'}, \tag{7}$$

we find after angular integration over $d\Omega_{1'}$ that terms in the squared matrix element which involve the denominator $(P_1-P_{1'})^4$ exceed all remaining terms by two powers of W_1/m. Since this denominator occurs only in the square of the direct term of the matrix element in (1) and arises solely from the plane-wave terms of the wave junctions describing the high-energy states 1 and 1′, it follows that the cross section is determined to the accuracy desired by the matrix element

$$M_{fi}=4\pi\alpha(P_1-P_{1'})^{-2}\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\times\int d\mathbf{r}\,\bar{\psi}_{2'}(\mathbf{r})\gamma_\mu\psi_2(\mathbf{r})e^{i(\mathbf{p}_1-\mathbf{p}_{1'})\cdot\mathbf{r}}. \tag{8}$$

The remaining integration of the matrix element is conveniently carried out in momentum space. Introducing the Fourier transforms

$$\psi_{2'}(\mathbf{r})=\int\chi_{2'}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{p},\qquad \psi_2(r)=\int\chi_2(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{q}, \tag{9}$$

the matrix element becomes

$$M_{fi}=2(2\pi)^4\alpha(P_1-P_{1'})^{-2}\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\times\int d\mathbf{p}\,\bar{\chi}_{2'}(\mathbf{p})\gamma_\mu\chi_2(\mathbf{p}+\mathbf{p}_{1'}-\mathbf{p}_1), \tag{10}$$

with

$$\chi_{2'}(\mathbf{p})=\delta(\mathbf{p}_{2'}-\mathbf{p})U(\mathbf{p}_{2'})+\frac{\alpha Z}{2\pi^2}\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m+W_{2'}}{[(\mathbf{p}_{2'}-\mathbf{p})^2+\mu^2](p^2-p_{2'}{}^2+i\delta)}U(\mathbf{p}_{2'}),$$

$$\chi_2(\mathbf{p})=\frac{N}{\pi^2}\frac{\lambda}{(p^2+\lambda^2)^2}U(\mathbf{p}_2)+\frac{N}{2\pi^2}\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{m}\frac{\lambda}{(p^2+\lambda^2)^2}U(\mathbf{p}_2).$$

The matrix element can now be written to first order in αZ in the form

$$M_{fi}=M_1+M_2+M_3, \tag{11}$$

where

$$M_1=\frac{32\pi^2\alpha N}{(P_1-P_{1'})^2}\frac{\lambda}{(K^2+\lambda^2)^2}\times\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\bar{U}(\mathbf{p}_{2'})\gamma_\mu U(\mathbf{p}_2), \tag{12a}$$

[4] Kirk W. McVoy and U. Fano, Phys. Rev. **116**, 1168 (1959).

[5] The over-all factors α and N which multiply the matrix element M_{fi} are disregarded in classifying terms in M_{fi} according to "orders of αZ."

$$M_2=-\frac{16\pi^2\alpha N}{m(P_1-P_{1'})^2}\frac{\lambda}{(K^2+\lambda^2)^2}\times\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\bar{U}(\mathbf{p}_{2'})\gamma_\mu\boldsymbol{\alpha}\cdot\mathbf{K}U(\mathbf{p}_2), \quad (12b)$$

and

$$M_3=\frac{16\alpha(\alpha Z)N}{(P_1-P_{1'})^2}\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\times\int d\mathbf{p}\frac{\lambda}{[(\mathbf{p}-\mathbf{p}_1+\mathbf{p}_{1'})^2+\lambda^2]^2}\times\frac{\bar{U}(\mathbf{p}_{2'})(-\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m+W_{2'})\gamma_\mu U(\mathbf{p}_2)}{[(\mathbf{p}_{2'}-\mathbf{p})^2+\mu^2](p^2-p_{2'}{}^2-i\delta)}, \quad (12c)$$

with

$$\mathbf{K}=\mathbf{p}_1-\mathbf{p}_{1'}-\mathbf{p}_{2'}.$$

M_3 is of the same order in αZ as M_1 and M_2, although it appears to be of higher order. This can be seen by noting that, in evaluation of M_3 to first order in αZ, the expansion to lowest order in λ of the factor $\lambda/[(\mathbf{p}-\mathbf{p}_1+\mathbf{p}_{1'})^2+\lambda^2]^2$ gives rise to a delta function via the relation

$$\lim_{\lambda\to 0}\lambda/(\mathbf{p}^2+\lambda^2)=\pi^2\delta(\mathbf{p}). \quad (13)$$

It follows that the integral in (12c) to zeroth order in λ is nonvanishing, and M_3 is of order αZ.

The use of relation (13) in evaluation of M_3 requires, however, an implicit expansion of the integrand in λ. Since an expansion in λ before integration is to be avoided, and since, moreover, a finite value of λ is required to insure convergence of the subsequent angular integration of $|M_3|^2$, we do not make use of (13) in the explicit evaluation of M_3. The exact integrals involved in (12c) are

$$(A_0,\mathbf{A})=\int d\mathbf{p}\frac{(1,\mathbf{p})}{[(\mathbf{p}-\mathbf{p}_{2'})^2+\mu^2][(\mathbf{p}-\mathbf{p}_1+\mathbf{p}_{1'})^2+\lambda^2]^2(p^2-p_{2'}{}^2-i\delta)}. \quad (14a)$$

To lowest order in αZ, these integrals are given by[6]

$$(A_0,\mathbf{A})=\frac{\pi^2}{\lambda K^2}\frac{(1,\mathbf{p}_1-\mathbf{p}_{1'})}{(\mathbf{p}_1-\mathbf{p}_{1'})^2-p_{2'}{}^2-2ip_{2'}\lambda}. \quad (14b)$$

With this result we find

$$M_3=\frac{8\pi^2\alpha N}{mp_{2'}(P_1-P_{2'})^2K^2}\frac{\lambda}{s-i\lambda}\bar{U}(\mathbf{p}_{1'})\gamma_\mu U(\mathbf{p}_1)\times\bar{U}(\mathbf{p}_{2'})[-\boldsymbol{\alpha}\cdot(\mathbf{p}_1-\mathbf{p}_{1'})+\beta m+W_{2'}]\gamma_\mu U(\mathbf{p}_2), \quad (15)$$

where

$$s=\frac{(\mathbf{p}_1-\mathbf{p}_{1'})^2-p_{2'}{}^2}{2p_{2'}}.$$

Carrying out the integration over $W_{1'}$, the cross section given by Eq. (2) can be rewritten as

$$d\sigma_B=(2\pi)^{-5}(W_1/p_1)p_{1'}p_{2'}W_{1'}W_{2'}dW_{2'}d\Omega_{2'}\times\int d\Omega_{1'}\tfrac{1}{4}\sum|M_{fi}|^2, \quad (16)$$

where to lowest order in αZ,

$$|M_{fi}|^2=|M_1|^2+|M_2|^2+|M_3|^2+2\,\mathrm{Re}(M_1M_2{}^\dagger+M_1M_3{}^\dagger+M_2M_3{}^\dagger). \quad (17)$$

All terms on the right-hand side of Eq. (17) apparently contribute to the same order of αZ in the cross section as expressed in Eq. (16). However, it is found that the contribution of the term $|M_3|^2$ is actually one order of αZ lower than that of the remaining terms in Eq. (17). This follows from the fact that as λ approaches zero, the factor

$$(1/\pi)[\lambda/(s^2+\lambda^2)] \quad (18)$$

which occurs in $|M_3|^2$ becomes the δ function, $\delta(s)$. Since the integration over $d\Omega_1$ in (16) is expressible as an integration over the argument s, the range of which includes the point $s=0$, the delta-function character of the factor (18) reduces the $|M_3|^2$ contribution to order αZ. The factors $\lambda^2/(\mathbf{K}^2+\lambda^2)^4$ which occur in (17) from the contributions of M_1 and M_2 do not cause a similar reduction in the order of αZ since $\mathbf{K}$ is never zero. Thus, only M_3 contributes to the cross section to the lowest order. Indeed, this fact is verified by a more detailed investigation of the terms in (17). As a consequence, we replace $|M_{fi}|^2$ in the cross section by $|M_3|^2$.

After evaluation of the sums over the spins in $|M_3|^2$, the cross section in this approximation becomes

$$d\sigma_B=\frac{\alpha^2(\alpha Z)|N|^2p_{1'}dW_{2'}d\Omega_{2'}}{\pi m^2p_1p_{2'}}\int d\Omega_{1'}\frac{1}{K^4(P_1-P_{1'})^4}\frac{\lambda}{s^2+\lambda^2}\times\{(\mathbf{p}_1-\mathbf{p}_{1'})^2[mW_{1'}P_1\cdot P_{2'}+mW_1P_{1'}\cdot P_{2'}+4mW_{1'}W_1W_{2'}-m^3W_{2'}-m^2P_1\cdot P_{2'}-2m^4]+\mathbf{p}_{2'}\cdot(\mathbf{p}_1-\mathbf{p}_{1'})[-2mW_1\mathbf{p}_{1'}\cdot(\mathbf{p}_1-\mathbf{p}_{1'})-2mW_{1'}\mathbf{p}_1\cdot(\mathbf{p}_1-\mathbf{p}_{1'})+4mW_{1'}W_{2'}W_1-2m^3W_{2'}+2m^2P_1\cdot P_{1'}+4m^4]-2(W_{2'}{}^2+m^2)[mW_1\mathbf{p}_{1'}\cdot(\mathbf{p}_1-\mathbf{p}_{1'})+W_{1'}m\mathbf{p}_1\cdot(\mathbf{p}_1-\mathbf{p}_{1'})]-p_{2'}{}^2[mW_1P_{1'}\cdot P_{2'}+mW_{1'}P_1\cdot P_{2'}]+P_1\cdot P_{2'}(3m^2W_{2'}{}^2+m^4)+m^2[8mW_1W_{1'}W_{2'}-3m^3W_{2'}+6m^2W_{2'}{}^2-mW_{2'}{}^3+2m^4]\}. \quad (19)$$

[6] See, for example, Mihai Gavrila, Phys. Rev. **113**, 514 (1959).

The angular integration in (19) can be simplified by elimination of angular-dependent terms from the numerator. This can be accomplished by expressing the latter in terms of factors occuring in the denominator; the resulting cancellations then leave the numerator independent of the $\mathbf{p}_{1'}$ direction. Using the relations

$$\begin{aligned}(\mathbf{p}_1-\mathbf{p}_{1'})^2&=(P_1-P_{1'})^2+x^2,\\ 2(\mathbf{p}_1-\mathbf{p}_{1'})\cdot\mathbf{p}_{2'}&=(P_1-P_{1'})^2-K^2+2xW_{2'},\\ 2(\mathbf{p}_1-\mathbf{p}_{1'})\cdot\mathbf{p}_{1'}&=2x(W_1-x)-(P_1-P_{1'})^2,\\ 2(\mathbf{p}_1-\mathbf{p}_{1'})\cdot\mathbf{p}_1&=2xW_1+(P_1-P_{1'})^2,\\ 2P_{1'}\cdot P_{2'}&=K^2-(P_1-P_{1'})^2+2P_1\cdot P_{2'},\\ 2P_1\cdot P_{1'}&=-(P_1-P_{1'})^2-2m^2,\end{aligned}$$

where $x=W_{2'}-m$, we obtain

$$\begin{aligned}d\sigma_B=&\frac{\alpha^2(\alpha Z)|N|^2p_{1'}dW_{2'}d\Omega_{2'}}{2\pi p_{2'}p_1}\{(x-W_1)I_{04}\\ &+(W_1+m-x)I_{22}+2(-2mW_1^2+mxW_1+m^2x)I_{42}\\ &+2[6mW_1^2+4xW_1^2+P_1\cdot P_{2'}(2W_1-x)-4x^2W_1\\ &-5mxW_1+2x^3+mx^2-4xm^2-4m^3]I_{24}\\ &+2[-2mxP_1\cdot P_{2'}(2W_1-x)+4m^2W_1^2(x+2m)\\ &-2m^2xW_1(3x+4m)-4m^3x^2-4m^2x^3]I_{44}\},\end{aligned}\tag{20}$$

with

$$I_{nm}=\int d\Omega_{1'}\,(s^2+\lambda^2)^{-1}(P_1-P_{1'})^{-n}K^{-m}.\tag{21}$$

It remains to evaluate the integrals I_{nm} to lowest order in αZ (or λ). Using $d\Omega_{1'}=p_{2'}(p_1p_{1'})^{-1}d\varphi_{1'}ds$ we can write

$$I_{nm}=p_{2'}(p_1p_{1'})^{-1}\int_0^{2\pi}d\varphi_{1'}\,\mathcal{S}_{nm},\tag{22}$$

where

$$\mathcal{S}_{nm}=\int_{-s_1}^{s_2}ds\,(s^2+\lambda^2)^{-1}(P_1-P_{1'})^{-n}K^{-m}$$

and

$$s_1=\frac{p_{2'}^2-(p_1-p_{1'})^2}{2p_{2'}}>0,$$

$$s_2=\frac{(p_1+p_{1'})^2-p_{2'}^2}{2p_{2'}}>0.$$

We wish to evaluate $\mathcal{S}_{nm}$ to lowest order in λ. However, λ^2 cannot be neglected compared to s^2, since s has a zero in the range of integration. In order to extract the lowest order term we express $\mathcal{S}_{nm}$ in the form

$$\mathcal{S}_{nm}=\mathcal{S}_{nm}'+\int_{-s_1}^{s_2}ds\,(s^2+\lambda^2)^{-1}[(P_1-P_{1'})^{-n}K^{-m}-(P_1-P_{1'})_0^{-n}K_0^{-m}],\tag{23}$$

where subscript 0 denotes the value of the quantity for $s=0$ and

$$\begin{aligned}\mathcal{S}_{nm}'&=(P_1-P_{1'})_0^{-n}K_0^{-m}\int_{-s_1}^{s_2}ds\,[s^2+\lambda^2]^{-1},\\ &=\lambda^{-1}(P_1-P_{1'})_0^{-n}K_0^{-m}\\ &\quad\times\left\{\pi+\tfrac{1}{2}i\ln\frac{(s_1+i\lambda)(s_2+i\lambda)}{(s_1-i\lambda)(s_2-i\lambda)}\right\}.\end{aligned}$$

The second integral in Eq. (23) is now finite as $\lambda\to0$ and the term of lowest order in λ is contained in $\mathcal{S}_{nm}'$. Therefore,

$$I_{nm}=(\lambda p_1p_{1'})^{-1}\pi p_{2'}(P_1-P_{1'})_0^{-n}\int_0^{2\pi}\frac{d\varphi_{1'}}{K_0^m}.$$

The remaining $\varphi_{1'}$ integration is easily carried out. Retaining only the highest power of W_1 we obtain

$$I_{n2}=\frac{1}{\lambda W_1^2}\frac{\pi^2}{[2m(W_{2'}-m)]^{n/2}(W_{2'}-m-p_{2'}\cos\theta_{2'})},\tag{24a}$$

$$I_{n4}=\frac{1}{\lambda W_1^2}\frac{\pi^2(W_{2'}-m)(W_{2'}+m-p_{2'}\cos\theta_{2'})}{2[2m(W_{2'}-m)]^{n/2}(W_{2'}-m-p_{2'}\cos\theta_{2'})^3p_{2'}^2}.\tag{24b}$$

Using these results in Eq. (20), the differential cross section to lowest order in αZ and highest order in W_1 is

$$\begin{aligned}d\sigma_B=&\frac{(\alpha Z)^4r_0^2m^3d\Omega_{2'}dW_{2'}}{2p_{2'}(W_{2'}-m)^2(W_{2'}-m-p_{2'}\cos\theta_{1'})}\\ &\times\left[\frac{2W_{2'}^2(W_{2'}+m-p_{2'}\cos\theta_{2'})}{(W_{2'}+m)(W_{2'}-m-p_{2'}\cos\theta_2')^2}-1\right],\end{aligned}\tag{25}$$

where $r_0=\alpha/m$, the classical radius of the electron. This cross section is independent of the incident energy W_1.

By retaining the terms M_1 and M_2 in the matrix element M_{fi}, it is possible to exhibit the energy dependence of the next lowest order term in the αZ expansion of the cross section $d\sigma_B$. Although terms proportional to the first and the second powers of the incident energy occur in this order in the squared matrix element, an exact cancellation of these terms takes place after angular integration over $d\Omega_{1'}$, and the cross section is found to have at most a logarithmic dependence on the energy W_1. A contribution to this next lowest order αZ term in the cross section can also arise, however, from the next αZ correction to the wave function of Eq. (3) as a result of a lowering in the order in αZ of this correction term after the angular integration over $d\Omega_{1'}$. This contribution can have again at most a logarithmic dependence on the energy.

III. DISCUSSION

A comparison of the value of $d\sigma_B$ given by Eq. (25) with the value obtained for this cross section in reference

2, shows the importance here of relativistic Coulomb field effects. The neglect of these effects in reference 2 is equivalent to the approximation of the matrix element M_{fi} by the term M_1 of Eq. (12a). This term gives a cross section proportional to $(\alpha Z)^5$ and W_1^2. The first-order relativistic correction to the wave function of the bound electron and the first-order relativistic correction to the wave function of the slow outgoing electron are associated, respectively, with the terms M_2 and M_3 in the matrix element. These terms make contributions to M_{fi} which are of the same order in αZ as M_1, and result in a cancellation of the terms involving W_1^2 and W_1 in the cross section. The term M_3 gives, in addition, the contribution of order $(\alpha Z)^4$ in the cross section which is calculated in Sec. II. Although it is at first surprising that this dominant contribution to the back scattering cross section arises from the Coulomb correction term in the wave function of the back scattered electron, this result is not unreasonable in view of the fact that it is the Coulomb coupling between the electron and the nucleus which allows momentum to be conserved for the large-angle scattering.

The energy spectrum of electrons which have lost energy in bremsstrahlung production has been given by McCormick, Keiffer, and Parzen.[1] Since the general result is rather complicated, only the result in the limit of high incident energy and low final electron energy will be given here.[7]

$$d\sigma_P=\frac{\alpha Z^2 r_0^2 m^2}{2\pi W_1(W_{2'}-p_{2'}\cos\theta_{2'})^2}\left(-\cos\theta_{2'}+\frac{p_{2'}+W_{2'}\cos\theta_{2'}}{p_{2'}}\ln\frac{W_{2'}+p_{2'}}{m}\right)dW_{2'}d\Omega_{2'}. \quad (26)$$

For purposes of comparison, we introduce the dimensionless quantity Σ related to the differential cross section $d\sigma$ by the definition

$$\Sigma=\frac{4\pi m}{r_0^2 Z^2}\frac{d\sigma}{dW_{2'}d\Omega_{2'}}. \quad (27)$$

Results of numerical evaluation of Σ_B and Σ_P for incident energies 4.123 and 60 m, for the scattering angle 120° and for $Z=13$ and 47 are presented graphically in Fig. 1. It is clear from the figure that the peak in the energy spectrum of bremsstrahlung-producing electrons is completely masked by the energy spectrum of electrons scattered on bound atomic electrons.

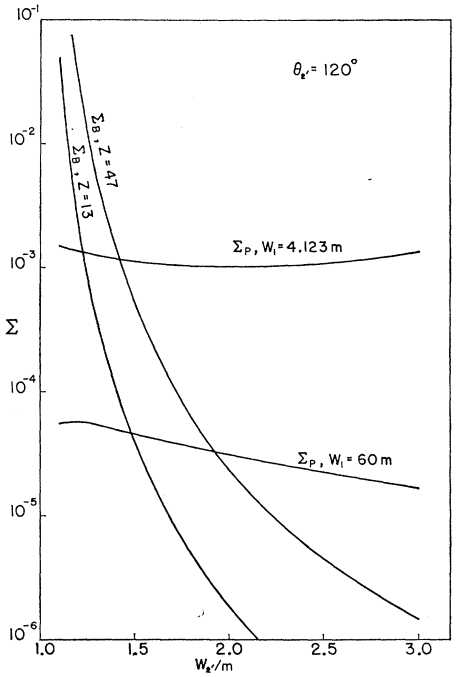

FIG. 1. Σ_P and Σ_B as functions of $W_{2'}$ for the scattering angle $\theta_{2'}=120°$. Σ_B is given for one K-shell electron.

The process of a Møller scattering followed by a nuclear Coulomb scattering which competes with the processes considered here is dependent on the target foil thickness, whereas $d\sigma_B$ and $d\sigma_P$ are not. The values of Σ for Møller scattering followed by a nuclear Coulomb scattering calculated in reference 2 for a foil thickness of 5×10^{18} atoms/cm^2 show that both Σ_B and Σ_P will be dominated by this process in the region of the Parzen peak. It is clear that the foil thickness must be drastically reduced in order that Σ_B and Σ_P be significant in this region.

The low-energy spectrum of positrons obtained in pair production by high-energy γ rays is closely related to the low-energy spectrum of bremsstrahlung-producing electrons. By using coincidence techniques the background due to various scattering processes can be greatly reduced and the low-energy region of the positron spectrum should be observable.[8]

[7] Here the notation is made to conform with that used in the calculation presented in this paper. Thus, W_1 refers to the energy of the electron in the initial state and $W_{2'}$ to the energy of the electron in the final state.

[8] C. L. Hammer (private communication).