possesses a continuous symmetry group under which the ground or vacuum state is not invariant, that state is, therefore, degenerate with other ground states. This implies a zero-mass boson. Thus, the solid crystal violates translational and rotational invariance, and possesses phonons; liquid helium violates (in a certain sense only, of course) gauge invariance, and possesses a longitudinal phonon; ferro-magnetism violates spin rotation symmetry, and possesses spin waves; superconductivity violates gauge invariance, and would have a zero-mass collective mode in the absence of long-range Coulomb forces.

It is noteworthy that in most of these cases, upon closer examination, the Goldstone bosons do indeed become tangled up with Yang-Mills gauge bosons and, thus, do not in any true sense really have zero mass. Superconductivity is a familiar example, but a similar phenomenon happens with phonons; when the phonon frequency is as low as the gravitational plasma frequency, $(4\pi G\rho)^{1/2}$ (wavelength$\sim 10^4$ km in normal matter) there is a phonon-graviton interaction: in that case, because of the peculiar sign of the gravitational interaction, leading to instability rather than finite mass.[12] Utiyama[13] and Feynman have pointed out that gravity is also a Yang-Mills field. It is an amusing observation that the three phonons plus two gravitons are just enough components to make up the appropriate tensor particle which would be required for a finite-mass graviton.

Spin waves also are known to interact strongly with magnetostatic forces at very long wavelengths,[14] for rather more obscure and less satisfactory reasons. We conclude, then, that the Goldstone zero-mass difficulty is not a serious one, because we can probably cancel it off against an equal Yang-Mills zero-mass problem. What is not clear yet, on the other hand, is whether it is possible to describe a truly strong conservation law such as that of baryons with a gauge group and a Yang-Mills field having finite mass.

I should like to thank Dr. John R. Klauder for valuable conversations and, particularly, for correcting some serious misapprehensions on my part, and Dr. John G. Taylor for calling my attention to Schwinger's work.

[12] J. H. Jeans, Phil. Trans. Roy. Soc. London **101**, 157 (1903).
[13] R. Utiyama, Phys. Rev. **101**, 1597 (1956); R. P. Feynman (unpublished).
[14] L. R. Walker, Phys. Rev. **105**, 390 (1957).

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Construction of Invariant Scattering Amplitudes for Arbitrary Spins and Analytic Continuation in Total Angular Momentum*

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From group-theoretical considerations, invariant scattering amplitudes for two-body reactions of particles with arbitrary spins and nonzero masses are constructed in various forms, including helicity amplitudes and amplitudes free of kinematical singularities. They are linear combinations of spin basis functions with scalar coefficients. In the process of construction the Pauli spin matrices are generalized for arbitrary spin. On the basis of a Mandelstam representation for the scalar coefficients, the unique analytic continuation of the amplitudes in total angular momentum is obtained. Possible kinematical singularities of the scalar amplitudes at the boundary of the physical region are discussed.

I. INTRODUCTION

THE basic quantities of S-matrix theory are the Lorentz-invariant scattering matrix elements (S functions), which depend on the spins and types of incoming and outgoing particles and on the mass shell values of their four-momenta. From the S functions, invariant scattering amplitudes (M functions) that have simpler transformation properties and that are expected to be free of kinematical singularities can be defined.[1] A general procedure has been given to construct the invariant amplitudes in terms of the irreducible unitary representations of the inhomogeneous proper Lorentz group, based on a two-component spinor formalism.[2]

Although the invariant scalar amplitudes for which the Mandelstam representation is expected to be valid have been known for some time in the simpler cases such as those of the pion-nucleon[3] and nucleon-nucleon[4]

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[1] H. P. Stapp, Phys. Rev. **125**, 2139 (1962); *Lectures on S-Matrix Theory* [W. A. Benjamin, Inc., New York (to be published)].
[2] A. O. Barut, Phys. Rev. **127**, 321 (1962).
[3] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).
[4] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960), (referred to hereafter as GGMW); D. Amati, E. Leader, and B. Vitale, Nuovo Cimento **17**, 68 (1960).

scattering systems, there is to our knowledge no systematic construction of such amplitudes for arbitrary spins.[5] The purpose of this paper is, first, to construct the invariant M functions of arbitrary spin for two-body reactions (two particles in, two particles out), and also to construct the S functions in various representations (for example, the helicity representation) in terms of scalar amplitudes and explicitly given basis functions.[6] Second, it is our purpose to define, on the basis of a Mandelstam representation for the two-body scalar amplitudes, an analytic continuation in total angular momentum that generalizes the recent work on simpler cases.[7] In pion-nucleon scattering, as already mentioned, there exist scalar amplitudes that are known to have no kinematical singularities. An investigation of this question for arbitrary spin will be reported in a separate paper. We proceed here on the assumption that one among a large class of possible bases will lead to scalar amplitudes without poles.

In this paper, we ignore isotopic spin and give no systematic discussion of C, P, and T transformations, but make only occasional comments where appropriate.

Apart from their theoretical interest, the considerations involving higher spins will be, we believe, of practical importance in connection with the new higher spin resonances, and perhaps in the problem of analytic continuation in spin of the S-matrix elements. Many of these considerations apply to processes involving arbitrary numbers of particles and are not restricted to two-body systems. For example, the spin matrices introduced in this paper generalizing the Pauli matrices to higher spins may be of interest in other applications. From these matrices we obtain the projection operators for the irreducible invariant subspaces of the tensors of arbitrary rank.

II. DEFINITION OF INVARIANT FUNCTIONS AND GENERAL PROCEDURE

The formulas developed in the succeeding sections are rather involved. To facilitate the reading, we outline in this section the procedure that we have followed; but first we define the transformation laws of the various invariant functions. It is often said that spin is only an inessential complication. Nevertheless, it appears that except in simple cases a certain amount of complication is, if not essential, at least unavoidable.

A. The Invariant Functions

We consider scattering processes for outgoing particles and incoming antiparticles with spins and four-momenta S_i, k_i, and incoming particles and outgoing antiparticles with spins and four-momenta S_j, k_j, all with nonzero rest masses. The invariant scattering functions (or S-matrix elements) have the following transformation property under representations of the inhomogeneous orthochronous proper Lorentz group[2,8]:

$$S(K)=\otimes_i \mathfrak{D}^{S_i}[A'(-k_i)] \otimes\otimes_j \mathfrak{D}^{S_j}[A'(k_j)]^* S[\Lambda^{-1}(A)K], \quad (2.1)$$

where

$$A'(k)=B_{k\leftarrow p}^{-1}AB_{q\leftarrow p} \quad \text{and} \quad \Lambda(A^{-1})k=q.$$

Here K stands for the set of incoming and outgoing four-momenta, k_n, with $\sum_n k_n=0$ from momentum conservation; and ΛK stands for the set of transformed momenta, Λk_n. Elements of the orthochronous proper homogeneous Lorentz group $L_+\uparrow$ are denoted by $\Lambda(A)$, where $\pm A$ are the corresponding elements of the two-by-two unimodular group. The spin indices of the S function, which have been suppressed, are transformed by direct products of the unitary matrices $\mathfrak{D}^{S_i}$ and $\mathfrak{D}^{S_j*}$, which are the well-known $[(2S_i+1),\,(2S_j+1)]$-dimensional irreducible representations of the three-dimensional proper real orthogonal group. An index transforming according to $\mathfrak{D}^S$ corresponds to an outgoing particle or incoming antiparticle and one transforming according to $\mathfrak{D}^{S*}$ corresponds to an incoming particle or outgoing antiparticle.[9] In the argument $A'(k)$ of $\mathfrak{D}^S$ or $\mathfrak{D}^{S*}$, the unimodular matrices B are so defined that

$$\Lambda(B_{k\leftarrow p})p=k,$$

and similarly for $B_{q\leftarrow p}$. The Lorentz transformation corresponding to the unitary-unimodular matrix $A'=B_{k\leftarrow p}^{-1}AB_{q\leftarrow p}$ transforms the vector p into itself (it is an element of the little group of the vector p), where $p=(m,0,0,0)$ is the rest-frame value of k; hence, this transformation is a rotation.

From the definition of p and Eq. (A1.1) in Appendix I, we have, in terms of Pauli matrices, σ_μ,[8]

$$B_{k\leftarrow p}B_{k\leftarrow p}^\dagger=k^\mu\sigma_\mu/m. \quad (2.2)$$

The general solution of this equation can be written in the form $B_{k\leftarrow p}=A_{k\leftarrow p}U$, where $A_{k\leftarrow p}$ is the Hermitian matrix $(k\cdot\sigma/m)^{1/2}$ and U is an arbitrary unitary matrix corresponding to the freedom of arbitrary rotations in the rest system of the particle. We use this freedom later in the construction of helicity amplitudes. An

[5] A. C. Hearn, Nuovo Cimento **21**, 333 (1961), discusses the amplitudes for spin ½ and photon processes, in a perturbation theory framework.

[6] A construction of the M functions from a somewhat different point of view has been given by Willard Miller, Mathematics Department, University of California, Berkeley, California.

[7] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962). For the N-N system, see V. N. Gribov, in, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki, (CERN, Geneva, 1962) and I. Muzinich, Phys. Rev. (to be published). See also V. Singh, Phys. Rev. **129**, 1889 (1963).

[8] For a list of conventions, notation, and various important relations involving two-component spinors, two-by-two matrices, and group representation matrices, see Appendix I.

[9] This choice is purely conventional, especially since the two representations are equivalent. It agrees with the usual convention in the four-component formalism, as will be seen incidentally in Appendix II. See also the references in footnote 1.

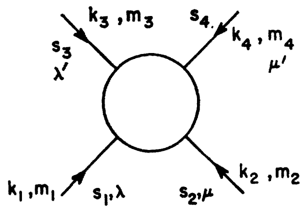

FIG. 1. Two-body scattering parameters.

important characteristic of the invariant M functions defined below is that their transformation property is independent of B.

The transformation law (2.1) also holds for the R functions,

$$R=S-I. \tag{2.3}$$

There is a natural way of simplifying this transformation law. Because the matrix $A'(k)=B_{k\leftarrow p}^{-1}AB_{q\leftarrow p}$ is unitary, we have the identity

$$\mathfrak{D}^S[A'(k)]=\mathfrak{D}^{(S,0)}[A'(k)]=\mathfrak{D}^{(0,S)}[A'(k)], \tag{2.4}$$

where $\mathfrak{D}^{(S,S')}$ are the irreducible, in general, nonunitary, representations of dimension $(2S+1)(2S'+1)$ of $L_+\uparrow$. We can, then, use the group property of $\mathfrak{D}^{(S,0)}$ and obtain

$$\mathfrak{D}^S(B_{k\leftarrow p}^{-1}AB_{q\leftarrow p})=\mathfrak{D}^{(S,0)}(B_{k\leftarrow p})^{-1} \times\mathfrak{D}^{(S,0)}(A)\mathfrak{D}^{(S,0)}(B_{q\leftarrow p}). \tag{2.5}$$

Thus, if we introduce M functions defined by

$$M(K)=\otimes_i\mathfrak{D}^{(S_i,0)}(B_{-k_i\leftarrow p_i}) \otimes\otimes_j\mathfrak{D}^{(S_j,0)}(B_{k_j\leftarrow p_j})^*R(K), \tag{2.6}$$

we see from (2.1) that they have the simple transformation law under $L_+\uparrow$,

$$M(K)=\otimes_i\mathfrak{D}^{(S_i,0)}(A) \otimes\otimes_j\mathfrak{D}^{(S_j,0)}(A)^*M[\Lambda(A^{-1})K]. \tag{2.7}$$

It is simpler to construct the solutions of this equation than those of (2.1). Equations (2.3), (2.6), and (2.7) are the basic formulas from which the construction of the M and S functions begins. For spin $\frac{1}{2}$ these are just the M functions introduced by Stapp.[1]

B. The Scalar Amplitudes

For practical purposes, such as the application of the Mandelstam representation, it appears convenient to use a representation of the invariant functions in which all of the dynamics is contained in a set scalar amplitudes. In a sense this removes spin from the problem. Our problem is, thus, to find a simple, explicit set of basis functions, $Y^{(i)}(K)$, in the spin space which have the same transformation property (2.7) as the M functions. Then we write

$$M(K)=\textstyle\sum_{(i)} A^{(i)}(K)Y^{(i)}(K), \tag{2.8}$$

where the $A^{(i)}(K)$ are Lorentz scalars and must, therefore, be functions of the scalar invariants formed from the four-momenta (and possibly of the signs of the energies). One can also require that the basis functions $Y^{(i)}(K)$ have definite transformation properties under P and T. Thus, if P and T are conserved, the total number of independent scalar amplitudes will be smaller than the $\prod_i(2S_i+1)\prod_j(2S_j+1)$ resulting from (2.7) and (2.8).

The essential requirement on the scalar amplitudes is that they shall have only the singularities of the M function itself, which on the basis of perturbation theory or of a pure S-matrix theory are expected to be only dynamical.[1] Furthermore, we wish to require that the basis functions themselves have no singularities. The simplest possibility is that the basis functions should be polynomials in the components of the linear momenta. To require that the basis functions have this form is not enough, however, for the scalar amplitudes could still have kinematical poles at various degenerate points where the basis functions become linearly dependent. Indeed, the question of whether there exists a set of basis functions that never induces kinematical poles in the scalar amplitudes already involves considerable subtlety in the case of two-body reactions; and therefore, we shall restrict ourselves primarily to this case in any discussion where the singularities are important.

The question of to what extent these various requirements determine a set of basis functions is not settled in this paper. Rather we seek to establish a basic formalism for arbitrary spins that can be used in the construction of a large class of basis functions. We follow a procedure that is natural and systematic, and that yields the usual analytic amplitudes in special cases. It consists first of building up in Sec. III a set of higher spin matrices from the spin-$\frac{1}{2}$ matrices, σ_μ, by using Clebsch-Gordan coefficients in a process corresponding to the addition of spins. For two-body reactions we then, in Sec. IV, combine the spin matrices with tensors formed from the four-momenta to obtain a set of basis functions, $Y^{(i)}(K)$; and we give a brief discussion of the question of kinematical poles in the resulting scalar amplitudes. If preliminary results are substantiated, a second paper showing how to eliminate the kinematical poles will be submitted by one of us (DNW).

C. Angular Momentum

In Sec. V we define an analytic continuation in total angular momentum for the scattering functions shown in Fig. 1. For this purpose it is convenient to use helicity amplitudes. Having constructed $Y^{(i)}(K)$ and, therefore, $M(K)$ by (2.8), we obtain the helicity amplitudes $H(K)$ from (2.6) by making the appropriate choice for B in the expression

$$R(K)=\mathfrak{D}^{(S_4,0)}(B_{-k_4\leftarrow p_4})^{-1}\otimes\mathfrak{D}^{(S_3,0)}(B_{-k_3\leftarrow p_3})^{-1} \otimes\mathfrak{D}^{(S_2,0)}(B_{k_2\leftarrow p_2})^{-1*}\otimes\mathfrak{D}^{(S_1,0)}(B_{k_1\leftarrow p_1})^{-1*} \times\textstyle\sum_{(i)} A^{(i)}(K)Y^{(i)}(K), \tag{2.9}$$

where now the $A^{(i)}(K)$ can be taken as functions of the Mandelstam variables,

$$s=(k_1+k_2)^2,\quad t=(k_1+k_3)^2,\quad u=(k_1+k_4)^2,$$

with $s+t+u=\sum_i m_i^2$. The helicity amplitudes $H(K)$ are defined to be $R(K)$ when

$$\begin{aligned}B_{k\leftarrow p}&=(k\cdot\sigma/m)^{1/2}\exp(-i\phi\sigma_3/2)\exp(-i\theta\sigma_2/2)\\&\qquad\times\exp(i\phi\sigma_3/2),\\&=\exp(-i\phi\sigma_3/2)\exp(-i\theta\sigma_2/2)\\&\qquad\times\exp(i\phi\sigma_3/2)(q\cdot\sigma/m)^{1/2},\end{aligned}\tag{2.10}$$

where $q^\mu=(k_0,0,0,|\mathbf{k}|)$, i.e., a velocity transformation from the rest frame to the z direction followed by a rotation to the direction (θ,ϕ), of $\mathbf{k}$.[10]

Without loss of generality we can put, in the center-of-mass frame of the s channel, $\phi=0$. It turns out that for any among a large class of basis functions the angular dependence (θ dependence) of the helicity amplitudes can be factored into a product of $d^S(\theta)$ functions in the form

$$H_{(\lambda)}(K)=\sum_{(i),R}A^{(i)}(s,t,u)Z_{(\lambda)R}{}^{(i)}d^R(\theta),\tag{2.11}$$

where $Z^{(i)}$ does not depend upon θ, and R is determined by the spins of the particles. Here (λ) stands for the indices $(\mu',\lambda',\dot\mu,\dot\lambda)$ and $d^S(\theta)\equiv\mathfrak{D}^{(S,0)}[\exp(i\theta\sigma_2/2)]$.

The projection over the total angular momentum J of $H_{(\lambda)}$ is defined by[10]

$$h_{(\lambda)}{}^J(s)=\frac{1}{2}(qq')^{1/2}\int_{-1}^{1}dz\,d^J{}_{\Delta\lambda'}{}^{\Delta\lambda}H_{(\lambda)},\tag{2.12}$$

where $z=\cos\theta$, $\Delta\lambda=\lambda-\mu$, $\Delta\lambda'=\lambda'-\mu'$, and q and q' are the magnitudes of the momenta of the initial and final particles, respectively.

We now write for the scalar amplitudes a partial-wave expansion in the s channel, for example,

$$A^{(i)}(s,t,u)=\sum_l(2l+1)A^{(i)}(l,s)d^l(\theta)_0{}^0,\tag{2.13}$$

where we put for the Legendre polynomials, $P_l(z)=d^l(\theta)_0{}^0$.

If we insert this into $H_{(\lambda)}$ and combine $d^R(\theta)$ with $d^l(\theta)_0{}^0$ into a single d function and perform the angular integration, which is of the form

$$\frac{1}{2}\int_{-1}^{1}dz\,d^J(\theta)_\mu{}^\nu d^R(\theta)_\mu{}^\nu=\frac{1}{2J+1}\delta_{JR},\tag{2.14}$$

we obtain

$$h_{(\lambda)}{}^J(s)=\sum_{l,(i)}A^{(i)}(l,s)\bar{Z}_{(\lambda)}{}^{(i)},\tag{2.15}$$

where $\bar{Z}$ contains a sum of the original Z times a number of Clebsch-Gordan coefficients. In the above sum the l values are restricted by the given J.

From the fixed-energy dispersion relation for $A^{(i)}(s,t,u)$ we express $A^{(i)}(l,s)$ in terms of the absorptive parts $A_t{}^{(i)}$ and $A_u{}^{(i)}$ of the amplitudes in the crossed channels and obtain

$$h_{(\lambda)}{}^J(s)=\sum_{l,(i)}\bar{Z}_{(\lambda)}{}^{(i)}\left[\int dz\,Q_l(z)A_t{}^{(i)}(s,z)+(-1)^l\int dz\,Q_l(z)A_u{}^{(i)}(s,z)\right],\tag{2.16}$$

where the $Q_l(z)$ are Legendre functions of the second kind. Assuming that the absorptive parts A_t and A_u are uniformly bounded in t and u by t^N (or u^N), we see that the expression (2.16) defines an analytic function of J for $\mathrm{Re}J>N'$, where N' is displaced from N by some integer determined by the spins of the particles.

Details are given in Sec. V.

III. CONSTRUCTION OF SPIN MATRICES

It is convenient to separate into two parts the construction of the basis functions $Y^{(i)}(K)$ for arbitrary spin. In this section we construct a set of matrices which span the spin space and which contain most of the complications in the transformation law due to spin. These matrices are independent of the four-momenta in the problem, except under special circumstances to be mentioned later; they have essentially no effect on the singularity structure of the scalar amplitudes. The results of this section apply to M functions that describe arbitrary numbers of particles.

The matrices that span a given spin space are labeled with tensor indices in addition to spin indices labeling their matrix elements. A complete set of basis functions $Y^{(i)}(K)$ is obtained by contracting the tensor indices of the spin matrices with a complete set of tensor functions which are polynomials in the components of the four-momenta. Given a spin basis, it is the construction of a basis for the space of tensor functions that can lead to possible kinematical poles in the scalar amplitudes. This question is discussed in Sec. IV.

A. Spin-$\frac{1}{2}$ Matrices

The basis for general spin is constructed from two-component Pauli spinors. Since the total number of incoming and outgoing fermions in any scattering process must be even,[11] the simplest case that we can consider involves two spin-$\frac{1}{2}$ particles, one incoming, the other outgoing.

Equation (2.7) then becomes

$$\begin{aligned}M(K)&=A\otimes A^*M[\Lambda(A^{-1})K]\\&=AM[\Lambda(A^{-1})K]A^\dagger,\end{aligned}\tag{3.1}$$

[10] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959). For rotations, an upper undotted index transforms as a lower dotted index.

[11] Because $\mathfrak{D}^{S0}(-A)=(-1)^{2S}\mathfrak{D}^{S0}(A)$ and $\Lambda(-A)=\Lambda(A)$, we have

$$M(K)=M(\Lambda(-I)K)=(-1)^{2\Sigma_iS_i}\otimes_j\mathfrak{D}^{S_j0}(I)M(K)=(-1)^{2\Sigma_iS_i}M(K).$$

Hence Σ_iS_i must be an integer if $M(K)$ is not to vanish identically.

or, writing the spinor indices,

$$M_{\alpha\dot{\beta}}(K)=A_{\alpha}{}^{\alpha'}A_{\dot{\beta}}{}^{\dot{\beta}'}M_{\alpha'\dot{\beta}'}[\Lambda(A^{-1})K]. \tag{3.2}$$

As usual, the dotted index (incoming particle or outgoing antiparticle) transforms according to A^* and the undotted index (outgoing particle or incoming antiparticle) according to A.[12] Any two-by-two matrix can be written as a linear combination of Pauli matrices, σ_μ. Hence, we can put

$$M(K)=f^\mu(K)\sigma_\mu. \tag{3.3}$$

From the transformation law of σ_μ given by (A1.1), it is clear that we must have

$$\Lambda_\mu{}^\nu f_\nu(\Lambda^{-1}K)=f_\mu(K), \tag{3.4}$$

if (3.1) is to be satisfied.

The four-vector function $f^\mu(K)$ can be expanded in terms of the four-momenta K, but that construction is reserved for Sec. IV.

If we define

$$\rho_\mu=(1/\sqrt{2})\sigma_\mu \quad \text{and} \quad \tilde{\rho}_\mu=(1/\sqrt{2})\tilde{\sigma}_\mu, \tag{3.5}$$

where $\tilde{\sigma}_\mu$ is defined in Appendix I, the orthogonality relations (A1.6) in Appendix I become

$$\rho_{\mu\alpha\dot{\beta}}\tilde{\rho}^{\mu\dot{\beta}'\alpha'}=\delta_\alpha{}^{\alpha'}\delta_{\dot{\beta}}{}^{\dot{\beta}'}, \quad \text{and} \quad \rho^\mu{}_{\alpha\dot{\beta}}\rho_{\mu\alpha'\dot{\beta}'}=C_{\alpha\alpha'}C_{\dot{\beta}\dot{\beta}'}, \tag{3.6}$$

where C is the "lowering" spinor defined in (A1.2).

The general formalism of the theory also requires basis spinors with two undotted or two dotted indices. Such spinors can be obtained in several different ways. For example, the matrices $\rho_\mu\tilde{\rho}_\nu C^{-1}$ have lower undotted indices, and they certainly span the space. There is a choice, however, introduced by Stapp,[1] that is natural and especially convenient for a discussion of crossing relations. It consists in defining the special spinors

$$g_{\alpha\dot{\beta}}(k)=k\cdot\sigma_{\alpha\dot{\beta}}/m, \tag{3.7}$$

which can be used to change a dotted index into an undotted one and vice versa, where k is taken to be the four-momentum of the particle whose spin index is to be operated upon.

We then define basis spinors

$$\omega_\mu(k)_{\alpha\beta}=g_{\alpha\dot{\alpha}'}\tilde{\rho}_\mu{}^{\dot{\alpha}'}{}_\beta=(k\cdot\sigma\tilde{\rho}_\mu C^{-1}/m)_{\alpha\beta},$$
$$\tilde{\omega}_\mu(k)_{\dot{\alpha}\dot{\beta}}=\tilde{\rho}_{\mu\dot{\alpha}}{}^{\beta'}g_{\beta'\dot{\beta}}=(C\tilde{\rho}_\mu k\cdot\sigma/m)_{\dot{\alpha}\dot{\beta}}. \tag{3.8}$$

These spinors transform according to $A\otimes A$ and $A^*\otimes A^*$, respectively. For example,

$$A\omega^\mu(k)A^T=\Lambda_\nu{}^\mu(A)\omega^\nu[\Lambda(A)k]. \tag{3.9}$$

[12] A review of spinor calculus with conventions for dotted and undotted indices is included in Appendix I.

They satisfy orthogonality relations

$$\omega^\mu(k)_{\alpha\beta}\omega_\mu(k)^{\alpha'\beta'}=\delta_\alpha{}^{\alpha'}\delta_\beta{}^{\beta'} \quad \text{and} \quad \omega^\mu(k)_{\alpha\beta}\omega_\mu(k)_{\alpha'\beta'}=C_{\alpha\alpha'}C_{\beta\beta'}, \tag{3.10}$$

with corresponding formulas for dotted indices.

A spin basis for arbitrarily many spin-$\frac{1}{2}$ particles is obtained by taking direct products of matrices chosen from among ρ_μ, $\omega_\mu(k)$, and $\tilde{\omega}_\mu(k)$, depending on the desired index types.

B. Properties of Matrices for Arbitrary Spin

Many of the characteristics of the spin matrices for higher spin are a straightforward generalization from the spin-$\frac{1}{2}$ matrices and can be understood without going through the details of a somewhat involved construction. Before proceeding to the actual construction, we shall, therefore, describe the essential results.

As already indicated, the fermion spin indices can always be paired; and we can also pair the boson indices by adding a dummy spin-0 index whenever the total number of particles is odd. Thus, we require a basis for matrices with two fermion or two boson spin indices; any spin space can be spanned with direct products of these. This basis is given by a set of rectangular matrices $\rho^{\mu_1\cdots\mu_{2M}}(SS')$, $\tilde{\rho}^{\mu_1\cdots\mu_{2M}}(SS')$, $\omega^{\mu_1\cdots\mu_{2M}}(SS';k)$, and $\tilde{\omega}^{\mu_1\cdots\mu_{2M}}(SS';k)$, where $M=\max(S,S')$, which span the spin-S, spin-S' space, and which reduce to (3.5) and (3.8) when $S=S'=\frac{1}{2}$. Here S and S' are the spins of the pair of bosons or fermions. The spin indices labeling the matrix elements have $2S+1$, $2S'+1$ values, respectively, ranging through S, $S-1$, $\cdots$, $-S$ and S', $S'-1$, $\cdots$, $-S'$.

1. Transformation Properties

The spin matrices just described are classified according to the representations of $L_+\uparrow$ of the types $\mathfrak{D}^{(S,0)}(A)$, $\mathfrak{D}^{(S,0)}(A^*)$, or the respective contragredient representations $\mathfrak{D}^{(S,0)}(A^{-1T})$, $\mathfrak{D}^{(S,0)}(A^{-1\dagger})$. The whole apparatus of the spinor calculus can be taken over for arbitrary spin. The spin indices will be written as lower undotted, lower dotted, upper undotted, and upper dotted, respectively, corresponding to the four representations listed above. The contraction of an upper with a lower index of the same type is then an invariant operation.

The raising of a spin-S index is accomplished by contracting on the right with the matrix

$$\mathfrak{D}^{(S,0)}(C^{-1})^{\alpha\beta}=\mathfrak{D}^{(S,0)}(C^{-1})^{\dot{\alpha}\dot{\beta}}=(-1)^{S-\alpha}\delta_{\alpha,-\beta}, \tag{3.11}$$

and lowering by contracting on the right with

$$\mathfrak{D}^{(S,0)}(C)_{\alpha\beta}=\mathfrak{D}^{(S,0)}(C)_{\dot{\alpha}\dot{\beta}}=(-1)^{S-\beta}\delta_{\alpha,-\beta}. \tag{3.12}$$

The spinor for changing dotted to undotted indices and vice versa, defined in (3.7) for spin $\frac{1}{2}$, becomes

$$\mathfrak{D}^{(S,0)}[g(k)]=\mathfrak{D}^{(S,0)}(k\cdot\sigma/m). \quad (3.13)$$

By convention we take the types of the indices of the matrices $\mathfrak{D}^{(S,0)}$ to be the same as those of their arguments.

The matrices $\rho_{(\mu)}(SS')$, where $(\mu)=(\mu_1\cdots\mu_{2M})$, are constructed to have a lower undotted spin-S index and a lower dotted spin-S' index, while the $\tilde{\rho}_{(\mu)}(SS')$ have an upper dotted spin-S index and an upper undotted spin-S' index. The construction is such that (A1.4) generalizes to

$$\tilde{\rho}_{(\mu)}(SS')=\mathfrak{D}^{(S,0)}(C^{-1})\rho_{(\mu)}(SS')^*\mathfrak{D}^{(S',0)}(C). \quad (3.14)$$

The ω matrices are defined by analogy with (3.8):

$$\omega_{(\mu)}(SS';k)=\mathfrak{D}^{(S,0)}(k\cdot\sigma/m)\tilde{\rho}_{(\mu)}(SS')\mathfrak{D}^{(S',0)}(C)^{-1}, \quad (3.15)$$

and similarly for the corresponding matrices with lower dotted indices.

The transformation laws are given explicitly by

$$\begin{aligned}\mathfrak{D}^{(S,0)}(A)\rho^{(\mu)}(SS')\mathfrak{D}^{(S',0)}(A)^\dagger&=\Lambda_{(\nu)}{}^{(\mu)}(A)\rho^{(\nu)}(SS'),\\ \mathfrak{D}^{(S,0)}(A)^{-1\dagger}\tilde{\rho}^{(\mu)}(SS')\mathfrak{D}^{(S',0)}(A)^{-1}&=\Lambda_{(\nu)}{}^{(\mu)}(A)\rho^{(\nu)}(SS'),\end{aligned} \quad (3.16)$$

and

$$\mathfrak{D}^{(S,0)}(A)\omega^{(\mu)}(SS';k)\mathfrak{D}^{(S',0)}(A)^T=\Lambda_{(\nu)}{}^{(\mu)}(A)\omega^{(\nu)}[SS';\Lambda(A)k],$$

where $\Lambda_{(\nu)}{}^{(\mu)}$ stands for a direct product of Lorentz transformations, one for each tensor index of (ν).

2. Orthogonality Relations

The fact that the spin matrices actually span the spin space is exhibited explicitly by the relations

$$\begin{aligned}\rho^{(\mu)}(SS')_{\alpha\dot{\beta}}\tilde{\rho}_{(\mu)}(S'S)^{\dot{\beta}'\alpha'}&=\delta_\alpha{}^{\alpha'}\delta_{\dot{\beta}}{}^{\dot{\beta}'}\\ \rho^{(\mu)}(SS')_{\alpha\dot{\beta}}\rho_{(\mu)}(SS')_{\alpha'\dot{\beta}'}&=\mathfrak{D}^{(S,0)}(C)_{\alpha\alpha'}\mathfrak{D}^{(S',0)}(C)_{\dot{\beta}\dot{\beta}'},\\ \omega^{(\mu)}(SS';k)_{\alpha\beta}\omega_{(\mu)}(SS';k)^{\alpha'\beta'}&=\delta_\alpha{}^{\alpha'}\delta_\beta{}^{\beta'},\end{aligned} \quad (3.17)$$

and those formulas obtained from these by raising and lowering indices. These relations are a special case of more general formulas given in Sec. III D.3.

3. Symmetry Properties

It will turn out that there is a connection between the ρ matrices and the irreducible subspaces of the tensors of rank $2M$. This connection induces various symmetries among the tensor indices of ρ, as well as making ρ traceless in the contraction of any pair of tensor indices. Actually, we have omitted an extra label in the description of the ρ matrices, expressing a freedom in their construction which corresponds to the fact that there are, in general, several irreducible subspaces of the same dimension in the space of tensors of rank $2M$. When $S=S'$, however, the $\rho^{(\mu)}(SS)$ are essentially unique; and they are symmetric in the interchange of any tensor indices. Similar results hold for $\tilde{\rho}$ and ω.

C. Spin-1 Matrices

Matrices for higher spin can be constructed from the ρ_μ matrices by a process of spin addition with the use of Clebsch-Gordan coefficients. Consider the quantities

$$\rho^{\mu\nu}(SS')_{\alpha\dot{\beta}}=\sum_{\gamma\gamma'\dot{\kappa}\dot{\kappa}'}C(\tfrac{1}{2},\tfrac{1}{2},S;\gamma,\gamma',\alpha)\times C(\tfrac{1}{2},\tfrac{1}{2},S';\dot{\kappa},\dot{\kappa}',\dot{\beta})\rho^\mu{}_{\gamma\dot{\kappa}}\rho^\nu{}_{\gamma'\dot{\kappa}'}, \quad (3.18)$$

where S, S' can have either of the values one or zero, and $C(j_1,j_2,j_3;\alpha_1,\alpha_2,\alpha_3)$ are the Clebsch-Gordan coefficients in Rose's notation.[13] The new quantities $\rho^{\mu\nu}$ transform according to the representation $\mathfrak{D}^{(S,0)}\otimes\mathfrak{D}^{(S',0)*}$. To prove this we start from the identity

$$\sum_{\gamma\gamma'\kappa\kappa'}C(S_1,S_2,S;\gamma,\gamma',\alpha)C(S_1,S_2,S';\kappa,\kappa',\beta)\times\mathfrak{D}^{(S_1,0)}(A)_\gamma{}^\kappa\mathfrak{D}^{(S_2,0)}(A)_{\gamma'}{}^{\kappa'}=\delta_{SS'}\mathfrak{D}^{(S,0)}(A)_\alpha{}^\beta, \quad (3.19)$$

which expresses the reduction of a direct product of representations into a direct sum. By using the orthogonality of the Clebsch-Gordan coefficients,

$$\sum_{\gamma\gamma'}C(S_1,S_2,S;\gamma,\gamma',\alpha)C(S_1,S_2,S';\gamma,\gamma',\alpha')=\delta_{SS'}\delta_\alpha{}^{\alpha'}$$

and $\quad (3.20)$

$$\sum_{S,\alpha}C(S_1,S_2,S;\gamma,\gamma',\alpha)C(S_1,S_2,S;\kappa,\kappa',\alpha)=\delta_\gamma{}^\kappa\delta_{\gamma'}{}^{\kappa'},$$

we get from (3.19) the identity

$$\sum_\beta\mathfrak{D}^{(S,0)}(A)_\alpha{}^\beta C(S_1,S_2,S;\kappa,\kappa',\beta)=\sum_{\gamma\gamma'}C(S_1,S_2,S;\gamma,\gamma',\alpha)\mathfrak{D}^{(S_1,0)}(A)_\gamma{}^\kappa\times\mathfrak{D}^{(S_2,0)}(A)_{\gamma'}{}^{\kappa'}. \quad (3.21)$$

This leads at once from (A1.1) to the transformation law

$$\mathfrak{D}^{(S,0)}(A)_\alpha{}^\beta\mathfrak{D}^{(S',0)}(A)_{\dot{\alpha}'}{}^{\dot{\beta}'}\rho^{\mu\nu}(SS')_{\beta\dot{\beta}'}=\Lambda_\sigma{}^\mu(A)\Lambda_\tau{}^\nu(A)\rho^{\sigma\tau}(SS')_{\alpha\dot{\alpha}'}, \quad (3.22)$$

where $\mathfrak{D}^{(S,0)}(A)_{\dot{\alpha}}{}^{\dot{\beta}}=\mathfrak{D}^{(S,0)}(A)^*{}_\alpha{}^\beta$. In matrix notation

[13] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

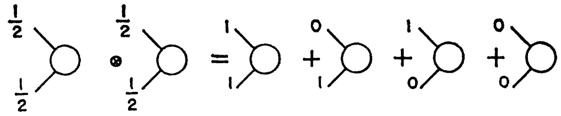

FIG. 2. Addition of spins.

this is the same as (3.16) when $(S,S')=(1,1)$; $(1,0)$; $(0,1)$.

According to the values of S, S', the $\rho^{\mu\nu}(SS')$ provide spin matrices for the four different situations shown schematically in Fig. 2. It is clear from the construction in (3.18) that because the direct product matrices $\rho^{\mu}\otimes\rho^{\nu}$ span the 16-dimensional product space, the matrices $\rho^{\mu\nu}(SS')$ must span the corresponding four direct-sum spaces of dimensions 9, 3, 3, and 1.

For any given pair of values S, S' there are 16 values of the tensor indices, and hence the $\rho^{\mu\nu}(SS')$ are not all linearly independent. In fact, various symmetry properties of the Clebsch-Gordan coefficients, for example,

$$C(S_1,S_2,S;\alpha_1,\alpha_2,\alpha)=(-1)^{S-S_1-S_2}C(S_2,S_1,S;\alpha_2,\alpha_1,\alpha), \qquad (3.23)$$

are reflected in symmetries of the tensor indices. A straightforward calculation gives

$$\begin{aligned}&\rho^{\mu\nu}(11)=\rho^{\nu\mu}(11),\\&\rho_{\mu}{}^{\mu}(11)=0,\\&\rho^{\mu\nu}(10)=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\rho_{\lambda\sigma}(10),\\&\rho^{\mu\nu}(01)=-\tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\rho_{\lambda\sigma}(01),\end{aligned} \qquad (3.24)$$

and

$$\rho^{\mu\nu}(00)=\tfrac{1}{2}g^{\mu\nu}.$$

The various symmetries follow from (3.23) and (3.18). That $\rho_{\mu}{}^{\mu}(11)=0$ follows from (3.6), (3.18), (3.23), and the fact that C is antisymmetric. We have used also (A1.5) and (A1.6) from Appendix I, as well as the often useful identity $\sqrt{2}C(\frac{1}{2},\frac{1}{2},0;\alpha,\beta,0)=C_{\alpha\beta}{}^{-1}$. There is a correspondence between the expressions (3.24) and the irreducible subspaces of the second-rank tensors of dimensions nine (symmetric and traceless), three (self-dual), three (anti-self-dual), and one (scalar proportional to $g_{\mu\nu}$), which will be further explained after orthogonality relations are obtained.

The matrices $\tilde{\rho}^{\mu\nu}(SS')$ can be obtained by replacing ρ with $\tilde{\rho}$ in the construction (3.18), or they can be obtained directly from $\rho^{\mu\nu}(SS')$ by the general procedure (3.14). Using (3.21) and (A1.4), we find that the two methods give the same result. Then, the orthogonality relations, (3.6) and (3.20), lead to

$$\rho^{\mu\nu}(SS')_{\alpha\dot{\beta}}\tilde{\rho}_{\mu\nu}(L'L)^{\dot{\beta}'\alpha'}=\delta_{SL}\delta_{S'L'}\delta_{\alpha}{}^{\alpha'}\delta_{\dot{\beta}}{}^{\dot{\beta}'},$$

and (3.25)

$$\rho^{\mu\nu}(SS')_{\alpha\dot{\beta}}\rho_{\mu\nu}(LL')_{\alpha'\dot{\beta}'}=\delta_{SL}\delta_{S'L'}\mathfrak{D}^{(S,0)}(C)_{\alpha\alpha'}\times\mathfrak{D}^{(S',0)}(C)_{\dot{\beta}\dot{\beta}'}.$$

Making use of the relation

$$\rho_{\mu\alpha\dot{\beta}}\tilde{\rho}_{\nu}{}^{\dot{\beta}\alpha}=\tfrac{1}{2}\operatorname{Tr}(\sigma_{\mu}\tilde{\sigma}_{\nu})=g_{\mu\nu} \qquad (3.26)$$

and the second relation in (3.20), we get

$$\begin{aligned}\textstyle\sum_{SS'}\tilde{\rho}^{\lambda\sigma}(S'S)^{\dot{\beta}\alpha}\rho^{\mu\nu}(SS')_{\alpha\dot{\beta}}&=\textstyle\sum_{SS'}\operatorname{Tr}[\tilde{\rho}^{\lambda\sigma}(S'S)\rho^{\mu\nu}(SS')],\\&=g^{\lambda\mu}g^{\sigma\nu}.\end{aligned} \qquad (3.27)$$

These relations can be used to get a compact characterization of the invariant subspaces of the second-rank tensors. An arbitrary tensor can be expanded:

$$T^{\mu\nu}=\textstyle\sum_{SS'}T^{\lambda\sigma}\operatorname{Tr}[\tilde{\rho}_{\lambda\sigma}(S'S)\rho^{\mu\nu}(SS')]. \qquad (3.28)$$

Clearly the "projected" tensors defined by

$$T^{\mu\nu}(SS')=T^{\lambda\sigma}\operatorname{Tr}[\tilde{\rho}_{\lambda\sigma}(S'S)\rho^{\mu\nu}(SS')] \qquad (3.29)$$

lie in the four invariant subspaces mentioned previously.

D. Matrices for Arbitrary Spin

For arbitrary pairs of boson or fermion spins we proceed inductively, generalizing the construction for spin 1. By addition of spins we reduce the direct-product space $\rho^{\mu_1}\otimes\cdots\otimes\rho^{\mu_N}\equiv\otimes_N\rho^{\mu}$ into a direct-sum space. There is, of course, a freedom in the order for coupling the spins. We shall follow the convention that the reduction is always carried out beginning at the left: $\{[\cdots(\rho^{\mu_1}\otimes\rho^{\mu_2})\cdots]\otimes\rho^{\mu_N}\}$. All other choices are related to this one by a unitary transformation.

1. Reduction of the Product Space

As an example, consider the reduction of the space $\otimes_3\rho^{\mu}$. We obtain a set of matrices

$$\begin{aligned}\rho^{\mu_1\mu_2\mu_3}[SS':(LL')]_{\alpha\dot{\beta}}=&\textstyle\sum_{\gamma\gamma'\dot{\kappa}\dot{\kappa}'}C(L,\frac{1}{2},S;\gamma,\gamma',\alpha)\\&\times C(L',\tfrac{1}{2},S';\dot{\kappa},\dot{\kappa}',\dot{\beta})\rho^{\mu_1\mu_2}(LL')_{\gamma\dot{\kappa}}\rho^{\mu_3}{}_{\gamma'\dot{\kappa}'},\end{aligned} \qquad (3.30)$$

where L, L' can have any combination of the values 1 and 0, and S, S' can have any combination of the values $\frac{3}{2}$ and $\frac{1}{2}$. The spin indices are labeled by S, S'; and L, L' label the intermediate spins that are added to $\frac{1}{2}$, $\frac{1}{2}$ to produce S, S'. The set of matrices $\rho^{\mu_1\mu_2\mu_3}[SS':(L)]$, where $(L)=(LL')$, will be called the "reduction" of the space $\otimes_3\rho^{\mu}$. In general, we shall use the notation $\rho^{\mu_1\cdots\mu_N}[SS':(L)]$ for the matrices that are the reduction of $\otimes_N\rho^{\mu}$. The spin indices are lower undotted for spin S and lower dotted for spin S', and (L) labels the set of pairs of intermediate spins that are the "path" by which the spins S, S' are reached.

The reduction is defined inductively by

$$\begin{aligned}\rho^{\mu_1\cdots\mu_{N+1}}&\{SS':[LL':(L'')]\}_{\alpha\dot{\beta}}\\&=\textstyle\sum_{\gamma\gamma'\dot{\kappa}\dot{\kappa}'}C(L,\frac{1}{2},S;\gamma,\gamma',\alpha)C(L',\frac{1}{2},S';\dot{\kappa},\dot{\kappa}',\dot{\beta})\\&\qquad\times\rho^{\mu_1\cdots\mu_N}[LL':(L'')]_{\gamma\dot{\kappa}}\rho^{\mu_{N+1}}{}_{\gamma'\dot{\kappa}'},\end{aligned} \qquad (3.31)$$

where $S=L\pm\frac{1}{2}$, $S'=L'\pm\frac{1}{2}$. The matrices

$$\tilde{\rho}^{\mu_1\cdots\mu_N}[SS':(L)],$$

which have an upper dotted spin-S index and an upper undotted spin-S' index, are defined inductively by replacing ρ with $\tilde{\rho}$ in (3.31).

2. Transformation Properties

We have already the transformation laws (A1.1) for ρ^μ and (3.22) for the reduction of $\otimes_2\rho^\mu$. A simple induction argument in which we use the identity (3.21) and the definition (3.31) then gives the general law,

$$\mathfrak{D}^{(S,0)}(A)\rho^{(\mu)}[SS':(L)]\mathfrak{D}^{(S',0)}(A)^\dagger = \Lambda_{(\nu)}{}^{(\mu)}(A)\rho^{(\nu)}[SS':(L)], \quad (3.32)$$

where we use the same notation as in (3.16) with $(\mu)=(\mu_1\cdots\mu_N)$. By the same kind of argument we can conclude that

$$\tilde{\rho}^{(\mu)}[SS':(L)]=\mathfrak{D}^{(S,0)}(C^{-1})\rho^{(\mu)}[SS':(L)]^* \times\mathfrak{D}^{(S',0)}(C), \quad (3.33)$$

and obtain the law

$$\mathfrak{D}^{(S,0)}(A)^{-1\dagger}\tilde{\rho}^{(\mu)}[SS':(L)]\mathfrak{D}^{(S',0)}(A)^{-1} = \Lambda_{(\nu)}{}^{(\mu)}(A)\tilde{\rho}^{(\nu)}[SS':(L)]. \quad (3.34)$$

Generalized ω matrices with two undotted or two dotted spin indices can be obtained by procedure in (3.15).

3. Orthogonality Relations

Again by induction, the orthogonality relations for spin $\frac{1}{2}$, (3.6), and for the reduction of $\otimes_2\rho^\mu$, (3.25), readily generalize to

$$\rho^{(\mu)}[SS':(L)]_{\alpha\dot{\beta}}\rho_{(\mu)}[JJ':(L')]_{\alpha'\dot{\beta}'} = \delta_{SJ}\delta_{S'J'}\delta_{(L)(L')}\mathfrak{D}^{(S,0)}(C)_{\alpha\alpha'}\mathfrak{D}^{(S',0)}(C)_{\dot{\beta}\dot{\beta}'} \quad (3.35)$$

for the reduction of $\otimes_N\rho^\mu$. Similarly,

$$\rho^{(\mu)}[SS':(L)]_{\alpha\dot{\beta}}\tilde{\rho}_{(\mu)}[J'J:(\tilde{L}')]^{\dot{\beta}'\alpha'} = \delta_{SJ}\delta_{S'J'}\delta_{(L)(L')}\delta_\alpha{}^{\alpha'}\delta_{\dot{\beta}}{}^{\dot{\beta}'}, \quad (3.36)$$

where $(\tilde{L}')$ is (L') with each pair of spins interchanged. Either of these equations proves that the $\rho^{(\mu)}[SS':(L)]$ span the spin-(S,S') space. Furthermore, from the first equation in (A1.6), from (3.27), and from the second orthogonality relation for Clebsch-Gordan coefficients, (3.20), one can show by induction that

$$\sum_{SS'(L)}\mathrm{Tr}\{\tilde{\rho}^{(\mu)}[S'S:(\tilde{L})]\rho^{(\nu)}[SS':(L)]\} = g^{(\mu)(\nu)}=g^{\mu_1\nu_1}\cdots g^{\mu_N\nu_N}, \quad (3.37)$$

where the trace is with respect to the matrix product in the spin-(S,S') space.

4. Irreducible Tensors

Equation (3.37) leads directly to an expansion for an arbitrary tensor of rank N into a sum of its irreducible parts. In fact, the tensors

$$\mathrm{Tr}\{\tilde{\rho}^{(\mu)}[S'S:(\tilde{L})]\rho^{(\nu)}[SS':(L)]\}$$

are for each label $[SS':(L)]$ projection operators into orthogonal, irreducible, invariant subspaces. It follows from (3.36) that they are projection operators into orthogonal subspaces; and the fact that they project into subspaces invariant under $L_+\uparrow$ follows from the transformation laws (3.32) and (3.34), which show that they are isotropic tensors with respect to $L_+\uparrow$. That they project into irreducible subspaces can be seen by noting that the ordinary Lorentz transformations, Λ, are equivalent to the representation $\mathfrak{D}^{(\frac{1}{2},0)}\otimes\mathfrak{D}^{(0,\frac{1}{2})}$, which is equivalent to $\mathfrak{D}^{(\frac{1}{2},0)}\otimes\mathfrak{D}^{(\frac{1}{2},0)*}$. Thus, the irreducible representations that occur in the reduction of the direct product $\Lambda^{(\mu)(\nu)}$ are equivalent to those that occur in the reduction of $\otimes_N\rho^\mu$. For any tensor of rank N we get

$$T^{(\mu)}=\sum_{SS'(L)}T^{(\nu)}\,\mathrm{Tr}\{\tilde{\rho}_{(\nu)}[S'S:(\tilde{L})] \times\rho^{(\mu)}[SS':(L)]\} =\sum_{SS'(L)}T^{(\mu)}[SS':(L)], \quad (3.38)$$

where $T^{(\mu)}[SS':(L)]$ are the irreducible parts of $T^{(\mu)}$.

5. The Spin Basis

In order to span the spin-(S,S') space, we can use any of the sets of matrices $\rho^{(\mu)}[SS':(L)]$ for N greater than or equal to the minimum integer such that the spins S, S' occur in the reduction of $\otimes_N\rho^{(\mu)}$. This freedom will be reduced by requiring that N actually be the minimum integer. Because at least $2S$ undotted spin-$\frac{1}{2}$ indices are needed to build up an undotted spin-S index and $2S'$ dotted spin-$\frac{1}{2}$ indices to build up a dotted spin-S' index, the minimum integer is $N=2M$, where $M=\max(S,S')$. Thus, we shall choose a set to span the spin space from among $\rho^{\mu_1\cdots\mu_{2M}}[SS':(L)]$.

In general there will still be a freedom in the choice of (L), the intermediate spins that are passed through in order to arrive at S, S'. This freedom is present, however, only in the set of left elements or the set of right elements of the pairs of spins in (L) and not in both; for either S or S' is the maximum spin that can occur in the reduction of $\otimes_{2M}\rho^\mu$; and the maximum spin can be reached in only one way. When $S=S'$ this discussion implies that (L) is uniquely determined. Furthermore, as we shall see, in that case $\rho(SS)$ is symmetric in all of its tensor indices, so that all possible orderings for carrying out the reduction give the same result. If $S\neq S'$, then there is a genuine freedom in the choice of (L) that corresponds to the occurrence of the same representation of $L_+\uparrow$ a number of times in the reduction. From the discussion in Sec. III D.4 on irreducible tensors, each choice corresponds to a particular symmetry character of the tensor indices.

A consequence of the requirement that the spin matrices have a minimum number of tensor indices is that they are traceless in the contraction of any pair of tensor indices. To see this, we suppose that $S \geqslant S'$ and write out the recursion (3.31) in full:

$$\rho^{\mu_1\cdots\mu_{2S}}[SS':(L)]_{\alpha\dot{\beta}} = \sum_{\alpha_i\gamma_i\dot{\kappa}_i\dot{\beta}_i} C(S-\tfrac{1}{2}, \tfrac{1}{2}, S; \gamma_{2S-1}, \alpha_{2S}, \alpha)C(S-1, \tfrac{1}{2}, S-\tfrac{1}{2}; \gamma_{2S-2}, \alpha_{2S-1}, \gamma_{2S-1}) \times\cdots\times C(\tfrac{1}{2},\tfrac{1}{2},1; \alpha_1,\alpha_2,\gamma_1) C(L_{2S-1},\tfrac{1}{2},S'; \dot{\kappa}_{2S-1},\dot{\beta}_{2S},\dot{\beta}) \times\cdots\times C(\tfrac{1}{2},\tfrac{1}{2},L_1; \dot{\beta}_1,\dot{\beta}_2,\dot{\kappa}_1)\rho^{\mu_1}{}_{\alpha_1\dot{\beta}_1}\cdots\rho^{\mu_{2S}}{}_{\alpha_{2S}\dot{\beta}_{2S}}. \quad (3.39)$$

There is a special symmetry of the Clebsch-Gordan coefficients; for $S_1 \geqslant S_2$, we have[14]

$$\begin{aligned}\sum_\beta C(S_1, S_2, S_1+S_2; \beta, \gamma, \alpha) &\times C(S_1-S_2, S_2, S_1; \delta, \kappa, \beta) \\ = \sum_\beta C(S_1, S_2, S_1+S_2; \beta, \kappa, \alpha) &\times C(S_1-S_2, S_2, S_1; \delta, \gamma, \beta). \quad (3.40)\end{aligned}$$

By substituting $S_2 = \frac{1}{2}$ into this equation and comparing with (3.39), we see that the sum of Clebsch-Gordan coefficients over γ_i is symmetric in the interchange of the α_i. [Symmetry in α_1, α_2 follows from (3.23).] Contraction on any pair of tensor indices gives a factor $C_{\alpha\alpha'}$, from (3.6), which is antisymmetric. Hence, the sum is zero. When $S=S'$ the sum of Clebsch-Gordan coefficients over γ_i, $\dot{\kappa}_i$ is symmetric in the interchange of the β_i indices as well, so that we then have symmetry in the interchange of the tensor indices. The symmetric and traceless property of the equal-spin matrices in their tensor indices can also be proved easily by induction, again, by the use of (3.40). Finally, it is clear that the proof of the vanishing of the traces is the same for $S < S'$.

IV. TENSOR BASES AND KINEMATICAL SINGULARITIES OF THE SCALAR AMPLITUDES

Having constructed a basis for the spin space, we can now expand any M function in the form

$$M_{(\alpha)}(K) = f_{(\mu)}(K) \prod_i \Gamma^{(\mu_i)}(S_i,S_i')_{(\alpha_i)}, \quad (4.1)$$

where the $\Gamma^{(\mu_i)}(S_i,S_i')_{(\alpha_i)}$ represent either ρ or ω spin matrices in the previous section, according to the index types; (μ_i) for fixed i represents a total of $2\max(S_i,S_i')$ tensor indices; (α_i) represents the two spin indices of the paired particles; (μ) represents the collection of all tensor indices; and (α) represents the collection of all spin indices. From the transformation laws (2.7) and (3.16), it is clear that

$$\Lambda_{(\mu)}{}^{(\nu)} f_{(\nu)}(K) = f_{(\mu)}(\Lambda K). \quad (4.2)$$

In general, the space of tensor functions $f^{(\mu)}(K)$ can be spanned by a set of tensors formed from the four-momenta K. For those cases where there are at least three linearly independent four-momenta in K, we can form a complete basis of arbitrary rank,[15] and we shall suppose for the purpose of the discussion immediately following that we have done so.

A. The Tensor Basis

We suppose that we have introduced a set of tensors of rank N, functions of the four-momenta K,

$$T^{\mu_1\cdots\mu_N}(K; i_1\cdots i_N) = T^{(\mu)}[K; (i)],$$
$$i_j = 1, 2, 3, 4, \quad (i) = i_1\cdots i_N,$$

and the reciprocal set $\hat{T}^{(\mu)}[K; (i)]$ defined by

$$\begin{aligned}\sum_{(i)} T^{(\mu)}[K; (i)]\hat{T}^{(\nu)}[K; (i)] &= g^{(\mu)(\nu)}, \\ T^{(\mu)}[K; (i)]\hat{T}_{(\mu)}[K; (j)] &= \delta_{(i)(j)},\end{aligned} \quad (4.3)$$

such that each tensor satisfies (4.2). We shall see how to form these tensors for two-body processes in Sec. IV C. To form a basis for the M functions, we combine the tensor and spin spaces to obtain functions,

$$Y^{(i)}(K) = T_{(\mu)}[K; (i)] \otimes_j \Gamma^{(\mu_j)}(S_j,S_j'), \quad (4.4)$$

which then transform as the M functions. Finally, we expand the M functions in terms of this basis,

$$M(K) = \sum_{(i)} A^{(i)}(K) Y^{(i)}(K), \quad (4.5)$$

where $A^{(i)}(K)$ are scalars under $L_+\uparrow$.

B. Determination of Scalar Amplitudes

In general, there are considerably more of the labels (i) than the dimension of the spin space. As we have seen in Secs. III D.4 and III D.5, there are symmetries among the tensor indices of $\Gamma^{(\mu)}$, which means that not all of the $T^{(\mu)}[K; (i)]$ are needed to span the tensor space. Thus, Eq. (4.5) does not determine a unique set of scalar coefficients. There is, however, a natural way to impose a set of subsidiary relations among the $A^{(i)}(K)$ so that they become determined.

First, we summarize the orthogonality relations among the spin matrices, (3.17), by the notation

$$\Gamma^{(\mu)}(S,S')_{(\alpha)} \tilde{\Gamma}_{(\mu)}(S,S')^{(\beta)} = \delta_{(\alpha)}{}^{(\beta)}, \quad (4.6)$$

where $\tilde{\Gamma}$ represents the appropriate $\tilde{\rho}$ or ω matrices with upper indices. Then, we define reciprocal basis functions

$$\tilde{Y}^{(i)}(K)^{(\alpha)} = \hat{T}_{(\mu)}[K; (i)] \prod_j \tilde{\Gamma}^{(\mu_j)}(S_j,S_j')^{(\alpha_j)}, \quad (4.7)$$

which satisfy orthogonality relations

$$\sum_{(i)} Y^{(i)}(K)_{(\alpha)} \tilde{Y}^{(i)}(K)^{(\beta)} = \delta_{(\alpha)}{}^{(\beta)}. \quad (4.8)$$

If we now require that the scalar amplitudes be defined

[14] This relation is probably known, although we have never run across it. In any event, it is a straightforward calculation from formula (11.18) in U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959).

[15] Given three independent four-vectors, one can always form a fourth by taking the skew product. See Sec. IV D.

by the equation

$$A^{(i)}(K)\equiv M_{(\alpha)}(K)\,\tilde{Y}^{(i)}(K)^{(\alpha)}, \quad (4.9)$$

we get an identity upon substituting into (4.5) and applying (4.8). This is precisely equivalent to requiring that the solutions for the scalar amplitudes in (4.8) satisfy the set of linear relations

$$A^{(i)}(K)=\sum_{(j)} A^{(j)}(K)Y^{(j)}(K)_{(\alpha)}\,\tilde{Y}^{(i)}(K)^{(\alpha)}, \quad (4.10)$$

which just suffices to determine them uniquely.

Equation (4.9) is basic for the study of kinematical singularities in the scalar amplitudes. It is clear that any singularities of $A^{(i)}(K)$ not possessed by $M(K)$ must come from $\tilde{Y}^{(i)}(K)$.[16] If the tensor basis $T^{(\mu)}[K;(i)]$ in (4.4) is constructed from polynomials in the momentum components, it will be holomorphic, and both $\hat{T}^{(\mu)}[K;(i)]$ and $\tilde{Y}^{(i)}(K)$ will be meromorphic with poles at those points where the $T^{(\mu)}[K;(i)]$ become linearly dependent. The question is whether there exists a set of $T^{(\mu)}[K;(i)]$ such that the poles do not appear in the $A^{(i)}(K)$. We do not attempt to give an answer for the many-particle case; and we give only a discussion for the case of two-body reactions, to which we restrict ourselves from now on.

C. Special Bases for Two-Particle Reactions

It is straightforward to obtain Y functions for two-particle scattering systems such as are described by Fig. 1. A method for constructing a tensor basis has been given by Hearn,[5] and several examples of spin-$\frac{1}{2}$ basis functions have been worked out by Stapp.[1] The simplest method is to construct a set of four independent four-vectors, $v^\mu(i)$, $i=1, 2, 3, 4$, in the region where at least three of the momenta are linearly independent, and then to construct a tensor basis

$$T^{\mu_1\cdots\mu_N}(i_1\cdots i_N)=v^{\mu_1}(i_1)\cdots v^{\mu_N}(i_N). \quad (4.11)$$

Unfortunately, as we shall see in Sec. IV D, this procedure appears to lead to kinematical poles for higher spins.

A special basis for spin-$\frac{1}{2}$, spin-0 scattering having definite signature under P and T is[1]

$$\begin{aligned}
Y^1&=[(k_1/m_1)-(k_3/m_3)]\cdot\sigma, \quad \sigma_P=+1, \quad \sigma_T=+1,\\
Y^2&=[(k_1/m_1)+(k_3/m_3)]\cdot\sigma, \quad \sigma_P=-1, \quad \sigma_T=-1,\\
Y^3&=n\cdot\sigma-(k_3\cdot\sigma/m_3)n\cdot\tilde{\sigma}(k_1\cdot\sigma/m_1), \quad \sigma_P=+1, \quad \sigma_T=+1,\\
Y^4&=n\cdot\sigma+(k_3\cdot\sigma/m_3)n\cdot\tilde{\sigma}(k_1\cdot\sigma/m_1), \quad \sigma_P=-1, \quad \sigma_T=+1,
\end{aligned} \quad (4.12)$$

where $n=k_2-k_4$, particles 1, 3 have spin, particles 1, 2 are incoming, and particles 3, 4 are outgoing. We show in Appendix II that the R amplitudes obtained with this basis are the same ones obtained from four-component spinors by

$$R=2u(-k_3)(A+\gamma\cdot nB)u(k_1). \quad (4.13)$$

In particular, the amplitudes A^1 and A^3 coincide with A and B.

We can define a set of four-vectors $s^\mu(i)_{3,1}$ so that the basis (4.12) becomes

$$Y^i{}_{3,1}=s(i)_{3,1}\cdot\rho, \quad (4.14)$$

where the subscripts, 3, 1, refer to the particles with spin. A basis for four spin-$\frac{1}{2}$ particles can then be obtained in the form

$$Y^{ij}=Y^i{}_{3,1}\otimes Y^j{}_{4,2}=s^\mu(i)_{3,1}s^\nu(j)_{4,2}\rho_\mu\otimes\rho_\nu, \quad (4.15)$$

where $s(j)_{4,2}$ is obtained from $s(j)_{3,1}$ by the interchange $k_1\leftrightarrow k_2$, $k_3\leftrightarrow k_4$. There will be an appropriate reduction in the number of basis functions if P and T symmetry are imposed.[1] This kind of basis is a slight generalization of the one obtained from (4.11).

Although the Y functions obtained from (4.11) or generalizations of (4.11), such as in the example (4.15), are not necessarily the best from the viewpoint of kinematical poles, they have certain advantages. The complete Y function defined by (4.4) splits into a tensor product of two Y functions for the two spin pairs. These functions will be described by the notation

$$Y(SS';i_1\cdots i_{2M})=v^{\mu_1}(i_1)\cdots v^{\mu_{2M}}(i_{2M})\Gamma_{(\mu)}(S,S'), \quad (4.16)$$

where $M=\max(S,S')$. They are multilinear in their four-vector arguments, and in Sec. IV D we shall see that they have simple inversion properties. When the four-vector arguments are expressed in relativistic spherical coordinates they become a generalization of spherical harmonics to many arguments and to the relativistic case, except for normalization and possible phases.[17] Because of the method of construction of the ρ and ω matrices, for example in (3.30), the two-spin Y functions satisfy recursion formulas analogous to those for spherical harmonics. If $S>S'>0$, we have as an example[18]

[16] This method of analyzing the kinematical singularities is the analog in the M-function formalism of the method given by GGMW.

[17] For a discussion of relativistic spherical functions see A. Z. Dolginov, Soviet Phys.—JETP **3**, 589 (1956); A. Z. Dolginov and I. N. Toptygin, *ibid.* **10**, 1022 (1960); and A. Z. Dolginov and A. N. Maskalev, *ibid.* **10**, 1202 (1960). Relativistic spherical coordinates are obtained from $t=\rho\cosh\alpha$, $r=\rho\sinh\alpha$, where $t^2-r^2>0$. We shall not give the details of the connection in this paper. For a discussion of this generalization in the nonrelativistic case, see R. Spitzer and H. P. Stapp, Phys. Rev. **109**, 540 (1958).

[18] Equations such as (4.17) implicitly express a spinor interpretation of the Clebsch-Gordan coefficients. In line with the discussion in Sec. III B.1 and Eq. (3.21) one can regard the

$$Y(S+\tfrac{1}{2},S'-\tfrac{1}{2};i_1\cdots i_{2S+1})_{\alpha\dot{\beta}}=\sum_{\gamma\gamma'\dot{\kappa}\dot{\kappa}'}C(S,\tfrac{1}{2},S+\tfrac{1}{2};\gamma,\gamma',\alpha)\times C(S',\tfrac{1}{2},S'-\tfrac{1}{2};\dot{\kappa},\dot{\kappa}',\dot{\beta})Y(SS';i_1\cdots i_{2S})_{\gamma\dot{\kappa}}\times Y(\tfrac{1}{2}\,\tfrac{1}{2};i_{2S+1})_{\gamma'\dot{\kappa}'}. \quad (4.17)$$

More generally, one can start from four-spin basis functions $Y(\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2};ij)$ that are not direct products of two $Y(\frac{1}{2}\,\frac{1}{2};i)$ functions and define recursive sum rules such as

$$Y[S_4S_3S_2S_1;(i)(j)]_{\alpha\gamma\dot{\beta}\dot{\kappa}}=\sum_{\delta\delta'\dot{\lambda}\dot{\lambda}'}\sum_{\epsilon\epsilon'\dot{\mu}\dot{\mu}'}C(J_4,L_4,S_4;\delta,\delta',\alpha)C(J_3,L_3,S_3;\epsilon,\epsilon',\gamma)C(J_2,L_2,S_2;\dot{\lambda},\dot{\lambda}',\dot{\beta})\times C(J_1,L_1,S_1;\dot{\mu},\dot{\mu}',\dot{\kappa})Y[J_4J_3J_2J_1;(i)]_{\delta\epsilon\dot{\lambda}\dot{\mu}}Y[L_4L_3L_2L_1;(j)]_{\delta'\epsilon'\dot{\lambda}'\dot{\mu}'}, \quad (4.18)$$

where one must, of course, keep track of the coupling scheme. We shall write this in the short form

$$Y[(S),(ij)]_{(\alpha)}=\mathcal{C}(J,L,S;\beta,\beta',\alpha)\times Y[(J),(i)]_{(\beta)}Y[(L),(j)]_{(\beta')}. \quad (4.19)$$

The reduction is shown schematically in Fig. 3 for a simple case; the right-hand side corresponds to the binomial expansion.

D. Kinematical Poles

A reciprocal basis to (4.11) is easily obtained from the reciprocals of the four-vectors $v(i)$, which are given by

$$\hat{v}(1)=[v(2)v(3)v(4)]/d,$$
$$\hat{v}(2)=-[v(1)v(3)v(4)]/d,$$
$$\hat{v}(3)=[v(1)v(2)v(4)]/d,$$
$$\hat{v}(4)=-[v(1)v(2)v(3)]/d,$$

with

$$d=\epsilon_{\mu\nu\lambda\rho}v^\mu(1)v^\nu(2)v^\lambda(3)v^\rho(4), \quad (4.20)$$

where $[xyz]_\mu=\epsilon_{\mu\nu\lambda\rho}x^\nu y^\lambda z^\rho$. From these definitions it is clear that

$$v(i)\cdot\hat{v}(j)=\delta_{ij},$$

and

$$\sum_i v^\mu(i)\hat{v}^\nu(i)=\sum_i \hat{v}^\mu(i)v^\nu(i)=g^{\mu\nu}. \quad (4.21)$$

The tensor basis reciprocal to (4.11) is then

$$\hat{T}^{\mu_1\cdots\mu_N}(i_1\cdots i_N)=\hat{v}^{\mu_1}(i_1)\cdots\hat{v}^{\mu_N}(i_N), \quad (4.22)$$

and the $\tilde{Y}$ functions are obtained by substituting into (4.7).

From (4.9) it is clear that any kinematical singularities in the scalar amplitudes must be poles coming from the vanishing of d, provided the $v(i)$ are polynomials in the components of the four-momenta. The determinant d vanishes if an only if the four-vectors $v(i)$ become linearly dependent.[19]

Four-momentum conservation implies that only three of the momenta can be independent for two-particle reactions. For the purpose of this discussion we shall choose

$$v(i)=k_i \quad \text{for} \quad i=2,3,4; \quad v(1)=[k_2k_3k_4]. \quad (4.23)$$

Clearly $v(1)$ is independent of k_2, k_3, k_4, and $v(1)=0$ if and only if k_2, k_3, k_4 are linearly dependent. Equation (4.20) implies that

$$d=v(1)\cdot v(1)=\det(k_i\cdot k_j) \quad \text{for} \quad i,j=2,3,4,$$

which is familiar in the analysis of scattering kinematics.[20] We shall write d in the form

$$d=\tfrac{1}{4}(stu-sa^2-tb^2-uc^2+2abc), \quad (4.24)$$

with

$$a=(m_1^2+m_2^2-m_3^2-m_4^2)/2,$$
$$b=(m_1^2+m_3^2-m_2^2-m_4^2)/2,$$
$$c=(m_1^2+m_4^2-m_2^2-m_3^2)/2,$$

where we have used the identity $s+t+u=\sum_i m_i^2$. This reduces in the equal-mass case, as usual, to $(stu/4)$. It can be shown that the basis in (4.12) or (4.14) has a determinant proportional to d.

To analyze the kinematical singularities induced in the scalar amplitudes by this basis, we consider first the case of spin-0, spin-$\frac{1}{2}$ scattering—of which pion-nucleon scattering is an example. It is well known that the scalar amplitudes in (4.13) have no kinematical singularities,[21] and our discussion should be regarded as an illustration of how such a proof goes in the M-function formalism.

The basis functions and their reciprocals are

$$Y^i=v(i)\cdot\rho, \quad \tilde{Y}^i=\hat{v}(i)\cdot\tilde{\rho}, \quad (4.25)$$

and the scalar amplitudes are

$$\mathrm{Tr}[M\tilde{\rho}^\mu]\hat{v}_\mu(i)=a^i(s,t,u)\equiv f^\mu(K)\hat{v}_\mu(i). \quad (4.26)$$

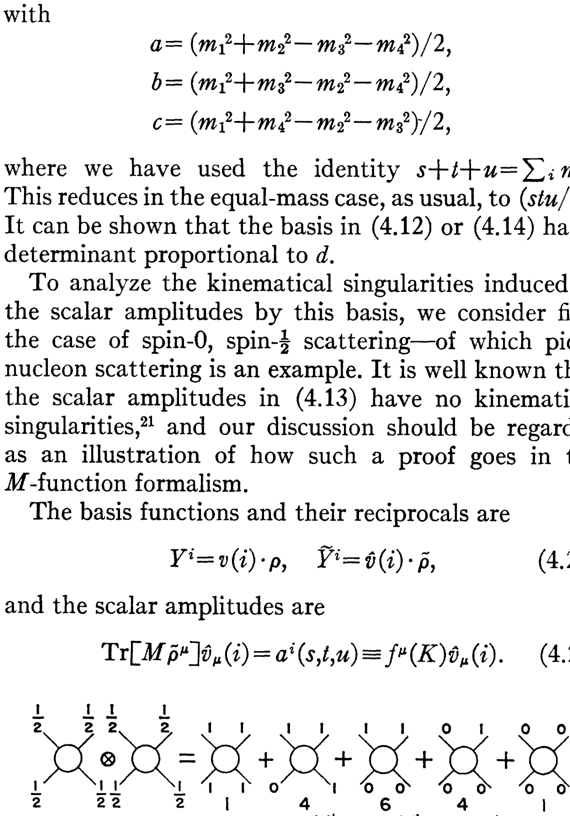

FIG. 3. Decomposition of direct products of spin-$\frac{1}{2}$ basis functions.

Clebsch-Gordan coefficients as spinors with the first two indices upper and the third lower, or with the first two indices lower and the third upper. This property is already familiar from the rotation group and is mentioned, for example, by E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), Chap. 24, pp. 292–296.

[19] In this statement we are taking it for granted that at least one of the four-vectors is inside the light cone and that all three of the vectors are real. When the vectors become complex the statement is no longer true in general, and our analysis of the kinematical singularities must, therefore, be considered heuristic. The essential result, however, survives a more careful treatment.

[20] See, e.g., T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

[21] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), Chap. 5.

We suppose that we can invoke the Hall-Wightman theorem[22] in a manner similar to that of GGMW[4] to infer that $a^i(s,t,u)$ is holomorphic except for the dynamical singularities from M and the kinematical poles in $\hat{v}$. We shall prove that the poles are not present.

Disregarding the dynamical singularities, we see by inspection of (4.20) and (4.26) that

$$\lim_{d\to 0} a^i(s,t,u)d=0, \qquad (4.27)$$

because $\lim v(1)=0$. Using (4.24) and eliminating u, we can write d in the form

$$d=-\tfrac{1}{4}t[s-s_+(t)][s-s_-(t)]. \qquad (4.28)$$

Thus, except for a finite number of values of t where $s_+(t)=s_-(t)$ or possibly $t=0$, $a^i(s,t)$ can have only simple poles in s. But from (4.27), $a^i(s,t)$ must have at least a simple zero in s for the same values. By applying the argument to each variable in turn, we find that at most $a^i(s,t,u)$ can have poles at a finite number of values of its argument. But this is impossible because a function of several complex variables cannot have isolated poles. [See, for example, the lecture notes of H. J. Bremermann, *Complex Analysis in Several Variables* (University of California Press, Berkeley, 1962) p. 91.]

This proof does not, however, generalize for a product basis to higher spin or to the case where more than two particles have spin. In general, if we write

$$\mathrm{Tr}[M\bar{\Gamma}^{(\mu_1)}(S_1,S_1')\otimes\bar{\Gamma}^{(\mu_2)}(S_2,S_2')]=f^{(\mu)}(K), \qquad (4.29)$$

which is holomorphic except for dynamical singularities, and

$$a(s,t,u;i_1\cdots i_N)=f_{(\mu)}(K)\hat{v}^{\mu_1}(i_1)\cdots\hat{v}^{\mu_N}(i_N), \qquad (4.30)$$

the best that we can conclude is that the coefficients (4.30) have kinematical poles of order at most d^{N-1}. That we can, in fact, get poles from a product basis is illustrated by the example $f_{\mu\nu}(K)=g_{\mu\nu}$. Then $a(s,t,u;ij)=\hat{v}(i)\cdot\hat{v}(j)$, and, in particular, $a(11)=\hat{v}(1)\cdot\hat{v}(1)=1/d$.

In general, considerable care must be exercised in the selection of a basis, as is already known by experience with the π-N and N-N cases. One expects from perturbation theory that such a basis exists.[5] In the N-N case, GGMW were able to obtain a proof only by doing a partial-wave analysis. Preliminary results obtained by one of us (DNW) indicate that this is not necessary, and that in fact a complete solution can be given for the problem of finding a basis leading to singularity-free amplitudes for two-body reactions. The details will be given in a second paper. For the purpose of the continuation in total angular momentum, we need only assume that there exists some basis formed from polynomials in the momentum components such that the scalar coefficients have only dynamical singularities; and that will be our procedure.

V. ANALYTIC CONTINUATION IN TOTAL ANGULAR MOMENTUM

The analytic continuation in total angular momentum J is most conveniently done in terms of the helicity amplitudes $H_{\mu'\lambda',\dot{\mu}\dot{\lambda}}$, which have simple projection properties in terms of the partial-wave helicity amplitudes $h_{\mu'\lambda',\dot{\mu}\dot{\lambda}}{}^J$.[10] In the s channel, for example, we have

$$H_{\mu'\lambda',\dot{\mu}\dot{\lambda}}=\frac{1}{2(qq')^{1/2}}\sum_{J=0}^{\infty}(2J+1)h_{\mu'\lambda',\dot{\mu}\dot{\lambda}}{}^J(s)\times\exp[i(\dot{\lambda}-\dot{\mu})\phi]\exp[-i(\lambda'-\mu')\phi]\times d^J(\theta)_{\lambda'-\mu',\dot{\lambda}-\dot{\mu}}, \qquad (5.1)$$

where q' and q are the magnitudes of the final and initial c.m. momenta, and where we have introduced the convention that the upper undotted index of the d^J matrix shall be written as lower dotted because both have the same transformation property for rotations.

Equation (5.1) can be formally transformed into a Sommerfeld-Watson representation

$$H_{(\lambda)}=-\frac{1}{2(qq')^{1/2}}\left\{\frac{1}{2i}\int_{\mathrm{Re}J=N}\frac{(2J+1)h_{(\lambda)}{}^J(s)d^J(\theta)_{\Delta\lambda',\Delta\dot{\lambda}}(-1)^J}{(\sin\pi J)}+\pi\sum_n\frac{[2\alpha(n)+1]\beta_{(\lambda)}(n,s)d^{\alpha(n)}(\pi-\theta)_{\Delta\lambda',-\Delta\dot{\lambda}}(-1)^{\Delta\lambda'}}{\sin\pi\alpha(n)}\right\}, \qquad (5.2)$$

where $\alpha(n)=\alpha(n,s)$ and $\beta(n,s)$ are the position and residue of the nth Regge pole of the partial-wave helicity amplitude $h^J(s)$. We wish now to establish that there is a unique analytic continuation of the partial-wave helicity amplitudes from the physical values of J.

Let us denote, in the center-of-mass system, the scattering angle of the outgoing particles 3 and 4 (Fig. 1) by θ and $\pi-\theta$. We shall put $\phi=0$ without any loss of generality. Also let us denote the helicities of particles 1, 2, 3, and 4 by $\dot{\lambda}$, $\dot{\mu}$, λ', and μ'. We write the M functions in the form of Eq. (4.5)

$$M_{(\alpha)}(K)=\sum_{(i)}A^{(i)}Y^{(S)(i)}(K)_{(\alpha)}. \qquad (5.3)$$

Here (i) labels the scalar amplitudes; $(S)=(S_4,\cdots,S_1)$ are the spins; and $(\alpha)=(\alpha_4,\alpha_3,\dot{\alpha}_2,\dot{\alpha}_1)$ are the spinor indices. The helicity amplitudes are given according to Eqs. (2.9) and (2.10), by

[22] D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **31**, No. 5 (1957).

$$H=\mathfrak{D}^{(S_4,0)}[\exp(i\theta'\sigma_2/2)(-k_4\cdot\tilde{\sigma}/m_4)^{1/2}]$$
$$\otimes\mathfrak{D}^{(S_3,0)}[\exp(i\theta\sigma_2/2)(-k_3\cdot\tilde{\sigma}/m_3)^{1/2}]$$
$$\otimes\mathfrak{D}^{(S_2,0)}[\exp(i\pi\sigma_2/2)(k_2\cdot\tilde{\sigma}/m_2)^{1/2}]^*$$
$$\otimes\mathfrak{D}^{(S_1,0)}[(k_1\cdot\tilde{\sigma}/m_1)^{1/2}]^*\sum_{(i)}A^{(i)}Y^{(S)(i)}, \quad (5.4)$$

where $\theta'=\pi-\theta$. The angles in the center-of-mass frame of particles 1 and 2 have been taken to be zero and π, respectively.

We want now to separate the θ-dependent part of $H_{(\lambda)}$. To see how the first two matrices act, we consider firstly the case $S_1=S_3=\frac{1}{2}$, $S_2=S_4=0$. The general case will be built up from here.

Consider the basis, Eq. (4.12), for the spin-$(\frac{1}{2},0)$ system. We first evaluate the R-basis functions, $R^{(i)}$,

$$R_{\lambda'\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}(i)}=\left(\frac{-k_3\cdot\tilde{\sigma}}{m_3}\right)^{\frac{1}{2}\,\alpha'}_{\lambda'}\left(\frac{k_1\cdot\tilde{\sigma}}{m_1}\right)^{\frac{1}{2}\,\dot{\alpha}}_{\dot{\lambda}}Y_{\alpha'\dot{\alpha}}{}^{\frac{1}{2}\frac{1}{2}(i)},$$

where $R=\sum_{(i)}A^{(i)}R^{(i)}$, or, in matrix form,

$$R^{\frac{1}{2}\frac{1}{2}(i)}=\left(\frac{-k_3\cdot\tilde{\sigma}}{m_3}\right)^{1/2}Y^{\frac{1}{2}\frac{1}{2}(i)}\left(\frac{k_1\cdot\tilde{\sigma}}{m_1}\right)^{1/2}. \quad (5.5)$$

We obtain in the center-of-mass frame (after some calculation):

$$R^{\frac{1}{2}\frac{1}{2}(1)}=(m_1m_3)^{-1/2}\{[(E_1+m_1)(E_3+m_3)]^{1/2}-\hat{q}'\cdot\boldsymbol{\sigma}\hat{q}\cdot\boldsymbol{\sigma}[(E_1-m_1)(E_3-m_3)]^{1/2}\},$$
$$R^{\frac{1}{2}\frac{1}{2}(2)}=(m_1m_3)^{-1/2}\{[(E_1-m_1)(E_3+m_3)]^{1/2}\hat{q}\cdot\boldsymbol{\sigma}-[(E_1+m_1)(E_3-m_3)]^{1/2}\hat{q}'\cdot\boldsymbol{\sigma}\},$$
$$R^{\frac{1}{2}\frac{1}{2}(3)}=(m_1m_3)^{-1/2}\{[E_2+E_4][(E_1+m_1)(E_3+m_3)]^{1/2}+q[(E_1-m_1)(E_3+m_3)]^{1/2}+q'[(E_1+m_1)(E_3-m_3)]^{1/2}$$
$$+\hat{q}'\cdot\boldsymbol{\sigma}\hat{q}\cdot\boldsymbol{\sigma}([(E_1-m_1)(E_3-m_3)]^{1/2}[E_2+E_4]+q'[(E_1-m_1)(E_3+m_3)]^{1/2}+q[(E_1+m_1)(E_3-m_3)]^{1/2})\}, \quad (5.6)$$

and

$$R^{\frac{1}{2}\frac{1}{2}(4)}=-(m_1m_3)^{-1/2}\{\hat{q}\cdot\boldsymbol{\sigma}(q[(E_1+m_1)(E_3+m_3)]^{1/2}+q'[(E_1-m_1)(E_3-m_3)]^{1/2}$$
$$+[E_2+E_4][(E_1-m_1)(E_3+m_3)]^{1/2})+\hat{q}'\cdot\boldsymbol{\sigma}(q[(E_1-m_1)(E_3-m_3)]^{1/2}+q'[(E_1+m_1)(E_3+m_3)]^{1/2}$$
$$+[E_2+E_4][(E_1+m_1)(E_3-m_3)]^{1/2})\}.$$

Here E_1, E_2, E_3, E_4 are the c.m. energies of the four particles, and $\hat{q}$ and $\hat{q}'$ are the unit initial and final momenta. In the equal-mass case these expressions simplify to

$$R^{(1)}=(1/m)[(E+m)-\hat{q}'\cdot\boldsymbol{\sigma}\hat{q}\cdot\boldsymbol{\sigma}(E-m)],$$
$$R^{(2)}=(q/m)\boldsymbol{\sigma}\cdot(\hat{q}-\hat{q}'),$$
$$R^{(3)}=(2/m)[(E+m)(2E-m)$$
$$+\hat{q}'\cdot\boldsymbol{\sigma}\hat{q}\cdot\boldsymbol{\sigma}(E-m)(2E+m)], \quad (5.7)$$

and

$$R^{(4)}=-(4qE/m)\boldsymbol{\sigma}\cdot(\hat{q}+\hat{q}').$$

These are just the standard expressions.

For the scattering of two spin-$\frac{1}{2}$ particles we have 16 combinations $R_{(\alpha)}{}^{\frac{1}{2}\frac{1}{2}(i)}(a)\ R_{(\alpha')}{}^{\frac{1}{2}\frac{1}{2}(j)}(a')$, where the argument a' indicates change of signs of $\hat{q}$ and $\hat{q}'$ and change of $E_1\leftrightarrow E_2$, $E_3\leftrightarrow E_4$, and $m_1\leftrightarrow m_2$, $m_3\leftrightarrow m_4$. There are, of course, other choices of basis functions possible that are not a direct product of two R functions.

For the spin-$(\frac{1}{2},0)$ system, we have from Eq. (5.4)

$$H_{\lambda'\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}}=\mathfrak{D}^{(\frac{1}{2},0)}[\exp(i\theta\sigma_2/2)]_{\lambda'}{}^{\beta}\sum_{(i)}A^{(i)}R_{\beta\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}(i)}. \quad (5.8)$$

Equations (5.6) show that $R^{\frac{1}{2}\frac{1}{2}(i)}$ are functions of $\boldsymbol{\sigma}\cdot\hat{q}$ and $\boldsymbol{\sigma}\cdot\hat{q}'$, which are the helicities of the two particles in the center-of-mass frame. The rotations merely diagonalize the helicities to their eigenvalues $\dot{\lambda}$ and λ'. This is, of course, precisely the meaning of the helicity amplitudes. Using the identity

$$[\exp(i\theta\sigma_2)]_{\lambda'}{}^{\alpha}(\boldsymbol{\sigma}\cdot\mathbf{q})_{\alpha\dot{\lambda}}=2\lambda' q[\exp(i\theta\sigma_2/2)]_{\lambda'\dot{\lambda}},$$

or in matrix notation

$$[\exp(i\theta\sigma_2/2)](\boldsymbol{\sigma}\cdot\mathbf{q})[\exp(-i\theta\sigma_2/2)]=\sigma_3 q, \quad (5.9)$$

we obtain from (5.8)

$$H_{\lambda'\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}}=\sum_{(i)}A^{(i)}Z^{\frac{1}{2}\frac{1}{2}(i)}(\dot{\lambda},\lambda')d^{1/2}(\theta)_{\lambda'\dot{\lambda}}, \quad (5.10)$$

where $Z^{\frac{1}{2}\frac{1}{2}(i)}$ is independent of the angle θ. It is obtained by replacing $\boldsymbol{\sigma}\cdot\hat{q}$ by 2λ and $\boldsymbol{\sigma}\cdot\hat{q}'$ by $2\lambda'$ in Eqs. (5.6). We have also used the identity

$$\mathfrak{D}^{(\frac{1}{2},0)}[\exp(i\theta\sigma_2/2)]=d^{1/2}(\theta).$$

The corresponding formula for the helicity amplitudes of four spin-$\frac{1}{2}$ particles is

$$H_{\mu'\lambda',\dot{\mu}\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}(i,j)}=\mathfrak{D}^{(\frac{1}{2},0)}\{\exp[i(\pi-\theta)\sigma_2/2]\}_{\mu'}{}^{\alpha}$$
$$\times\mathfrak{D}^{(\frac{1}{2},0)}[\exp(i\theta\sigma_2/2)]_{\lambda'}{}^{\beta}$$
$$\times R_{\alpha,-\dot{\mu}}{}^{\frac{1}{2}\frac{1}{2}(i)}R_{\beta\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}(j)}(-1)^{\frac{1}{2}-\dot{\mu}}$$

where we have used

$$\mathfrak{D}^{(J,0)}[\exp(i\pi\sigma_2/2)]_{\alpha}{}^{\beta}=(-1)^{J-\alpha}\delta_{\alpha}{}^{-\beta}.$$

Then,

$$H_{\mu'\lambda',\dot{\mu}\dot{\lambda}}{}^{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}(i,j)}=Z_{(\lambda)}{}^{\frac{1}{2}\frac{1}{2}(i,j)}d^{1/2}(\theta)_{\lambda'\dot{\lambda}}$$
$$\times d^{1/2}(\pi-\theta)_{\mu',-\dot{\mu}}. \quad (5.11)$$

The separation of the angular part in the above manner can be performed for any basis that is a polynomial in the momenta. Such a basis can be reduced to a sum of terms such as $k_i{}^\mu k_j{}^\nu$, $k_i\cdot k_j g^{\mu\nu}$, and $\epsilon_{\lambda\sigma\mu\nu}k_i{}^\mu k_j{}^\nu$, for example, in the case of second-rank tensors; and these multiplied with the spin basis give eventually terms like $k_i{}^\mu\sigma_\mu\otimes k_j{}^\nu\sigma_\nu$ from which the angular parts can be obtained by use of Eq. (5.9). The angular parts of the scalar products $k_i\cdot k_j$, which are polynomials in s, t, and u, are obtained by Legendre expansion. Thus, if a general polynomial basis is used, we obtain a sum of terms of the type (5.11).

The form of the transformation properties of the R and H functions is the same as that of the M functions; only the argument of $\mathfrak{D}^{(S_i,0)}$ is different. Consequently, the basis functions of the higher spin R and H functions are constructed from Eqs. (5.6) and Eq. (5.11) by means of Clebsch-Gordan coefficients in exactly the same manner as the higher spin M functions. We have, therefore, corresponding to Eqs. (4.18) and (4.19) the recursion formulas

$$R_{(\gamma)}{}^{(S)(i,j)}=\mathcal{C}(J,L,S;\alpha,\beta,\gamma)R_{(\alpha)}{}^{(J)(i)}R_{(\beta)}{}^{(L)(j)}, \quad (5.12)$$

and

$$H_{(\lambda)}{}^{(S)(i,j)}=\mathcal{C}(J,L,S;\sigma,\tau,\lambda)H_{(\sigma)}{}^{(J)(i)}H_{(\tau)}{}^{(L)(j)}. \quad (5.13)$$

We shall now exhibit the angular dependence of the higher spin helicity amplitudes. First, let us consider the spin-1 helicity amplitudes.

$$H_{(\lambda)}{}^{(1111)(ijkl)}=\sum C(\tfrac{1}{2},\tfrac{1}{2},1;\alpha',\beta',\lambda')C(\tfrac{1}{2},\tfrac{1}{2},1;\dot\alpha,\dot\beta,\dot\lambda)C(\tfrac{1}{2},\tfrac{1}{2},1;\kappa',\rho',\mu')C(\tfrac{1}{2},\tfrac{1}{2},1;\dot\kappa,\dot\rho,\dot\mu)H_{\alpha'\dot\alpha}{}^{\frac{1}{2}\frac{1}{2}(i)}(\theta)$$
$$\times H_{\beta'\dot\beta}{}^{\frac{1}{2}\frac{1}{2}(j)}(\theta)H_{\kappa'\dot\kappa}{}^{\frac{1}{2}\frac{1}{2}(k)}(\pi-\theta)H_{\rho'\dot\rho}{}^{\frac{1}{2}\frac{1}{2}(l)}(\pi-\theta). \quad (5.14)$$

From Eq. (5.10) we have

$$H_{(\lambda)}{}^{(1111)(ijkl)}=\sum C(\tfrac{1}{2},\tfrac{1}{2},1;\alpha',\beta',\lambda')C(\tfrac{1}{2},\tfrac{1}{2},1;\dot\alpha,\dot\beta,\dot\lambda)C(\tfrac{1}{2},\tfrac{1}{2},1;\kappa',\rho',\mu')C(\tfrac{1}{2},\tfrac{1}{2},1;\dot\kappa,\dot\rho,\dot\mu)Z^{\frac{1}{2}\frac{1}{2}(i)}(\alpha',\dot\alpha)Z^{\frac{1}{2}\frac{1}{2}(j)}(\beta',\dot\beta)$$
$$\times Z^{\frac{1}{2}\frac{1}{2}(k)}(\kappa',\dot\kappa)Z^{\frac{1}{2}\frac{1}{2}(l)}(\rho',\dot\rho)d^{1/2}(\theta)_{\alpha'\dot\alpha}d^{1/2}(\theta)_{\beta'\dot\beta}d^{1/2}(\pi-\theta)_{\kappa',-\dot\kappa}d^{1/2}(\pi-\theta)_{\rho',-\dot\rho}. \quad (5.15)$$

Since the Z's depend upon the helicities but not the angles, we can write the Eq. (5.15) in the form

$$H_{(\lambda)}{}^{1\cdots1}=Z^{(1111)(i)}[\alpha',\dot\alpha;\beta',\dot\beta;\kappa',\dot\kappa;\rho',\dot\rho;(\lambda)]d^{1/2}(\theta)_{\alpha'\dot\alpha}d^{1/2}(\theta)_{\beta'\dot\beta}d^{1/2}(\pi-\theta)_{\kappa',-\dot\kappa}d^{1/2}(\pi-\theta)_{\rho',-\dot\rho} \quad (5.16)$$

with the obvious definition of $Z^{(1111)(i)}$.

In this form, the equation can be generalized, and we obtain for the higher spin case

$$H_{(\lambda)}{}^{(S)(i)}=Z^{(S)(i)}[\alpha'_1,\dot\alpha_1;\cdots,\alpha'_N,\dot\alpha_N;\beta'_1,\dot\beta_1;\cdots;\beta'_M,\dot\beta_M;(\lambda)]d^{1/2}(\theta)_{\alpha'_1\dot\alpha_1}\cdots d^{1/2}(\theta)_{\alpha'_N\dot\alpha_N}$$
$$\times d^{1/2}(\pi-\theta)_{\beta'_1,-\dot\beta_1}\cdots d^{1/2}(\pi-\theta)_{\beta'_M,-\dot\beta_M}, \quad (5.17)$$

where $N=\max(2S_1,2S_3)$ and $M=\max(2S_2,2S_4)$. The $d^{1/2}(\theta)$ and $d^{1/2}(\pi-\theta)$ functions can now be recombined into a sum of single d functions multiplied by Clebsch-Gordan coefficients by using the relations

$$d^J(\theta)_{\lambda\dot\mu}=(-1)^{\lambda-\dot\mu}d^J(\theta)_{-\lambda,-\dot\mu}=(-1)^{\dot\lambda-\mu}d^J(\theta)_{\mu\dot\lambda}=(-1)^{J-\lambda}d^J(\pi-\theta)_{\lambda,-\dot\mu}$$

and

$$d^J{}_{\lambda'\dot\lambda}d^L{}_{\mu'\dot\mu}=\textstyle\sum_I C(J,L,I;\lambda',\mu',\lambda'+\mu')C(J,L,I;\dot\lambda,\dot\mu,\dot\lambda+\dot\mu)d^I{}_{\lambda'+\mu',\dot\lambda+\dot\mu}. \quad (5.18)$$

For example, in the case of spin 1, Eq. (5.16), we find

$$H_{(\lambda)}{}^{(1111)(i)}=Z^{(1111)(i)}[\alpha',\dot\alpha;\beta',\dot\beta;\kappa',\dot\kappa;\rho',\dot\rho;(\lambda)](-1)^{1-\dot\kappa-\dot\rho}\textstyle\sum_{I,I',k}C(\tfrac{1}{2},\tfrac{1}{2},I;\alpha',\beta',\alpha'+\beta')C(\tfrac{1}{2},\tfrac{1}{2},I;\dot\alpha,\dot\beta,\dot\alpha+\dot\beta)$$
$$\times C(\tfrac{1}{2},\tfrac{1}{2},I';-\kappa',-\rho',-\kappa'-\rho')C(\tfrac{1}{2},\tfrac{1}{2},I';-\dot\kappa,-\dot\rho,-\dot\kappa-\dot\rho)C(I,I',k;\alpha'+\beta',-\kappa'-\rho',\alpha'+\beta'-\kappa'-\rho')$$
$$\times C(I,I',k;\dot\alpha+\dot\beta,-\dot\kappa-\dot\rho,\dot\alpha+\dot\beta-\dot\kappa-\dot\rho)d^k(\theta)_{\alpha'+\beta'-\kappa'-\rho',\dot\alpha+\dot\beta-\dot\kappa-\dot\rho} \quad (5.19)$$

or

$$H_{(\lambda)}{}^{(1111)(i)}=\textstyle\sum_k W^{(i)}[\alpha',\dot\alpha;\beta',\dot\beta;\kappa',\dot\kappa;\rho'\dot\rho;(\lambda);k]d^k(\theta)_{\alpha'+\beta'-\kappa'-\rho',\dot\alpha+\dot\beta-\dot\kappa-\dot\rho}, \quad (5.20)$$

where

$$W^{(i)}[\alpha',\dot\alpha;\beta',\dot\beta;\kappa',\dot\kappa;\rho',\dot\rho;(\lambda);k]$$

is the coefficient of $d^k(\theta)$ in (5.19).

The general case is also of this form but with a more complicated lower index of the same form.

We are now in the position to discuss the partial-wave helicity amplitudes, which are defined by Eq. (2.12). We have

$$h_{(\lambda)}{}^J(s)=\tfrac{1}{2}(qq')^{1/2}\int_{-1}^{1}d(\cos\theta)d^J(\theta)_{\Delta\lambda',\Delta\dot\lambda}H_{(\lambda)}$$
$$=\tfrac{1}{2}(qq')^{1/2}\int_{-1}^{1}dz\,d^J(\theta)_{\Delta\lambda',\Delta\dot\lambda}\sum_{(i),l}(2l+1)A^{(i)}(l,s)d^l(\theta)_{0\dot0}\sum_k W^{(i)}[\alpha',\dot\alpha;\beta',\dot\beta;\kappa',\dot\kappa;\rho',\dot\rho;(\lambda);k]$$
$$\times d^k(\theta)_{\alpha'+\beta'-\kappa'-\rho',\dot\alpha+\dot\beta-\dot\kappa-\dot\rho}, \quad (5.21)$$

where

$$d^k(\theta)_{\alpha'+\beta'-\kappa'-\rho',\dot\alpha+\dot\beta-\dot\kappa-\dot\rho}=d^k(\theta)_{\Delta\lambda',\Delta\dot\lambda}$$

from the Clebsch-Gordan coefficients in Eq. (5.19) with the indices on d^k just $\Delta\dot\lambda$ and $\Delta\lambda'$. Finally, we combine $d^l(\theta)_{0\dot0}$ with $d^J(\theta)_{\Delta\lambda',\Delta\dot\lambda}$ to obtain

$$d^l{}_{0\dot0}d^J{}_{\Delta\lambda',\Delta\dot\lambda}=\sum_{L=|J-l|}^{J+l}C(l,J,L;0,\Delta\lambda',\Delta\lambda')C(l,J,L;\dot0,\Delta\dot\lambda,\Delta\dot\lambda)d^L{}_{\Delta\lambda',\Delta\dot\lambda},$$

and integrate each term $d^L d^k$ by using Eq. (2.14); hence,

$$h_{(\lambda)}{}^J(s)=\tfrac{1}{2}(qq')^{1/2}\sum_{(i)}\sum_{l}(2l+1)A^{(i)}(l,s)\sum_{k=|J-l|}^{J+l}W^{(i)}[(\lambda);k]C(l,J,k;0,\Delta\lambda',\Delta\lambda')C(l,J,k;0,\dot{\Delta\lambda},\Delta\lambda), \quad (5.22)$$

and, because the k values are restricted by the Clebsch-Gordan coefficients in their definition, Eq. (5.20), only a restricted number of l values of $A^{(i)}(l,s)$ contribute to each $h_{(\lambda)}{}^J$ with a given J. For example, in the case of spin-1 particles, $k=0, 1, 2$, and only three l values contribute; thus, $l=J$, $J-1$, and $J-2$, respectively. Writing the terms separately, we have

$$\begin{aligned}h_{(\lambda)}{}^J=\tfrac{1}{2}(qq')^{1/2}\textstyle\sum_{(i)}\{&(2J+1)A^{(i)}(J,s)W^{(i)}[(\lambda),0]C(J,J,0;0,\Delta\lambda',\Delta\lambda')C(J,J,0;\dot{0},\Delta\dot{\lambda},\Delta\dot{\lambda})\\&+(2J-1)A^{(i)}(J-1,s)W^{(i)}[(\lambda),1]C(J-1,J,1;0,\Delta\lambda',\Delta\lambda')C(J-1,J,1;\dot{0},\Delta\dot{\lambda},\Delta\dot{\lambda})\\&+(2J-3)A^{(i)}(J-2,s)W^{(i)}[(\lambda),2]C(J-2,J,2;0,\Delta\lambda',\Delta\lambda')C(J-2,J,2;\dot{0},\Delta\dot{\lambda},\Delta\dot{\lambda})\}. \quad (5.23)\end{aligned}$$

In the higher spin case we will have, in general, more terms of this form.

This equation and the generalization of it will now be used to define an analytic continuation of h^J in J. The Clebsch-Gordan coefficients can be continued analytically in J in terms of their closed-form expression. Note that this continuation is naturally not unique. We can take one that does not change the asymptotic behavior of $A(J,s)$ for large $|J|$ in order to make the Sommerfeld-Watson transformation (5.2) possible.

Assuming that the scalar amplitudes $A^{(i)}(s,t,u)$ satisfy the Mandelstam representation (see previous section), we obtain, in the usual way,[23] an expression for each term $A^{(i)}(J,s)$ in (5.19) suitable for analytic continuation in J:

$$A^{(i)}(J,s)=\frac{1}{\pi}\int dz\,A_t{}^{(i)}(s,z)Q_J(z)+(-1)^J\frac{1}{\pi}\int dz\,A_u{}^{(i)}(s,z)Q_J(z), \quad (5.24)$$

where $A_t{}^{(i)}$ and $A_u{}^{(i)}$ are the absorptive parts of the scalar amplitudes, $A^{(i)}$, in the t and u channels, respectively. These absorptive parts are assumed to be bounded uniformly in s by t^N and u^N, so that $A(J,s)$ is a holomorphic function of J for $\mathrm{Re}J>N$. Equation (5.24) inserted in (5.23) together with the analytically continued Clebsch-Gordan coefficients defines finally the analytic continuation of the partial-wave helicity amplitudes. Note that in various terms of h^J, J occurs in the argument of A displaced by integer units, so that the poles will occur displaced in $A(J,s)$.

VI. CONCLUSION

By an application of the theory of representations of the Lorentz group, we have shown in some detail how to extend the two-component S-matrix formalism to describe nonzero-mass particles of arbitrary spin. In the process we have obtained the generalization of the Pauli spinors to arbitrary spin and the projection operators for the irreducible subspaces of the tensors of arbitrary rank.

Although we have given a general prescription for expanding the S matrix for two-body reactions in terms of a set of basis functions, we have not given in this paper specifications for choosing the basis functions for the general case in such a way as to avoid possible kinematical poles at the boundary of the physical region. With the assumption that there exist scalar amplitudes that satisfy the Mandelstam representation, we have obtained the unique continuation in total angular momentum.

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APPENDIX I: NOTATION, CONVENTIONS, PROPERTIES OF SPINORS

Our Lorentz metric is $g_{00}=1=-g_{11}=-g_{22}=-g_{33}$; also, $\epsilon_{0123}=-1$. For matrices we use the notation M^T for transpose, $M^\dagger$ for Hermitian conjugate, M^* for complex conjugate.

A brief review of spinor calculus,[24] leads us to note a number of relations involving the Pauli matrices, σ_μ, where

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$

and

$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and the space-inverted matrices, $\tilde{\sigma}_\mu=(\sigma_0,-\boldsymbol{\sigma})$. The one-to-two homomorphism between $L_+\uparrow$ and the two-by-two unimodular group is expressed by

$$\Lambda_{\mu\nu}(\pm A)=\tfrac{1}{2}\,\mathrm{Tr}(\tilde{\sigma}_\mu A\sigma_\nu A^\dagger),$$

[23] M. Froissart, in *Proceedings of the La Jolla Conference on the Theory of Weak and Strong Interactions* (unpublished); V. N. Gribov, Soviet Phys.—JETP **14**, 1395 (1962).

[24] For these and other formulas from the spinor calculus, see, e.g., W. L. Bade and H. Jehle, Rev. Mod. Phys. **25**, 714 (1953); E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Blackie & Son Ltd., London, 1953).

and the transformation character of σ_μ, $\tilde{\sigma}_\mu$ is expressed by

$$Ax^\mu\sigma_\mu A^\dagger=[\Lambda(A)x]^\mu\sigma_\mu$$

and (A1.1)

$$A^{-1\dagger}x^\mu\tilde{\sigma}_\mu A^{-1}=[\Lambda(A)x]^\mu\tilde{\sigma}_\mu,$$

where A is a two-by-two unimodular matrix, and $A=\mathfrak{D}^{(\frac{1}{2},0)}(A)$.

For any spinor, those indices transforming according to A, A^* are written as lower undotted, lower dotted, respectively, and those transforming according to the contragredient transformations A^{-1T}, $A^{-1\dagger}$ are written as upper undotted, upper dotted, respectively. Thus, from (A1.1), σ_μ, $\tilde{\sigma}_\mu$ have indices $\sigma_{\mu\alpha\dot{\beta}}$, $\tilde{\sigma}_\mu{}^{\dot{\alpha}\beta}$. Contraction of relatively upper and lower indices of the same type is an invariant operation. We use the summation convention throughout for repeated relatively upper and lower tensor or spinor indices.

If the matrix C is defined by

$$C^{-1}=-C=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \text{(A1.2)}$$

we have the general matrix equation for any M,

$$C^{-1}M^TC=M^{-1}\det M. \qquad \text{(A1.3)}$$

The spinor indices are taken to be $C^{-1\alpha\beta}=C^{-1\dot{\alpha}\dot{\beta}}$ and $C_{\alpha\beta}=C_{\dot{\alpha}\dot{\beta}}$; and these matrices are used as raising and lowering spinors, contracting always on the right index. The matrices $\tilde{\sigma}_\mu$ satisfy the identities

$$\tilde{\sigma}_\mu=C^{-1}\sigma_\mu{}^TC=C^{-1}\sigma_\mu{}^*C. \qquad \text{(A1.4)}$$

We write the indices of the Kronecker's δ symbol in two different ways, for example, $\delta_{SS'}$, $\delta_\alpha{}^{\alpha'}$. Both mean the same thing. The indices are written as relatively upper and lower when we wish to emphasize the spinor character of the symbol.

The following equations and orthogonality relations are often useful:

$$\sigma_\mu\tilde{\sigma}_\nu=g_{\mu\nu}+\tfrac{1}{2}i\epsilon_{\mu\nu\lambda\rho}\sigma^\lambda\tilde{\sigma}^\rho,$$

and

$$\tilde{\sigma}_\mu\sigma_\nu=g_{\mu\nu}-\tfrac{1}{2}i\epsilon_{\mu\nu\lambda\rho}\tilde{\sigma}^\lambda\sigma^\rho; \qquad \text{(A1.5)}$$

$$\tfrac{1}{2}\operatorname{Tr}(\sigma_\mu\tilde{\sigma}_\nu)=g_{\mu\nu},$$

$$\tfrac{1}{2}\sigma^\mu{}_{\alpha\dot{\beta}}\sigma_\mu{}^{\dot{\beta}'\alpha'}=\delta_\alpha{}^{\alpha'}\delta_{\dot{\beta}}{}^{\dot{\beta}'},$$

and

$$\tfrac{1}{2}\sigma^\mu{}_{\alpha\dot{\beta}}\sigma_{\mu\alpha'\dot{\beta}'}=C_{\alpha\alpha'}C_{\dot{\beta}\dot{\beta}'}. \qquad \text{(A1.6)}$$

For any four-vector x we have $(x\cdot\sigma)(x\cdot\tilde{\sigma})=x\cdot x$. The Hermitian matrix $(k\cdot\sigma/m)^{1/2}$ corresponds to a Lorentz transformation from rest to the four-momentum k: $(k\cdot\sigma/m)^{1/2}=\cosh(\chi/2)+\hat{k}\cdot\boldsymbol{\sigma}\sinh(\chi/2)$, where $\hat{k}$ is the unit three-vector and χ is the "angle" of the Lorentz transformation; also, $\mathbf{k}=\hat{k}m\sinh\chi$, $k_0=m\cosh\chi$.

The representation matrices for the proper rotation group and the proper homogeneous orthochronous Lorentz group, $\mathfrak{D}^S(A)$, $\mathfrak{D}^{(S,S')}(A)$, are defined for unitary-unimodular and unimodular two-by-two matrices A, respectively, with S, S' half-integers. The matrices $\mathfrak{D}^S(A)$ are unitary; and the representation $\mathfrak{D}^S$ is unitary-equivalent to $\mathfrak{D}^{S*}$, which follows from (A1.3) and the group property. But $\mathfrak{D}^{(S,S')}(A)$ is in general not unitary, and the representation $\mathfrak{D}^{(S,S')}$ is inequivalent to $\mathfrak{D}^{(S',S)}$ unless $S=S'$. The following identities hold: $\mathfrak{D}^{(S,0)}(A)=\mathfrak{D}^{(0,S)}(A)^{-1\dagger}$; $\mathfrak{D}^{(S,0)}(A^*)=\mathfrak{D}^{(S,0)}(A)^*$; $\mathfrak{D}^{(S,0)}(A^T)=\mathfrak{D}^{(S,0)}(A)^T$. The choice $\mathfrak{D}^{(\frac{1}{2},0)}(A)=A$ is a convention. The opposite convention, $\mathfrak{D}^{(0,\frac{1}{2})}(A)=A$, is often used. If the latter convention is used, $\mathfrak{D}^{(S,0)}$ in our formulas should be replaced by $\mathfrak{D}^{(0,S)}$.

APPENDIX II: RELATION TO FOUR-COMPONENT FORMALISM

The customary introduction of the invariant scattering amplitudes has been in terms of four-component spinors. Stapp has already given the relation between his two-component M-function formalism and the four-component formalism.[1] We give here a demonstration that exhibits the relation between the corresponding scalar amplitudes for pion-nucleon scattering without isotopic spin.

According to (2.6), the M function for the situation described in (4.12) is

$$M=B_{-k_3\leftarrow p_3}RB_{k_1\leftarrow p_1}{}^\dagger. \qquad \text{(A2.1)}$$

The positive-energy solutions of the free-particle Dirac equation in momentum space can be written in the form

$$u_{\dot{\alpha}}(k)=\begin{pmatrix}B_{k\leftarrow p} & \phi_{\dot{\alpha}}\\ B_{k\leftarrow p}{}^{-1\dagger} & \phi_{\dot{\alpha}}\end{pmatrix}, \qquad \text{(A2.2)}$$

where $\phi_{\dot{\alpha}}$ represents two two-component vectors, which we take to be

$$\phi_{1/2}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix},\quad \phi_{-1/2}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}. \qquad \text{(A2.3)}$$

We use the following representation for the Dirac matrices:

$$\gamma_\mu=\begin{pmatrix}0 & \sigma_\mu\\ \tilde{\sigma}_\mu & 0\end{pmatrix}. \qquad \text{(A2.4)}$$

Then, writing

$$\bar{u}_\alpha=u_{\dot{\alpha}}{}^\dagger\gamma_0, \qquad \text{(A2.5)}$$

$$R_{\alpha\dot{\beta}}=2\bar{u}_\alpha(-k_3)Tu_{\dot{\beta}}(k_1), \qquad \text{(A2.6)}$$

and evaluating (A2.1) using (A2.2) and (A2.5), we obtain

$$M=T_{11}\frac{k_1\cdot\sigma}{m_1}-\frac{k_3\cdot\sigma}{m_3}T_{22}+T_{12}-\frac{k_3\cdot\sigma}{m_3}T_{21}\frac{k_1\cdot\sigma}{m_1}, \qquad \text{(A2.7)}$$

where T_{ij} are the two-by-two blocks of the T matrix. From (4.13) and (A2.4), these are given by

$$T_{11}=T_{22}=A,\quad T_{12}=B\sigma\cdot n,\quad T_{21}=B\tilde{\sigma}\cdot n. \qquad \text{(A2.8)}$$

The M function (A2.7) thus agrees completely with the M function given by the basis (4.12) in the P- and T-conserving case, where $A=A^1$ and $B=A^3$.