Extension of the Regge Representation*

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(Received 19 November 1962)

The Regge formula is modified in such a way as to exhibit the "full" contribution of each Regge pole to the scattering amplitude. In this modified form both the contribution from each pole and the new background term have the correct cuts in the z plane. In the case where the partial wave amplitude is meromorphic in the whole l plane we show that, under certain assumptions, the scattering amplitude can be represented by a series sum of contributions from the Regge poles. Each contribution has the correct cut in the z plane, and the series converges for all z in the cut plane. An approximation of the scattering amplitude at low energies in terms of a few contributions from leading poles is discussed. Finally, it is shown that this modified Regge formula leads to a relatively simple bootstrap procedure for constructing the scattering amplitude from unitarity and analyticity.

I. INTRODUCTION

THE nature of the behavior of the scattering amplitude in potential scattering for large values of momentum transfer was first given by the work of Regge.[1] The conjecture has been made that results similar to Regge's may also be true in relativistic elementary particle scattering where now the momentum transfer of one channel is the total energy in the crossed channel.[2,3]

Regge's method consisted of examining the analytic properties of the partial wave amplitude as a function of angular momentum. He showed that the amplitude was a meromorphic function of l in the half-plane $\mathrm{Re}\,l \geq -\frac{1}{2}$. Using this result and the Watson-Sommerfeld transformation he was able to modify the partial wave series and obtain the representation given in Eq. (4) below. This representation consists of two terms. The first a background integral which for large z vanishes as $z^{-1/2}$. The second is a sum of contributions from the poles in the angular momentum plane which are proportional to $P_{l_n}(-z)$ and which determine the behavior for large z. The position of the poles l_n is dependent on the energy.

One can show that bound states and resonances are associated with poles in the angular momentum plane. This fact leads to the conjecture that all elementary particles and resonances are associated with moving poles in the angular momentum plane.[2] Following this conjecture it would be interesting to consider the possibility of approximating the scattering amplitude by contributions from a few poles for all z and not just large z.

For any such investigation the Regge representation in its usual form is not very useful because one knows very little about the so-called background term. Furthermore, the contributions of the poles as given in Regge's formula have the cut in the z plane starting at the wrong threshold.

In this paper we modify the Regge formula in such a way so as to exhibit the "full" contribution of each Regge pole. This contribution is shown to have the correct branch point in the z plane. The new background term in this modified Regge representation will also have the correct threshold in the z plane.

In the cases where the partial wave amplitude is meromorphic in the whole l plane one can push the contour of integration for the modified background integral to the left and replace it by a series of contributions from the left-half plane poles. The new representation thus enables us to do what in the original form could not be carried out as can be seen from Mandelstam's paper on the extension of the Regge formula.[4] What we finally achieve is a representation for the scattering amplitude as a series sum of contributions from the Regge poles where the contribution from each pole has the right threshold in z.

In Sec. III we discuss the possibility of whether the scattering amplitude for low energies can be approximated by the contributions from few leading poles. With such an approximation one can try to fit the low-energy data with a few poles and thus obtain some information about the nonresonant Regge trajectories.

Finally in Sec. IV we show that the modified form of the Regge representation leads to a simplification of the unitarity condition. It is also shown that unitarity and analyticity lead to a bootstrap procedure for calculating the weight function that appears in the new background term. This bootstrap procedure is much simpler than that connected with the Mandelstam representation.[5]

Before we get into the details we remark that many of the results of this paper can be trivially generalized to the relativistic case if one makes the necessary assumptions of meromorphy of the partial wave amplitude in the angular momentum plane.

* Work supported in part by the National Science Foundation.
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[1] T. Regge, Nuovo Cimento **14**, 951 (1959). See also A. Bottino, A. M. Longoni, and T. Regge, *ibid.* **23**, 954 (1962).
[2] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962). G. F. Chew, S. C. Frautschi, and S. Mandelstam, *ibid.* **126**, 1202 (1962). G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962).
[3] V. N. Gribov, Soviet Phys.—JETP, **14**, 1395 (1962).
[4] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 254 (1962).
[5] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N.Y.) **10**, 62 (1960).

II. EXTENSION OF THE REGGE FORMULA

Our starting point is the Regge representation of the scattering amplitude in potential scattering. Regge[1] showed that for potentials which are superpositions of Yukawa potentials; i.e.,

$$rV(r)=\int_m^\infty \sigma(\mu)e^{-\mu r}d\mu, \quad (1)$$

the partial wave scattering amplitude $A(l,s)$ is meromorphic in l in the half-plane $\mathrm{Re}l>-\frac{1}{2}$. Here s is the usual energy variable and in this section we are concerned only with physical, real and positive values of s. Furthermore, Regge proved that as $|l|\to\infty$ in the half plane A takes the following asymptotic form:

$$A(\lambda,s)\sim(C(s)/\sqrt{\lambda})e^{-\lambda\xi},\quad |\lambda|\to\infty. \quad (2)$$

Here we have used $\lambda\equiv l+\frac{1}{2}$ and ξ is given by

$$\cosh\xi=1+m^2/2s, \quad (3)$$

where m is the lower limit in (1). Using these two results and the Watson transform of the partial wave expansion, Regge obtained for the scattering amplitude $f(s,z)$ the representation

$$f(s,z)=-i\int_{-i\infty}^{+i\infty}\lambda d\lambda\, P_{\lambda-\frac{1}{2}}(-z)\frac{A(\lambda,s)}{\cos\pi\lambda} -\pi\sum_{n=1}^{N}\frac{\beta_n(s)P_{\lambda_n-\frac{1}{2}}(-z)2\lambda_n}{\sin\pi l_n}, \quad (4)$$

where β_n is the residue of $A(\lambda,s)$ at the pole $\lambda=\lambda_n=l_n+\frac{1}{2}$. The number of poles for $\mathrm{Re}\lambda>0$ is finite and is denoted here by N.

If one now continues (4) to unphysical or complex values of z then each of the two terms in (4) will have a cut starting at $z=1$. However, it is known that $f(s,z)$ has a cut which starts at $z=\cosh\xi=1+m^2/2s$.[5] Evidently, some cancellation must occur between the two terms in (4) and we shall seek a representation which among other things explicitly exhibits this cancellation.

For this purpose we note the following representation for the Legendre functions $P_{\lambda-\frac{1}{2}}(z)$,[6]

$$\frac{\pi P_{\lambda-\frac{1}{2}}(z)}{\cos\pi\lambda}=\sqrt{2}\int_0^\infty\frac{\cosh\lambda x}{(\cosh x+z)^{1/2}}dx =\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}\frac{e^{\lambda x}}{(\cosh x+z)^{1/2}}dx. \quad (5)$$

This representation holds only in the restricted region $-\frac{1}{2}<\mathrm{Re}\lambda<\frac{1}{2}$. We limit ourselves for the moment to physical z, $-1<z<+1$, and integrate by parts to get

$$\frac{\pi\lambda P_{\lambda-\frac{1}{2}}(z)}{\cos\pi\lambda}=\frac{1}{(2)^{3/2}}\int_{-\infty}^{+\infty}\frac{e^{\lambda x}\sinh x}{(\cosh x+z)^{3/2}}dx. \quad (6)$$

The integral in (4) runs along the line $\mathrm{Re}\lambda=0$ and hence we can substitute the representation (6) for the Legendre function appearing in the integrand. We obtain

$$f(s,z)=\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}\frac{B(x,s)\sinh x}{(\cosh x-z)^{3/2}}dx +2\pi\sum_{n=1}^{N}\frac{\beta_n(s)P_{\lambda_n-\frac{1}{2}}(-z)\lambda_n}{\cos\pi\lambda_n}, \quad (7)$$

where now $B(x,s)$ is given by

$$B(x,s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}d\lambda\, e^{\lambda x}A(\lambda,s). \quad (8)$$

The integral defining B is essentially a Fourier transform and it is clear from (2) that it exists for all x.

For $x<\xi$, one can use (2) and the fact that $A(\lambda,s)$ is meromorphic in the right half λ plane to express B in terms of the right-hand poles of $A(\lambda,s)$. For this purpose we write

$$B(x,s)\equiv B_L(x,s)\theta(x-\xi)+B_R(x,s)\theta(\xi-x). \quad (9)$$

For $x<\xi$ we can close the contour in (8) in the right half plane and obtain

$$B(x,s)\equiv B_R(x,s)=-\sum_{n=1}^{N}\beta_n(s)e^{\lambda_n x},\quad x<\xi. \quad (10)$$

Thus (7) can now be written as

$$f(s,z)=\frac{1}{\sqrt{2}}\int_\xi^\infty\frac{B_L(x,s)\sinh x dx}{(\cosh x-z)^{3/2}} +\sum_{n=1}^{N}\beta_n(s)\left[\frac{-1}{\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{\lambda_n x}\sinh x}{(\cosh x-z)^{3/2}}dx +\frac{2\pi\lambda_n P_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}\right]. \quad (11)$$

So far we have considered only physical z, however each term in (11) can be analytically continued in z. For the first term one can easily see that the cut will start at $z=\cosh\xi=1+m^2/2s$. To avoid the difficulty with the $\frac{3}{2}$ power in the denominator one can integrate the first term by parts and obtain

$$f_L(s,z)=\sqrt{2}\int_\xi^\infty\frac{B_L'(x,s)dx}{(\cosh x-z)^{1/2}}-\sqrt{2}\frac{B_L(\xi,s)}{(\cosh\xi-z)^{1/2}}. \quad (12)$$

The last expression can now be easily continued in z

[6] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 156, (11).

and has the correct branch point. It follows now that since the second term in (11) is just a finite sum then it also must have the correct branch cut in z. In the Appendix we shall show explicitly how the cuts of the two terms in the square brackets cancel each other in the region $1<z<[1+m^2/2s]$.

It is tempting at this point to identify each term in the summation of (11) as the full contribution to $f(s,z)$ of each Regge pole in the right half plane. If we denote by $R(s,z;\lambda_n)$ the contribution of the pole at $\lambda=\lambda_n$, we have

$$R(s,z;\lambda_n)=\beta_n(s)\left[\frac{-1}{\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{\lambda_n x}\sinh x dx}{(\cosh x-z)^{3/2}}+\frac{2\pi\lambda_n P_{\lambda_n-\frac{1}{2}}(-z)}{\cos\pi\lambda_n}\right],\quad \mathrm{Re}\lambda_n>0. \quad (13)$$

For large $|z|$ the second term in (13) is the one that dominates. Also near a resonance the second term dominates. However, away from resonances or large values of $|z|$ the first term, which can be considered as the background term of a specific Regge pole, is of comparable value as the usual Regge term. For large z this background term behaves as $|z|^{-1/2}$.

The association of (13) with the full contribution of the nth Regge pole will become more meaningful if we can show that $B_L(x,s)$, i.e., the background term in (11), is determined by the left-hand singularities of $A(\lambda,s)$. That this is the case, we shall demonstrate for the case of potentials for which $A(\lambda,s)$ is meromorphic in the whole λ plane.

The properties of $A(\lambda,s)$ for $\mathrm{Re}\lambda<0$ are more complicated than those in the right half plane. In general, it is not true that $A(\lambda,s)$ is meromorphic in the half plane $\mathrm{Re}\lambda<0$. However, Mandelstam[4] and Froissart[7] have shown that for a subclass of the potentials defined by (1), $A(\lambda,s)$ is actually meromorphic in the left half plane. Let us for the moment limit ourselves to this subclass. Then one can for $x>\xi$ move the contour in (8) to the left and get

$$B_L(x,\xi)=B(x,\xi)=\sum_{n=1}^{N(L)}\beta_n(s)e^{\lambda_n x}+\frac{1}{2\pi i}\int_{-L-i\infty}^{-L+i\infty}d\lambda\, e^{\lambda x}A(\lambda,s),\quad x>\xi. \quad (14)$$

The sum represents the contributions from the poles lying in the strip, $-L<\mathrm{Re}\lambda<0$. The exponential in the integral in (14) is now a decreasing exponential and if $A(\lambda,s)$ does not blow up fast as $\lambda\to-\infty$ then one can let $L\to\infty$ and obtain for B_L a series representation in terms of contributions from the left-hand poles.

The exact behavior of $A(\lambda,s)$ as $|\lambda|\to\infty$ in the left half plane is not known. However, one can make convincing, though not fully rigorous, arguments to show it is at least bounded by the Born approximation in that region.[8] The Born term blows up exponentially in the left half plane as $\lambda\to-\infty$. We shall assume here that, excluding the neighborhoods of the poles, $A(\lambda,s)$ is bounded by an increasing exponential in λ as $|\lambda|\to\infty$ in the left half λ plane. We write

$$|A(\lambda,s)|\leq(|C'(s)|/\sqrt{|\lambda|})e^{|\mathrm{Re}\lambda|\xi}, \quad (15)$$

where ξ is positive and given by (3). Such a bound seems to be consistent at least for pure Yukawa potentials.

When (15) holds one can, for $x>\xi$, close the contour in (8) to the left and obtain

$$B_L(x,\xi)\equiv B(x,\xi)=\sum_{n'=1}^{\infty}\beta_{n'}(s)e^{\lambda_{n'}x},\quad x>\xi,\quad \mathrm{Re}\lambda_{n'}<0. \quad (16)$$

The convergence of the series in (16) is, of course, intimately connected with the validity of the inequality (15). At the end of this section we shall show that in the case of a pure Yukawa potential and for high enough energies this series does indeed converge if $x>\xi$ and diverges for $x<\xi$.

Substituting (14) or (16) in (11) one can now identify the contribution of a left-hand Regge pole to $f(s,z)$ as

$$R(s,z;\lambda_n)=\frac{\beta_n(s)}{\sqrt{2}}\int_{\xi}^{\infty}\frac{e^{\lambda_n x}\sinh x}{(\cosh x-z)^{3/2}}dx;\quad \mathrm{Re}\lambda_n<0. \quad (17)$$

This again has the correct cut in the z plane. It is also easy to check by using (6) that in the strip $-\frac{1}{2}<\mathrm{Re}\lambda_n<+\frac{1}{2}$ the representations (13) and (17) are identical. In fact, $R(s,z;\lambda)$ as given in (17) defines a function of λ which is regular in the half-plane $\mathrm{Re}\lambda<\frac{1}{2}$. The Eq. (13) provides an analytic continuation of $R(s,z;\lambda)$ to the right half plane, and $R(s,z;\lambda)$ is thus an entire function of λ. The function R does not, to the best of our knowledge, have a simple representation in terms of Legendre functions, although its integral representation (17) is very similar to that of the Legendre functions the only difference being in the limits of integration.

For potentials for which (15) holds the scattering amplitude can thus be written as

$$f(s,z)=\sum_{n=1}^{\infty}R(s,z,\lambda_n). \quad (18)$$

Here the sum extends over all the poles in the right and left half plane. For $\mathrm{Re}\lambda_n>0$, R is given by (13), and for $\mathrm{Re}\lambda_n<0$, R is given by (17). The series in (18) will converge for all physical $s\neq0$ and all z in the cut plane.

[7] M. Froissart, J. Math. Phys. 3, 922 (1962).

[8] See, for example, B. R. Desai and R. G. Newton (to be published).

We now take the partial wave projection of $R(s,z;\lambda_n)$ to obtain the contribution of the nth pole to the lth partial wave. We write

$$r(\lambda,s;\lambda_n)=\frac{1}{2}\int_{-1}^{+1}P_{\lambda-\frac{1}{2}}(z)R(s,z;\lambda_n)dz. \qquad (19)$$

Here λ is half-integral. The integral above can be easily performed if one uses the inverse of the representation (6), namely,

$$\frac{\sinh x}{(\cosh x-z)^{3/2}}=-i\sqrt{2}\int_{-i\infty}^{+i\infty}\lambda' d\lambda'\, P_{\lambda'-\frac{1}{2}}(-z)\frac{e^{-\lambda' x}}{\cos\pi\lambda'}. \qquad (20)$$

The partial wave projection turns out to be the same for (13) and for (17) and is independent of whether $\mathrm{Re}\lambda_n$ is greater than or less than zero. The result of the integration gives

$$r(\lambda,s;\lambda_n)=-\beta_n(s)e^{-(\lambda-\lambda_n)\xi}/(\lambda_n-\lambda),\quad \lambda=l+\tfrac{1}{2}. \qquad (21)$$

This result can be obtained directly when (15) holds by applying the Cauchy theorem to the function $F(\lambda,s)=A(\lambda,s)e^{\lambda\xi}$ and taking an arbitrarily large circle for the contour. However, (21) will still hold for the right-hand poles even if (15) does not hold.

It is interesting to note that $r(\lambda,s;\lambda_n)$ has the same analytic properties in s as the full partial wave amplitude for physical l. Namely, it is analytic in the cut s plane with the cuts on the real axis. The right-hand cut extends from zero to infinity and the left-hand one from $-\infty$ to $-m^2/4$. The left-hand cut is due to $\xi=\cosh^{-1}(1+m^2/2s)$. In general, $\beta_n(s)$ and $\lambda_n(s)$ have only right-hand cuts.

Another property of $r(\lambda,s;\lambda_n)$ is that, for the $\mathrm{Re}\lambda_n>0$ poles, it has the correct threshold behavior as $s\to 0$. It is known that for $\mathrm{Re}\lambda_n>0$, $\beta_n(s)\sim s^{l_n(0)}$ as $s\to 0$, where $l_n=\lambda_n-\frac{1}{2}$.[9] Substituting this in (21) and using (3), we can easily check that $r(\lambda,s;\lambda_n)\sim s^l$ as $s\to 0$. This threshold behavior is of course the same as that of $A(\lambda,s)$ for integer l, $l=\lambda-\frac{1}{2}$.

Finally, we make a few remarks about the high-energy behavior for the case of pure Yukawa potentials. In general $A(\lambda,s)$ approaches the Born approximation as $|s|\to\infty$ for $\mathrm{Re}\lambda\geq 0$. For pure Yukawa potentials the Regge trajectories, $\lambda_n(s)$, approach as $|s|\to\infty$ the trajectories of the corresponding Coulomb potentials, and we have

$$\lim_{s\to\infty}\lambda_n(s)=-n+\tfrac{1}{2},\quad n=1,2,3,\cdots. \qquad (22)$$

For a simple Yukawa potential, $-ge^{-mr}/r$, $A(\lambda,s)$ takes the following asymptotic form for $\mathrm{Re}\lambda>0$,

$$A(\lambda,s)\simeq\frac{g}{2s}Q_{\lambda-\frac{1}{2}}\left(1+\frac{m^2}{2s}\right),\quad |s|\to\infty. \qquad (23)$$

[9] V. N. Gribov and I. Ya Pomeranchuk, Phys. Rev. Letters 9, 238 (1962).

The Legendre function $Q_{\lambda-\frac{1}{2}}$ has the following expansion

$$Q_l(\eta)=-e^{-l\alpha}\sum_{n=1}^{\infty}[P_{n-1}(\eta)e^{-n\alpha}/n+l],\ \cosh\alpha=\eta. \qquad (24)$$

Substituting this in (23), and comparing the result with the sum of the contributions (21) to a given partial amplitude, we get

$$\beta_n(s)\simeq(-g/2s)P_{n-1}(1+m^2/2s),\quad |s|\to\infty. \qquad (25)$$

This asymptotic behavior of the residues β_n gives us a consistency check on (15) for high energies. From (22) and (25) one can easily see that the series (16) will converge absolutely for high energies as long as $x>\xi$, and the series will diverge for $x<\xi$. The convergence of (16) for $x>\xi$ is as we have mentioned earlier directly connected with the validity of the conjecture (15) on the asymptotic behavior of $A(\lambda,s)$ as $\mathrm{Re}\lambda\to-\infty$.

III. APPROXIMATION OF SCATTERING AMPLITUDE BY CONTRIBUTIONS FROM LEADING POLES

The technique of using complex angular momenta was first applied by Sommerfeld to the problem of the scattering of radio waves by the earth. In that problem the partial wave series was converted into a series sum of contributions from complex angular momentum poles. The latter series turned out to converge much faster than the original partial wave expansion. The question arises whether a similar situation holds for the series (18). Of course, it is well known that for large $|z|$ one term dominates in (18) and that is the one with the largest $\mathrm{Re}\lambda_n$. It is also true that if we have a resonance, i.e., a pole $l_n(s_R)$ with $\mathrm{Re}l_n$ near an integer and $\mathrm{Im}l_n$ small, then as $s\to s_R$ the contribution from the pole giving the resonance dominates. We shall investigate below whether the results of the previous section can lead to an approximation of $f(s,z)$ by one or several terms of the series (18) for any z and s lying in a low-energy domain.

The basis of such an approximation is the representation (17). The presence of the exponential in the integrand of (17) leads us to the conclusion that each Regge pole with $\mathrm{Re}\lambda_n\ll 0$ gives a small contribution to $f(s,z)$ in the domain where $\xi>1$. This will be true if the residues $\beta_n(s)$ do not grow as $\mathrm{Re}\lambda_n$ becomes large and negative. Let us for the moment assume that the β_n's do not grow for large λ_n. In fact, reference (8) contains plausible arguments which show that the β_n's decrease fast as λ_n becomes large in the left half plane. We shall return to a discussion of the β_n's at the end of this section.

Under these assumptions about the β_n's, we can see that in the series (18), for $\xi>1$, we have to take the poles in the right half plane and only those left-hand poles near the imaginary axis to get a good approximation to the amplitude.

TABLE I. Location of the leading Regge poles at low energies for a pure Yukawa potential.[a]

	$s=0.01$	$s=0.5$
$\mathrm{Re}\lambda_1$	0.6	0.5
$\mathrm{Re}\lambda_2$	−0.4	−0.8
$\mathrm{Re}\lambda_3$	−1.0	−1.5
$\mathrm{Re}\lambda_4$	−1.5	−2.6
$\mathrm{Re}\lambda_5$	−2.5	−2.9
$\mathrm{Re}\lambda_6$	−2.1	−3.7

[a] See reference 10.

The usefulness of this approximation will depend on the distribution of the poles near the imaginary axis for $\xi>1$. At first sight it might look that this procedure is doomed since ξ is given by

$$\xi=\ln\{(1+m^2/2s)+[(1+m^2/2s)^2-1]^{1/2}\}.$$

This means that ξ becomes large only for very small s. It is known that as $s\to 0$ there are an infinite number of poles near the line $\mathrm{Re}\lambda=0$.[9] However, these poles fall away rapidly from $\mathrm{Re}\lambda=0$ as s starts to increase from zero, and for pure Yukawa potentials there is a domain in s with $\xi>1$ and with the poles well away from $\mathrm{Re}\lambda=0$.

Let us take a specific example of a Yukawa potential with $m=1$, $V(r)=-ge^{-r}/r$. Ahmadzadeh, Burke, and Tate[10] have computed the first six Regge trajectories for this potential for several values of the coupling constant g. As in (22) the poles were identified by their high-energy limits.

In the domain $0.01\leq s\leq 0.5$, ξ is larger than unity and $1.3\lesssim\xi\lesssim 4.6$, where the upper limit goes with the lower value of s. We consider the case $g=2$, and tabulate the results of reference 10 for $\mathrm{Re}\lambda_n$ at $s=0.01$ and $s=0.5$. For $n>6$ the trajectories will lie farther to the left. One sees from Table I that already at $s=0.01$ the poles have moved away from the line $\mathrm{Re}\lambda=0$. It is evident from (17) that the contributions for $n\geq 4$ would be small compared to the contributions from the first three poles. For example the contribution of λ_4 at $s=0.5$ will be proportional to a factor $e^{-4.5}$.

In the region $0.01\leq s\leq 0.5$ we can thus write the following approximation

$$f(s,z)\cong\sum_{n=1}^{3} R(s,z;\lambda_n). \qquad (18')$$

Here R is given by (13) if $\mathrm{Re}\lambda_n>0$ and by (17) if $\mathrm{Re}\lambda_n<0$.

A similar approximation holds for the other two values of g calculated in reference (10), $g=0.05$ and $g=5$. For the weak coupling case one pole contribution suffices. On the other hand, for the stronger coupling, $g=5$, the situation is worse and one might need an extra term in (18′) to keep the same accuracy. For the strong coupling cases we could conclude that a one pole approximation would not be valid.

An actual calculation of the residues $\beta_n(s)$ for the trajectories computed in reference 10 would be necessary before one could make a more definite statement on the number of terms needed in (18′). If it turns out that for certain energies a few terms are enough, we would then have a basis for experimentally determining some properties of the trajectories that are not associated with any resonance.

If an approximation like (18′) holds for any physical scattering process, then one can by comparing with the data perform a pole fit at different energies instead of the usual partial wave analysis. Such a pole analysis would be more difficult than the partial wave analysis, would require better data, and probably would not lead to unique results. However, it might still give us some information on the existence and position of the invisible trajectories which do not demonstrate their presence by producing resonances at some energy or dominating the amplitude for asymptotic values of the energy in the crossed channel.

Finally, we discuss the behavior of $\beta_n(s)$ for small s. Again the region near the threshold might seem to be a danger zone because of the centrifugal barrier. As $s\to 0$, $A(\lambda,s)\sim s^{\lambda-\frac{1}{2}}$. This is true for physical λ and it is also true for all $\mathrm{Re}\lambda>0$. From this one can conclude that if $\mathrm{Re}\lambda_n>0$, $\beta_n(s)\sim s^{\lambda_n(0)-\frac{1}{2}}$ as $s\to 0$. If this same threshold behavior holds when $\mathrm{Re}\lambda_n(0)<0$, then we would be in trouble for then $\beta_n(s)$ will start growing as $s\to 0$ for the left-hand poles. However, this is not the case and in fact, for $\mathrm{Re}\lambda<0$, $A(\lambda,s)$ cannot blow up like $s^{\lambda-\frac{1}{2}}$ near threshold.

It is easy to see from the unitarity condition for $A(\lambda,s)$, given in Eq. (28) below, that for real λ and excluding the neighborhoods of the poles, $|s^{\frac{1}{2}}A(\lambda,s)|<1$. This inequality holds for all real λ where $A(\lambda,s)$ is regular and for all real $s>0$. Thus $|A(\lambda,s)|$ never increases faster than $s^{-\frac{1}{2}}$ as $s\to 0$, no matter whether λ is positive or negative. This is most likely true for complex λ also, and $\beta_n(s)$ will have the corresponding threshold behavior.

We stress that in our proposed approximation we have anyway to take s sufficiently large to allow the trajectories that go to $\mathrm{Re}\lambda=0$ as $s\to 0$ to move away appreciably from that line.

IV. AN ITERATION SCHEME FOR CONSTRUCTING THE WEIGHT FUNCTION $B(x,s)$ FROM UNITARITY AND ANALYTICITY

We consider again the case of potentials satisfying (1). We shall show in this section that unitarity gives us an iterative procedure for calculating the weight function $B(x,s)$. The nonlinear equation that one obtains is much simpler than the corresponding equation for the weight function of the Mandelstam representation.[5]

For large enough s all the Regge poles move to the

[10] A. Ahmadzadeh, P. G. Burke, and C. Tate (to be published).

left and the representation (7) becomes

$$f(s,z)=\frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}\frac{B(x,s)}{(\cosh x-z)^{3/2}}\sinh x dx. \quad (26)$$

Here we have

$$B(x,s)=B_L(x,s),\quad x>\xi$$
$$=0,\quad x<\xi.$$

The second equality follows from the fact that we are considering s large enough so that all the poles are on the left.

A representation similar to (26) holds in every order of perturbation theory for all values of $s>0$. If we write f_n and B_n for the nth Born terms of f and B, respectively, we get

$$f_n(s,z)=\frac{1}{\sqrt{2}}\int_{\xi}^{\infty}\frac{B_n(x,s)}{(\cosh x-z)^{3/2}}\sinh x dx. \quad (27)$$

For any n and physical s, $f_n(s,z)$ vanishes at least as fast as z^{-1} as $z\to\infty$.[5] This leads to the conclusion that $A_n(\lambda,s)$, the nth Born approximation of the partial wave amplitude, is analytic for $\mathrm{Re}\lambda>0$, and has no poles in the right half plane.

We now write down the unitarity condition for $B(x,s)$. To do that we have to recall the unitary condition for $A(\lambda,s)$,

$$A(\lambda,s)-A^*(\lambda^*,s)=2is^{1/2}A(\lambda,s)A^*(\lambda^*,s). \quad (28)$$

Noting that $B(x,s)$ as defined in (8) is essentially a Fourier transform of $A(\lambda,s)$, we get

$$\mathrm{Im}B(x,s)=s^{1/2}\int_{-\infty}^{+\infty}B(x',s)B^*(x-x',s)dx'. \quad (29)$$

This last equation is the unitarity relation for B. It can as we shall see below be used to effect a bootstrap procedure for the calculation of B from the first-order B_1. Before we can do that, however, we have to discuss the analytic properties of B in s in order to obtain a way for calculating B from $\mathrm{Im}B$ in each successive order.

For s large enough so that all the Regge poles are in the left half plane, the partial wave amplitude for $\mathrm{Re}\lambda\geq 0$ can be written as

$$A(\lambda,s)=\frac{1}{\pi}\int_{m^2}^{\infty}Q_{\lambda-\frac{1}{2}}\left(1+\frac{t'}{2s}\right)D(s,t')\frac{dt'}{2s}. \quad (30)$$

Here $D(s,t)$ is the discontinuity of $f(s,t)$ across the cut in the t plane with $z=1+t/2s$. Substituting this in (8), we obtain after performing the λ integration

$$B(x,s)=\frac{1}{\sqrt{2}\pi}\int_{m^2}^{\infty}\frac{dt'}{2s}\frac{D(s,t')}{(\cosh x-1-t'/2s)^{1/2}}\times\theta\left(\cosh x-1-\frac{t'}{2s}\right). \quad (31)$$

Here we have used the following integral:

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}e^{\lambda x}Q_{\lambda-\frac{1}{2}}(z)d\lambda=\frac{1}{\sqrt{2}}\frac{1}{(\cosh x-z)^{1/2}}\theta(\cosh x-z). \quad (32)$$

From (31) one can see that for a fixed x, B is not analytic in s, since the θ function depends on s. However, by a simple change of variables one can get a closely related function which is regular in the cut s plane. Let us introduce the new variable y given by

$$y=2s(\cosh x-1), \quad (33)$$

and write

$$b(y,s)=B(\cosh^{-1}(1+y/2s),\,s). \quad (34)$$

Substituting in (31), we obtain

$$b(y,s)=\frac{1}{\sqrt{2}\pi}\int_{m^2}^{y}\frac{dt'}{(2s)^{1/2}}\frac{D(s,t')}{(y-t')^{1/2}}. \quad (35)$$

For any finite $y>m^2$ the integral above converges, and this representation for $b(y,s)$ holds for all s. It is well known that $D(s,t)$ is analytic in the cut s plane for fixed $t>m^2$ and has only a right-hand cut. Thus, for any finite $y>m^2$, we can analytically continue (35) in s and $b(y,s)$ would be regular in the cut plane with only a right-hand cut starting at $s=0$. The discontinuity of the function $b(y,s)$ across the cut in the s plane is not related to the imaginary part but to the real part of $D(s,t)$. This is due to the factor $s^{1/2}$ in the integrand. In order to obtain a useful dispersion relation for our iteration procedure we have to define

$$\tilde{b}(y,s)=s^{1/2}b(y,s). \quad (36)$$

The function $\tilde{b}$ now satisfies the following simple dispersion relation

$$\tilde{b}(y,s)=\tilde{b}_1(y)+\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{Im}\tilde{b}(y,s')}{s'-s-i\epsilon}ds', \quad (37)$$

where $\tilde{b}_1$ is the first Born term for $\tilde{b}$ and is given by

$$\tilde{b}_1(y)=\frac{-1}{2}\int_{m^2}^{y}dt'\frac{\sigma(\sqrt{t'})}{(y-t')^{1/2}}\frac{1}{2\sqrt{t'}}. \quad (38)$$

Here σ is the weight function of the superposition of Yukawa potentials defined in (1). We have used the fact that $\lim_{|s|\to\infty}D(s,t)=-\pi\sigma(\sqrt{t})/2\sqrt{t}$.

Equations (37) and (29) are enough to define our bootstrap procedure and to determine the scattering amplitude completely. Returning to the original variables we get from (38) for $B(x,s)$ in first order

$$B_1(x,s)=\frac{-1}{2}\int_{m^2}^{\infty}\frac{dt'}{(2t')^{1/2}}\frac{\sigma(\sqrt{t'})}{2s(\cosh x-1-t'/2s)^{1/2}}\times\theta\left(\cosh x-1-\frac{t'}{2s}\right). \quad (39)$$

It is evident that $B_1(x,s)=0$ for $x<\xi$. Substituting B_1 into the unitarity equation (29), we immediately see that $\mathrm{Im}B_2(x,s)=0$ for $x<2\xi$. We can now use (34), (36), and (37) to get $B_2(x,s)$. It turns out that $B_2(x,s)=0$ for $x<\cosh^{-1}(1+2m^2/s)$. Thus in the region $\xi<x<\xi_1$, where $\cosh\xi_1=1+2m^2/s$, $B(x,s)$ is identical with the first Born term $B_1(x,s)$. Similarly we can show that in the region $\xi<x<\xi_2$, where $\cosh\xi_2=1+9m^2/2s$, only $B_1(x,s)$ and $B_2(x,s)$ contribute. Thus in general for $x<\xi_n$, $\cosh\xi_n=1+(n+1)^2m^2/2s$, $B(x,s)$ is given exactly by the sum of the first n Born terms.

The above iteration scheme for $B(x,s)$ has two advantages over the iteration procedure for the Mandelstam weight function. First, the integral equation (29) is much simpler than the one in the Mandelstam case and contains only one integration. Secondly, the function $B(x,s)$ is more directly related to the Regge poles than $\rho(s,t)$.

ACKNOWLEDGMENTS

A major part of this work was done while the author was visiting Brookhaven National Laboratory during the summer of 1962. The author wishes to thank Professor G. C. Wick for the hospitality of the theoretical group at Brookhaven. He also thanks Professor J. R. Oppenheimer for the hospitality extended to him at the Institute for Advanced Study.

It is a pleasure to thank Professor S. B. Treiman for several helpful discussions.

APPENDIX

We shall show here that $R(s,z;\lambda)$ has no branch cut in the region $1\leq z<\cosh\xi$, for the case $\mathrm{Re}\lambda>0$. We have from (13)

$$R(s,z;\lambda)=\beta(s)\left[\frac{-1}{\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{\lambda x}\sinh x}{(\cosh x-z)^{3/2}}dx+\frac{2\pi\lambda P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\right]. \quad \text{(A1)}$$

We want to show that

$$\Delta R(s,z;\lambda)=\frac{1}{2i}[R(s,z+i\epsilon;\lambda)-R(s,z-i\epsilon;\lambda)]=0;\quad 1\leq z\leq\cosh\xi. \quad \text{(A2)}$$

Let us take the second term first. The discontinuity in the Legendre function is given by

$$\frac{1}{2i}[P_{\lambda-\frac{1}{2}}(-z-i\epsilon)-P_{\lambda-\frac{1}{2}}(-z+i\epsilon)]=\cos\pi\lambda P_{\lambda-\frac{1}{2}}(z). \quad \text{(A3)}$$

Therefore for the second term we have

$$\Delta\left[\beta(2\pi\lambda)\frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\right]=2\pi\lambda\beta P_{\lambda-\frac{1}{2}}(z);\quad z\geq 1. \quad \text{(A4)}$$

We now use the following integral representation for $P_{\lambda-\frac{1}{2}}(z)$,[11] for $z>1$,

$$P_{\lambda-\frac{1}{2}}(z)=\frac{\sqrt{2}}{\pi}\int_0^{\cosh^{-1}z}\frac{\cosh\lambda x}{(z-\cosh x)^{1/2}}dx. \quad \text{(A5)}$$

This gives for the discontinuity of the second term for $z>1$

$$\Delta\left[\beta(2\pi\lambda)\frac{P_{\lambda-\frac{1}{2}}(-z)}{\cos\pi\lambda}\right]=(2)^{3/2}\beta\lambda\int_0^{\cosh^{-1}z}\frac{\cosh\lambda x}{(z-\cosh x)^{1/2}}dx. \quad \text{(A6)}$$

Before we continue the first term to unphysical z we have to do an integration by parts and write

$$I(s,z;\lambda)\equiv\frac{-\beta}{\sqrt{2}}\int_{-\infty}^{\xi}\frac{e^{\lambda x}\sinh x}{(\cosh x-z)^{3/2}}dx=\sqrt{2}\beta\frac{e^{\lambda\xi}}{(\cosh\xi-z)^{1/2}}-\sqrt{2}\beta\lambda\int_{-\infty}^{\xi}\frac{e^{\lambda x}dx}{(\cosh x-z)^{1/2}}. \quad \text{(A7)}$$

Now for $1\leq z<\cosh\xi$ we have

$$\Delta I=-\sqrt{2}\beta\lambda\int_{-\alpha}^{+\alpha}\frac{e^{\lambda x}}{(z-\cosh x)^{1/2}}dx=-(2)^{3/2}\beta\lambda\int_0^{\alpha}\frac{\cosh\lambda x}{(z-\cosh x)^{1/2}},\quad \cosh\alpha=z. \quad \text{(A8)}$$

This is identical with (A6) except for the sign, and hence (A2) is valid. In continuing the square roots in (A7) we have used $(\cosh x-z\mp i\epsilon)^{1/2}=\mp i(z-\cosh x)^{1/2}$ in the region where $z>\cosh x$. This choice of branch is determined by the fact that the representations (5) and (A5) should satisfy the relation (A3).

[11] See reference 6, same page, formula (8).