Elastic Nucleon-Deuteron Scattering*

K. L. KOWALSKI† AND D. FELDMAN
Department of Physics, Brown University, Providence, Rhode Island
(Received 20 August 1962)

Some formal and practical problems concerning the effects of the internal target nucleon motion and of the multiple scattering on the elastic scattering of high-energy nucleons by deuterons are considered. In order to provide a foundation for the examination of these effects, two well-known forms of the impulse approximation are studied within the context of a multiple-scattering formalism, and it is found that the form due to Watson appears to be the most convenient and consistent in its application. Some methods for solving the multiple-scattering equations are investigated. The direct use of the optical-model approach is shown to be impractical for very light nuclei and, in particular, for the deuteron. An alternative means of obtaining solutions of the multiple-scattering equations (when the number of target nucleons is small) which permits the exact treatment of the ground-state scattering while allowing a systematic treatment of the contributions due to the excited intermediate nuclear states is discussed. A practical technique is developed for generating approximate solutions of the two-body integral equations which occur in the various multiple-scattering formalisms.

In the study of the consequences of the internal target nucleon motion, the impulse approximation is used to express the complete nucleon-deuteron transition matrix element in terms of two-nucleon transition (t) matrices in the form of integrals over the internal-momentum distribution of the target nucleons; these integrals are then evaluated under the assumption that the principal contribution to the scattering occurs for those values of the relative target nucleon momentum, $\mathbf{q}$, such that $\mathbf{q}=z\boldsymbol{\kappa}$, where $\boldsymbol{\kappa}$ is one-half the momentum transfer and $0\leqslant z\leqslant 1$. The variation of the off-the-energy-shell t matrix elements over this range of $\mathbf{q}$ is taken into account for the Hamada potential. The integrals are then employed to calculate (in the single-scattering approximation) the cross sections and polarizations for elastic nucleon-deuteron scattering for incident nucleon (lab) energies of 40, 95, and 150 MeV and c.m. scattering angles of 30° to 150°. A comparison is made between the present calculation and results obtained with the customary procedures of either neglecting entirely the $\mathbf{q}$ dependence of the t matrices (Chew approximation) or assuming that these matrices depend only linearly on $\mathbf{q}$. At 150 MeV, where the off-the-energy-shell effects are largest, the results are shown to depend significantly on the type of t operator used, especially at large angles. The best fit to the polarization at 150 MeV is obtained with a t operator which corresponds approximately to Watson's form of the impulse approximation.

INTRODUCTION

THERE has been a considerable amount of effort devoted to the study of high-energy[1] elastic N-(nucleon)-d(deuteron) scattering.[2,3] In most investigations of this problem up to the present time the impulse approximation (IA) has been employed in the form originally proposed by Chew.[4] That is, the scattering process is viewed as a simple superposition of the (single) scatterings of the incident nucleon from each of the (quasi-free) target nucleons. Usually, the internal motion of the target nucleons has also been neglected. This picture of the scattering appears to be quite accurate for small momentum transfers.

When the momentum transfer is large, the preceding picture of the scattering process fails, for in this case several effects, whose relative importance is only poorly understood, can no longer be neglected. These effects include the internal motion of the target nucleons, the multiple scattering, the binding forces, and the pickup scattering. In the present work, we are concerned with some of the formal and practical problems related to the evaluation of the first two of these effects within the context of the IA, where we take the IA to imply only the neglect of the binding forces. We do not consider the pickup scattering.[5–7]

The problem of accounting for the internal motion of the target nucleons appears, at first glance, to offer no difficulty except as to computational labor. In principle, all that is required is the proper evaluation of the (nuclear) matrix elements of the two-nucleon transition operators with respect to eigenstates of the noninteracting system composed of the target and the incident nucleon. However, there does exist some ambiguity in the choice of these transition operators.

This is, in part, due to the existence of two somewhat different formulations of the IA. The first, due to Chew and others,[4,8–10] has been applied repeatedly, in various

* This work was supported, in part, by the U. S. Atomic Energy Commission.

† Present address: the Department of Physics, Case Institute of Technology, Cleveland, Ohio.

[1] By *high-energy* we mean the circumstance where the kinetic energy of the incident particle is much greater than the average kinetic energy of a target nucleon.

[2] *Proceedings of the Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959*, edited by T. C. Griffith and E. A. Power (Pergamon Press, Inc., London, 1960), Vol. I.

[3] M. Verde, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 142.

[4] G. F. Chew, Phys. Rev. **80**, 196 (1950).

[5] We will ignore those identity effects which result in the pickup scattering [as in reference 6]. Therefore, we formally regard the incident nucleon as distinguishable from the target nucleons. It appears that for reasonably high energies the pickup process is significant only for extreme backward scatterings (cf. reference 7).

[6] G. Takeda and K. Watson, Phys. Rev. **97**, 1336 (1955).

[7] H. Postma and R. Wilson, Phys. Rev. **121**, 1229 (1961).

[8] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[9] J. Ashkin and G. C. Wick, Phys. Rev. **85**, 686 (1952).

[10] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

degrees of approximation, to N-d scattering.[11–14] The second was introduced by Watson[15,16] within the context of a multiple-scattering theory. The two-body transition operators which appear in these alternative formulations of the IA have quite distinct mathematical properties.[17] Even the Chew *et al.* formalism itself is not without ambiguity.[9]

Now, given some definite choice of a two-nucleon transition operator, the evaluation of the nuclear matrix elements of this operator is an exceedingly involved computational problem. An essentially exact calculation has been carried out by Fulton and Schwed[12] using a simplified model of the N-N interaction. However, the necessity of performing a nearly exact evaluation may be questioned from several points of view. Such a computation is somewhat inconsistent in view of the uncertainties existing in the present-day knowledge of the deuteron wave function and the N-N potential. Secondly, the calculations of Fulton and Schwed and others[7,11,14] indicate that the approximation of completely neglecting the internal motion of the target nucleons is valid for a rather large range of angles in the forward scattering hemisphere. Finally, there is evidence that a simple modification of the preceding approximation yields improved results for large momentum transfers.[3,14,18,19] It would, therefore, appear that it should be possible to devise a method for evaluating the nuclear matrix elements of the two-nucleon transition operators which accounts for the dominant effects of the internal target nucleon motion (including off-the-energy-shell effects) and yet which does not involve either a prohibitive and/or an unnecessary amount of computational labor.

Next, let us consider some of the problems associated with the determination of the multiple scattering. There exist a variety of ways in which one can construct a multiple-scattering theory for the scattering of particles from complex nuclei. A formulation which seems to permit the most convenient and consistent application of the IA has been developed by Watson.[15,16] When the pickup scattering is neglected, this formalism is directly applicable to the N-d problem. Nevertheless, even with the application of the IA, the solution of the multiple-scattering equations is a complicated many-body[20] problem.

The inherent many-body character of the multiple-scattering equations arises from intermediate scatterings which involve excited nuclear states.[21] About the only possibility for solving these equations in any simple way occurs when the contribution from the excited states is small enough to be considered as a perturbation upon the scattering which takes place with the nucleus always in its ground state.[22,23] A method for systematically evaluating these corrections to the ground-state scattering is the so-called optical-model formalism.[24–26]

From a formal standpoint, it is easy to establish the existence of an operator, the optical potential, which comprises the inhomogeneous term and the kernel (exclusive of a two-body propagator) of an integral equation involving only the ground-state matrix elements of the complete transition operator.[24] On the other hand, the problem of finding an integral equation for the optical potential which is suitable for the approximate treatment of the excited states is considerably more difficult. Unfortunately, the existing optical-model formalisms do not seem to be directly applicable to the case when the target consists of a small number of nucleons. Some studies make the explicit assumption that the number of target nucleons is large.[15,25] Other treatments introduce pseudo-two-body transition operators which contain projection operators whose quantitative effects are not very well understood.[26,27]

It is evident that it is generally desirable to have an optical-model formalism which is valid independently of the number of target nucleons and which contains only quantities which can be calculated without the introduction of a subsidiary many-body problem. It should be stressed that in the case of a deuteron target the only reason for considering the optical-model formalism is to permit the systematic evaluation of the contributions arising from the excited intermediate nuclear states. If one is only interested in the ground-state scattering, there is no advantage over the direct consideration of the multiple-scattering equations.

In Sec. I of this paper we discuss the relationship between the two forms of the IA in terms of a multiple-scattering theory. Section II is devoted to some aspects of the solution of the multiple-scattering equations with particular regard to the derivation and application of an optical-model formalism when the target consists of a small number of nucleons. In Secs. III-VI we consider the evaluation of the ground-state nuclear matrix elements of the various two-nucleon transition operators, taking into account the internal motion of the target nucleons. These matrix elements are used to

[11] Y. Sakamoto and T. Sasakawa, Progr. Theoret. Phys. (Kyoto) **21**, 879 (1959).
[12] T. Fulton and P. Schwed, Phys. Rev. **115**, 973 (1959).
[13] L. Favella and M. Olivetti, Nuovo Cimento **11**, 679 (1959).
[14] L. Castillejo and L. S. Singh, reference 2, p. 193.
[15] K. M. Watson, Phys. Rev. **89**, 575 (1953).
[16] K. M. Watson, Phys. Rev. **105**, 1388 (1957).
[17] K. L. Kowalski and D. Feldman, J. Math. Phys. (to be published).
[18] P. B. Daitch and J. B. French, Phys. Rev. **85**, 695 (1952).
[19] O. Chamberlain and M. O. Stern, Phys. Rev. **94**, 666 (1954).
[20] We apply the term *many-body* to any system containing as few as three particles.
[21] We assume throughout this work that the target is initially and finally in its ground state.
[22] Some work has been devoted to the case when the target particles can be regarded as fixed. A comprehensive review is contained in reference 23 along with additional references. See also Sec. 6 of reference 16.
[23] L. H. Schick, Rev. Mod. Phys. **33**, 608 (1961).
[24] H. Feshbach, Ann. Rev. Nucl. Sci. **8**, 49 (1958).
[25] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
[26] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).
[27] See the Appendix to reference 25.

calculate the cross sections and polarizations for N-d scattering, in the single-scattering approximation, over a representative range of energies and momentum transfers. The results are discussed in Sec. VII, and some concluding remarks are made in the last section.

I. FORMULATION OF THE SCATTERING IN TERMS OF TWO-BODY PROCESSES

The scattering of a high-energy particle by a nucleus is conveniently studied by formulating the problem in terms of successive collisions of the incident particle with individual target nucleons. One such multiple-scattering theory has been developed by Watson.[15] We now study some aspects of this and similar formalisms with regard to the application of the IA.

The Schrödinger equation for the scattering problem is written as

$$H|\Psi_a\rangle=E_a|\Psi_a\rangle, \tag{1.1}$$

where H is the complete Hamiltonian, viz.,

$$H=K_1+H_N+V. \tag{1.2}$$

Here, H_N is the complete target Hamiltonian with eigenstates, $|\gamma\rangle$, defined by

$$H_N|\gamma\rangle=\epsilon_\gamma|\gamma\rangle, \tag{1.3}$$

and K_1 is the kinetic-energy operator of the incident particle. We assume that the particle-nucleus interaction, V, can be represented by a sum of two-body potentials,

$$V=\sum_\alpha v(\alpha), \tag{1.4}$$

where $v(\alpha)$ denotes the interaction between the incident particle and the target nucleon α; the sum in (1.4) is over all the target nucleons. Finally, the total energy, E_a, is given by

$$E_a=E_1+\epsilon_0, \tag{1.5}$$

where E_1 is the initial energy of the incident particle, and ϵ_0 is the total ground-state energy of the target (which includes any possible initial c.m. motion).[21]

The outgoing-wave solution, $|\Psi_a^{(+)}\rangle$, of Eq. (1.1) can be expressed in terms of the wave operator $\Omega_a^{(+)}$, i.e.,

$$|\Psi_a^{(+)}\rangle=\Omega_a^{(+)}|a\rangle=(1+G_aV\Omega_a^{(+)})|a\rangle, \tag{1.6}$$

where

$$G_a=(E_a-H_0+i\epsilon)^{-1}, \quad (\epsilon\to+0), \tag{1.7}$$

and[28]

$$H_0|a\rangle\equiv(K_1+H_N)|a\rangle=E_a|a\rangle. \tag{1.8}$$

The transition probability is calculated from the energy-conserving matrix elements of the transition operator

$$T_a^{(+)}=V\Omega_a^{(+)}=V+VG_aT_a^{(+)}. \tag{1.9}$$

[28] The index a actually refers to a complete set of commuting observables of the noninteracting system composed of the incident particle and the target. However, we use a primarily as an energy index.

Similar relations can be written down for the incoming-wave solution of (1.1).

The solution of the integral equation (1.9) can be written in the form[15]

$$T_a^{(+)}=\sum_\alpha T(\alpha), \tag{1.10}$$

where the $T(\alpha)$ satisfy the set of coupled integral equations

$$T(\alpha)=t(\alpha)+t(\alpha)G_a\sum_{\beta\neq\alpha}T(\beta), \tag{1.11}$$

and

$$t(\alpha)=v(\alpha)+v(\alpha)G_at(\alpha)=v(\alpha)+t(\alpha)G_av(\alpha). \tag{1.12}$$

A solution of Eq. (1.9) which is equivalent to the preceding is

$$T_a^{(+)}=\sum_\alpha T_0(\alpha), \tag{1.13}$$

where

$$T_0(\alpha)=t_0(\alpha)+t_0(\alpha)G_a\sum_{\beta\neq\alpha}T_0(\beta)+\bar{\Delta}(\alpha)T_0(\alpha). \tag{1.14}$$

Here,

$$t_0(\alpha)=v(\alpha)+v(\alpha)G(\alpha)t_0(\alpha)=v(\alpha)+t_0(\alpha)G(\alpha)v(\alpha), \tag{1.15}$$

$$G(\alpha)=[E_1-(K_1+K_\alpha)+i\epsilon]^{-1}, \tag{1.16}$$

$$\bar{\Delta}(\alpha)=t_0(\alpha)G(\alpha)\Delta(\alpha)G_a=t_0(\alpha)G_a\Delta(\alpha)G(\alpha), \tag{1.17}$$

$$\Delta(\alpha)=-[\epsilon_0-(H_N-K_\alpha)]; \tag{1.18}$$

K_α is the kinetic-energy operator of particle α. It should be noted that $t_0(\alpha)$, in contrast to $t(\alpha)$, is a two-body operator; the connection between the two is given by

$$t(\alpha)=[1-\bar{\Delta}(\alpha)]^{-1}t_0(\alpha). \tag{1.19}$$

The solution (1.13) permits a somewhat more descriptive introduction of the IA than does the solution (1.10); otherwise, the formulation of Watson, Eq. (1.10), is definitely more useful. In the IA it will be seen that the two solutions are identical in form.

It is apparent from Eq. (1.19) that the operator $\bar{\Delta}(\alpha)$ contains the effects of the nucleus on the scattering of the incident particle from the target nucleon α. The presence of the last term on the right-hand side of Eq. (1.14) and the structure of $\bar{\Delta}(\alpha)$ indicate that these effects manifest themselves as "rescatterings"[29] of the incident particle from the same target nucleon. This sort of rescattering can only occur because of the binding forces acting on α. However, the IA is commonly understood to mean that for sufficiently high incident energies the binding forces have a negligible dynamical effect on the scattering of the incident particle from the individual target nucleons. Accordingly, we can formally define the IA by the requirement that the operator $\bar{\Delta}(\alpha)$ be set equal to zero in Eqs. (1.14) and (1.19).[30]

It is interesting to observe just how the binding terms, $\bar{\Delta}(\alpha)$, enter into the integral equation for a

[29] Each of the individual scatterings is represented by t_0.

[30] We call this the *Watson* form of the IA.

particular $T_0(\alpha)$. In the case of a deuteron target the set (1.14) can be decoupled and one obtains, for each α,

$$T_0(\alpha)=t_0(\alpha)+t_0(\alpha)G_a[1-\bar{\Delta}(\alpha')]^{-1}t_0(\alpha') +t_0(\alpha)G_a\{[1-\bar{\Delta}(\alpha')]^{-1}t_0(\alpha')G_a +\Delta(\alpha)G(\alpha)\}T_0(\alpha), \quad (1.20)$$

where $\alpha'\neq\alpha$. Therefore, in order that the IA be valid, it is necessary that

$$|\langle a''|\bar{\Delta}(\alpha)|a'\rangle|\ll 1, \quad (1.21a)$$

and

$$|\langle a''|\bar{\Delta}(\alpha')|a'\rangle|\ll|\langle a''|t_0(\alpha)G_at_0(\alpha')G_a|a'\rangle|, \quad (1.21b)$$

at least for those states a', a'' with energies $E_{a'}$, $E_{a''}$ near the energy shell (E_a). The condition (1.21b) is simply a consistency requirement since it asserts that the binding corrections be much smaller than the multiple-scattering corrections. It is clear from Eq. (1.19) that (1.21a) is a necessary condition for the replacement of $t(\alpha)$ by $t_0(\alpha)$ in (1.11). Henceforth, we assume that the inequalities (1.21) are satisfied and, moreover, that they are sufficient for the validity of the IA. Some examples have been considered which indicate that (1.21a) should be satisfied for sufficiently high incident-particle energies.[8]

The IA was originally introduced in a manner slightly different from that of the preceding discussion.[4,8–10] We now examine this alternative (Chew) form of the IA and study its relationship to the Watson formulation.

Let us introduce the two-body transition operators

$$t^{(+)}(\alpha)=v(\alpha)[1+\mu^{(+)}(\alpha)], \quad t^{(-)}(\alpha)=[1+\mu^{(-)}(\alpha)]v(\alpha), \quad (1.22)$$

where

$$\mu^{(+)}(\alpha)=\sum_n G_nt_n^{(+)}(\alpha)|n\rangle\langle n|, \quad \mu^{(-)}(\alpha)=\sum_n|n\rangle\langle n|t_n^{(-)}(\alpha)G_n. \quad (1.23)$$

Here, $|n\rangle$ denotes an eigenvector (plane-wave state) of the complete kinetic-energy operator, K, viz.,

$$K|n\rangle=E_n|n\rangle; \quad (1.24)$$

also,

$$G_n=(E_n-K+i\epsilon)^{-1}. \quad (1.25)$$

The operators $t_n^{(\pm)}(\alpha)$ satisfy

$$t_n^{(+)}(\alpha)=v(\alpha)+v(\alpha)G_nt_n^{(+)}(\alpha), \quad t_n^{(-)}(\alpha)=v(\alpha)+t_n^{(-)}(\alpha)G_nv(\alpha), \quad (1.26)$$

where it is to be understood that $t_n^{(+)}[t_n^{(-)}]$ always operates to the left (right) of states with energy E_n. It can then be readily verified that[10]

$$G_at^{(+)}(\alpha)=\mu^{(+)}(\alpha)+G_aA^{(+)}(\alpha), \quad t^{(-)}(\alpha)G_a=\mu^{(-)}(\alpha)+A^{(-)}(\alpha)G_a, \quad (1.27)$$

where

$$A^{(+)}(\alpha)=A_0^{(+)}(\alpha)-\mu^{(+)}(\alpha)G_a^{-1}, \quad A^{(-)}(\alpha)=A_0^{(-)}(\alpha)-G_a^{-1}\mu^{(-)}(\alpha), \quad (1.28)$$

and

$$A_0^{(\pm)}(\alpha)=\pm[\Delta(\alpha),\mu^{(\pm)}(\alpha)]. \quad (1.29)$$

The relations between $t^{(\pm)}(\alpha)$ and $t(\alpha)$ are obtained from Eq. (1.12) with the aid of the identities (1.27), viz.,

$$t(\alpha)=t^{(+)}(\alpha)[1-G_aA^{(+)}(\alpha)]^{-1}, \quad t(\alpha)=[1-A^{(-)}(\alpha)G_a]^{-1}t^{(-)}(\alpha). \quad (1.30)$$

It is evident from Eq. (1.19) that similar relations hold between $t^{(\pm)}(\alpha)$ and $t_0(\alpha)$.

It follows from Eqs. (1.30) that

$$t(\alpha)|a\rangle=t^{(+)}(\alpha)|a\rangle+t(\alpha)G_aA_0^{(+)}(\alpha)|a\rangle, \quad \langle a|t(\alpha)=\langle a|t^{(-)}(\alpha)+\langle a|A_0^{(-)}(\alpha)G_at(\alpha). \quad (1.31)$$

The terms which involve $A_0^{(\pm)}(\alpha)$ in (1.31), and which evidently represent binding corrections, were estimated by Chew and Wick to be negligible for high incident energies.[8] The Chew form of the IA can then be defined by the stipulation that the quantities $t(\alpha)G_aA_0^{(+)}(\alpha)$ and $A_0^{(-)}(\alpha)G_at(\alpha)$ be set equal to zero in Eqs. (1.31).

In the (Chew) IA, Eqs. (1.31) can be used in conjunction with (1.11) to construct an expression for $T_a^{(+)}$ which involves only the operators $t^{(\pm)}(\alpha)$ provided one is concerned only with the single- and double-scattering terms. For higher order multiple scatterings one is forced to employ Eqs. (1.30) as they stand. However, it is not at all evident that the operator $G_aA^{(+)}(\alpha)$, for example, is small even if the IA is valid. Therefore, the replacement of $t(\alpha)$ or $t_0(\alpha)$ by $t^{(+)}(\alpha)$ in (1.11) appears to be unjustified, unless the intermediate multiple scatterings take place primarily on the energy shell. This observation emphasizes the difficulty in constructing a multiple-scattering formalism in terms of the $t^{(\pm)}(\alpha)$ operators which is convenient for the application of the IA.

Moreover, there exist difficulties with the Chew form of the IA even if one is only concerned with the single-scattering terms. If the IA is valid and the multiple scattering is neglected, we see from Eqs. (1.11) and (1.31) that one can apparently write, with equal validity,

$$T_a^{(+)}\approx\sum_\alpha t^{(+)}(\alpha), \quad T_a^{(+)}\approx\sum_\alpha t^{(-)}(\alpha). \quad (1.32)$$

However, $t^{(+)}(\alpha)$ and $t^{(-)}(\alpha)$ are not identical; for example, on the energy shell,[9]

$$t^{(+)}(\alpha)-t^{(-)}(\alpha)=A_0^{(+)}(\alpha)-A_0^{(-)}(\alpha). \quad (1.33)$$

The terms on the right-hand side of (1.33) do not have the form characteristic of quantities which are small when the IA is valid. Therefore, there is no reason to expect the difference between $t^{(+)}(\alpha)$ and $t^{(-)}(\alpha)$ to be especially small, even for high incident energies, except in the case of vanishingly weak binding forces.

It is completely consistent with the Chew approach to represent the single-scattering terms by the mean of $t^{(+)}(\alpha)$ and $t^{(-)}(\alpha)$, viz.,[10]

$$\bar{t}(\alpha)=\tfrac{1}{2}[t^{(+)}(\alpha)+t^{(-)}(\alpha)]. \tag{1.34}$$

The use of the operator $\bar{t}(\alpha)$ instead of $t^{(+)}(\alpha)$ or $t^{(-)}(\alpha)$ in (1.32) has the advantage of preserving the reciprocity property of $T_a^{(+)}$. Unfortunately, one cannot generally substitute $\bar{t}(\alpha)$ for $t(\alpha)$ in Eq. (1.11). Also, since $\bar{t}(\alpha)$ apparently does not satisfy an integral equation, it is difficult to see how one can develop a multiple-scattering formalism in terms of this operator.

It is evident that the Chew form of the IA possesses many undesirable features, particularly with respect to the description of multiple-scattering processes. Although the multiple scattering (in the IA) can be elegantly expressed in terms of the operators $t_0(\alpha)$, these two-body operators have a considerably more complicated mathematical structure than the $t^{(\pm)}(\alpha)$ operators which appear in the Chew formalism. These mathematical differences have been discussed elsewhere.[17]

II. SOLUTION OF THE MULTIPLE-SCATTERING EQUATIONS

The solution of the multiple-scattering equations (1.11) is still a many-body problem even after the application of the IA. However, it is evident that, if the contribution arising from excited intermediate nuclear states can be neglected, the solution of the set (1.11) becomes simply a two-body problem. If one is interested only in the multiple scattering which occurs with the nucleus always in its ground state, and if the number of target nucleons is small, then it is practical to consider Eqs. (1.11) directly.

On the other hand, the use of these equations as they stand is inconvenient when one wishes to account for the effects of the excited states in an approximate way while treating the ground-state scattering exactly or else when the number of target nucleons is not very small. Under either or both of these circumstances different ways of solving (1.11) must be sought. One such method is the optical-model formalism and most of this section is devoted to an elaboration of this technique.

Now the integral equation[31]

$$T=V+VG_aT=V+TG_aV \tag{1.9a}$$

can be rewritten in the form

$$T=U+UPG_aT, \tag{2.1}$$

where

$$U=V+VQG_aU, \tag{2.2}$$

and

$$P+Q=1. \tag{2.3}$$

[31] We replace $T_a^{(+)}$ by T in order to emphasize that T is defined with respect to states of arbitrary energy. The symbol $T_a^{(+)}$ will be reserved for the situation when T operates to the left of states corresponding to the energy E_a.

We define a particular P and Q, namely, P_0 and Q_0, by

$$P_0\equiv\textstyle\sum_{\gamma_0}|\gamma_0\rangle\langle\gamma_0|, \tag{2.4}$$

where $|\gamma_0\rangle$ is a nuclear ground state which includes the c.m. motion. In this case, Eq. (2.1) is a two-body equation for the ground-state matrix elements of T. All the effects of the excited states are contained in the *optical potential*, U.

Equation (2.2) for U is not very useful as it stands. A great deal of effort has been devoted to the problem of finding an alternative integral equation for U in a form suitable for approximate calculations.[24] One method[26,27] is of particular interest, even though it departs from the form (2.1)–(2.2) slightly, since it makes no explicit assumption as to the number of target nucleons in contrast to other studies.[15,16,25] We discuss this method in detail in order to ascertain whether or not its applicability in the case of a very light nucleus is justified. Our treatment follows closely the work of Kerman, McManus, and Thaler.[26]

Now, if an isotopic-spin notation is employed, all of the target nucleons can be regarded as identical. Then, the only matrix elements of T which are of interest in calculating observable quantities are those defined with respect to the completely antisymmetrized target states $|\gamma'\rangle$. We may, therefore, write

$$T=Av+AvG_a'T, \tag{2.5}$$

with

$$G_a'=\mathcal{A}G_a, \tag{2.6}$$

where (2.5) is assumed to be defined with respect to the states $|\gamma'\rangle$, $\mathcal{A}$ is the antisymmetrization operator in all the nuclear variables, and A denotes the number of target nucleons. The particle index α in $v(\alpha)$ need no longer be explicitly indicated.

It is a simple matter to show that

$$T=At'[1-(A-1)G_a't']^{-1}, \tag{2.7}$$

where t' satisfies

$$t'=v+vG_a't'. \tag{2.8}$$

From (2.7) we deduce that

$$T'=(A-1)t'(1+G_a'T'), \tag{2.9}$$

with

$$T'=[(A-1)/A]T. \tag{2.10}$$

In view of Eqs. (2.9) and (2.2), the optical potential, U', corresponding to T' satisfies

$$U'=(A-1)t'(1+Q_0G_a'U'). \tag{2.11}$$

The principal difficulty with the preceding formalism is the occurrence of the operator t' whose defining integral equation (2.8) contains the propagator G_a' instead of G_a, and so we must consider the question of how to approximate t' by a somewhat more manageable quantity (say, t or t_0). Kerman, McManus, and Thaler[26] interpret the replacement of t' by t_0 as a form of the IA

and estimate, in a rough way, that the corresponding error should be small, in the case of a deuteron target, at sufficiently high incident energies. However, this replacement does seem somewhat unnatural.[32] More generally, we examine the possibility of substituting t for t'.

Now t' is related to t by

$$t'=t+t(G_a'-G_a)t'. \tag{2.12}$$

Equation (2.12) can be used in conjunction with (2.7) to obtain the solution for T' in terms of t, viz.,

$$T'=(A-1)t[1-(A-1)\bar{G}_a t]^{-1}, \tag{2.13}$$

where

$$\bar{G}_a=(AG_a'-G_a)(A-1)^{-1}. \tag{2.14}$$

In order to obtain an equivalence[33] between (2.13) and (2.7) (with t' replaced by t in the latter equation), it is necessary that $\bar{G}_a$ be set equal to G_a'. This is possible only if $A\gg 1$, as is evident from (2.14). We conclude that the replacement of t' by t (and also t' by t_0) is a $(1/A)$ correction and, therefore, is unjustified for very light nuclei.

In view of the preceding, we now turn our attention to the possible development of an optical-model formalism which is valid for light nuclei. The solution of Eq. (2.1) can be written as

$$T=\sum_\alpha \bar{T}(\alpha), \tag{2.15}$$

where $\bar{T}(\alpha)$ satisfies

$$\bar{T}(\alpha)=t''(\alpha)+t''(\alpha)P_0G_a\sum_{\beta\neq\alpha}\bar{T}(\beta), \tag{2.16}$$

and

$$t''(\alpha)=U(\alpha)+U(\alpha)P_0G_at''(\alpha). \tag{2.17}$$

We have expressed the solution of Eq. (2.2) in the form

$$U=\sum_\alpha U(\alpha). \tag{2.18}$$

It may be expected from the structure of Eqs. (2.15)–(2.18) that the quantities $U(\alpha)$ will correspond to a multiple-scattering-type solution of Eq. (2.2). Obviously, one such solution is

$$U(\alpha)=u(\alpha)+u(\alpha)Q_0G_a\sum_{\beta\neq\alpha}U(\beta), \tag{2.19}$$

where

$$u(\alpha)=v(\alpha)+v(\alpha)Q_0G_au(\alpha). \tag{2.20}$$

The relationship between $u(\alpha)$ and $t(\alpha)$ is

$$t(\alpha)=u(\alpha)+u(\alpha)P_0G_at(\alpha). \tag{2.21}$$

The quantity $u(\alpha)$ is clearly the "optical potential" for the scattering of the incident particle from the target nucleon α. If the number of target nucleons is very large, it then follows from Watson's considerations on projection operators[16] that Eqs. (2.19) are valid to within terms of order A^{-1} if $u(\alpha)$ is replaced by $t(\alpha)$. In this case one obtains the usual[16,25] optical-model formalism which holds for heavy nuclei.

If the contribution to $U(\alpha)$ arising from the excited intermediate nuclear states is neglected, viz.,

$$U(\alpha)\approx u(\alpha), \tag{2.22a}$$

we see from Eqs. (2.17) and (2.21) that

$$t''(\alpha)\approx t(\alpha). \tag{2.22b}$$

This is, of course, just what one would obtain from Eqs. (1.11), again with the neglect of the excited states. Equations (2.22) demonstrate why it is much more practical, for very light nuclei, to consider the multiple-scattering equations (2.16) directly rather than to calculate T from (2.1) with the use of the optical potential. Even in the lowest order approximation to U given by Eq. (2.22a), the quantities $u(\alpha)$ must be calculated from $t(\alpha)$ [or $t_0(\alpha)$ in the IA]. It is clear from (2.21) that this is a two-body problem; but since the solution will inevitably be approximate, even the single-scattering terms will not be treated exactly. We conclude that the direct use of the optical potential is apparently not practical in the case of very light nuclei.

We, therefore, return to a consideration of Eqs. (2.16) with particular emphasis on obtaining an improved evaluation of $t''(\alpha)$. The corrections to $t''(\alpha)$ arising from the excited nuclear states can be carried out in a systematic, if somewhat cumbersome, manner with the use of the iteration solution of Eq. (2.19). For the sake of illustration, we consider the first-order correction to $t''(\alpha)$. Let

$$t''(\alpha)=t(\alpha)+\Delta_1t''(\alpha) \tag{2.23}$$

and

$$U(\alpha)=u(\alpha)+\Delta_1U(\alpha), \tag{2.24}$$

where

$$\Delta_1U(\alpha)=u(\alpha)Q_0G_a\sum_{\beta\neq\alpha}u(\beta). \tag{2.25}$$

Then, to this order, Eq. (2.17) becomes

$$\Delta_1t''(\alpha)=\Delta_1U(\alpha)[1+P_0G_at(\alpha)]+u(\alpha)P_0G_a\Delta_1t''(\alpha), \tag{2.26}$$

where use has been made of Eq. (2.21). The solution of Eq. (2.26) is

$$\Delta_1t''(\alpha)=t(\alpha)Q_0G_a\sum_{\beta\neq\alpha}u(\beta)[1+P_0G_at(\alpha)]. \tag{2.27}$$

The higher order corrections to $t''(\alpha)$, namely, $\Delta_2t''(\alpha)$, $\Delta_3t''(\alpha),\cdots$, where $\Delta_nt''(\alpha)$ contains only terms of order n in the excited states, can be found in an analogous manner.

The expression (2.27) for $\Delta_1t''(\alpha)$ is somewhat more complicated than what one usually expects. However,

[32] A. Herzenberg and E. J. Squires, Nucl. Phys. **19**, 280 (1960).

[33] It should be remarked that Eq. (2.13) cannot be used to deduce an integral equation for T' in terms of t and $\bar{G}_a$ as was done with Eq. (2.7). This is due to the fact that $\bar{G}_a$ permits intermediate nuclear states of arbitrary symmetry.

it appears that this additional complexity is the price one has to pay in order to avoid errors of order A^{-1} or the appearance of many-body projection operators such as $\mathcal{A}$.

We have seen how it is possible to describe the dominant features of the multiple scattering by means of two-body integral equations, the prototype of which is Eq. (2.1). These equations are still rather difficult to solve in practice for several reasons. First, it is inconvenient to employ a partial-wave analysis due to the fact that the quantities of interest are nuclear matrix elements rather than matrix elements defined with respect to plane-wave states. Secondly, the "potential" terms appearing in these equations are non-Hermitian so that methods, such as conventional variational principles, which assume that the potential is Hermitian are inapplicable. Finally, it is well known that the solutions obtained by iteration (Neumann) or Fredholm expansions are usually either unreliable or very slowly convergent.

We accordingly discuss a very simple method for obtaining approximate solutions to equations such as (2.1) which avoids the preceding difficulties. Let us consider, in the c.m. system,[34] the integral equation (2.1) for the ground-state matrix elements of T, viz.,[35]

$$\langle \mathbf{k}_f | T | \mathbf{k}_i \rangle = \langle \mathbf{k}_f | U | \mathbf{k}_i \rangle + \int d\mathbf{k}_p \frac{\langle \mathbf{k}_f | U | \mathbf{k}_p \rangle \langle \mathbf{k}_p | T | \mathbf{k}_i \rangle}{E_i - E_p + i\epsilon}, \quad (2.28)$$

where the $\mathbf{k}$'s denote wave vectors of the particle originally incident on the nucleus, and the E's are the corresponding energies.

Now suppose that U yields a fairly good representation of T, but we wish to obtain a somewhat better approximation. It is consistent with our supposition to write

$$\langle \mathbf{k}_p | T | \mathbf{k}_i \rangle \equiv T_{p,i} \approx U_{p,i} C, \quad (2.29)$$

where, for example,

$$C = [U_{f,i}]^{-1} T_{f,i}; \quad (2.30)$$

here $\mathbf{k}_p$ and $\mathbf{k}_f$ are completely arbitrary as to both direction and magnitude. If the expression (2.29) for $T_{p,i}$ is inserted into the right-hand side of Eq. (2.28) with C given by (2.30), one obtains an algebraic equation for $T_{f,i}$ which is easily solved, viz.,

$$T_{f,i} = U_{f,i} \left[U_{f,i} - \int d\mathbf{k}_p \frac{U_{f,p} U_{p,i}}{E_i - E_p + i\epsilon} \right]^{-1} U_{f,i}. \quad (2.31)$$

This approximation technique can be continued indefinitely. In general, after N repetitions of the procedure, one obtains

$$T_{f,i}^{(N+1)} = T_{f,i}^{(N)} \left[T_{f,i}^{(N)} - \int d\mathbf{k}_p \frac{U_{f,p} T_{p,i}^{(N)}}{E_i - E_p + i\epsilon} \right]^{-1} U_{f,i}, \quad (2.32)$$

where $N = 0, 1, \cdots$, and

$$T_{f,i}^{(0)} = U_{f,i}. \quad (2.33)$$

The method just outlined for solving the two-body integral equations has a number of desirable features. For example, the form of Eqs. (2.31) and (2.32) suggests that the procedure may have the convergence properties expected of a variational formulation in contrast to successive iterations of (2.28). Furthermore, each order of approximation requires no more computational labor than do corresponding orders of approximation in the direct iteration approach.

The approximate procedure [Eqs. (2.32) and (2.33)] can easily be adapted to the coupled set of two-body equations (2.16). In the concluding section of this paper we comment upon the usefulness of Eq. (2.31) in obtaining multiple-scattering corrections in the N-d problem.

III. NUCLEAR MATRIX ELEMENTS OF THE TWO-BODY TRANSITION OPERATORS

The work of the preceding two sections has been largely independent of the nature of the target nucleus. We now confine ourselves exclusively to the case of a deuteron target and consider the problem of evaluating the ground-state matrix elements, $\langle a' | t | a \rangle$, of the two-nucleon transition operators.[36] Here, $|a\rangle$ is the state composed of the free incident particle and the deuteron ground state including its c.m. motion. For the sake of simplicity, we neglect the D state of the deuteron and any possible Coulomb effects.

The states $|a\rangle$ are written as

$$|a\rangle = |\mathbf{k}_1, \zeta\rangle, \quad (3.1)$$

where $\mathbf{k}_1$ is the wave vector of the incident nucleon (1) and ζ refers to a three-body spin state; it is not necessary to mention explicitly the deuteron variables other than the spin since we consider only the case when the target is in its ground state. In the coordinate representation,

$$\langle \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 | \mathbf{k}_1, \zeta \rangle = (2\pi)^{-3/2} e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} \Psi(\mathbf{r}_2, \mathbf{r}_3) | \zeta \rangle, \quad (3.2)$$

where $\Psi(\mathbf{r}_2, \mathbf{r}_3)$ is the deuteron S-state wave function including its c.m. motion and $|\zeta\rangle$ is any one of the six three-nucleon spin functions which are symmetrical in particles (2) and (3) which comprise the deuteron. The position vectors of the various particles are $\mathbf{r}_j$ ($j = 1, 2, 3$). In what follows the three-body plane-wave states are denoted by $|\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \zeta\rangle$.

[34] The transformation to the c.m. system is particularly simple in the case of a deuteron target; otherwise, cf., R. Lipperheide, Ann. Phys. (N.Y.) **17**, 114 (1962).

[35] Throughout this discussion we suppress any spin dependences for the sake of simplicity.

[36] The symbol t is here used in a generic sense to refer to $t^{(\pm)}$ or t_0.

The three-body spin functions can be decomposed into the sum[37]

$$|\zeta\rangle=\sum_{\nu,s} b(\zeta,\nu,s)|\nu\rangle|s\rangle, \tag{3.3}$$

where $|\nu\rangle$ and $|s\rangle$ represent two- and one-nucleon spin functions, respectively. The indices ζ, ν, and s serve to label the eigenvalues of both the total spin as well as the component of the spin in some direction for the system in question.

We confine ourselves, for the sake of definiteness, to two-body transition operators involving only the target nucleon (2). We then have

$$\begin{aligned}\langle \mathbf{k}_f,\zeta_f|t(2)|\mathbf{k}_i,\zeta_i\rangle &= \sum_{\zeta',\zeta''}\int d\tau_{\mathbf{k}'}d\tau_{\mathbf{k}''}\langle \mathbf{k}_f,\zeta_f|\mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3'',\zeta''\rangle \\ &\times\langle \mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3'',\zeta''|t(2)|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3',\zeta'\rangle \\ &\times\langle \mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3',\zeta'|\mathbf{k}_i,\zeta_i\rangle, \end{aligned}\tag{3.4}$$

where

$$d\tau_{\mathbf{k}}\equiv d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \tag{3.5}$$

and $\mathbf{k}_i$, $\mathbf{k}_f$ are two different wave vectors of particle (1). It follows as a consequence of translational and Galilean invariance that

$$\begin{aligned}&\langle \mathbf{k}_1'',\mathbf{k}_2'',\mathbf{k}_3'',\zeta''|t(2)|\mathbf{k}_1',\mathbf{k}_2',\mathbf{k}_3',\zeta'\rangle \\ &\quad=\delta(\mathbf{k}_3''-\mathbf{k}_3')\delta(\mathbf{k}_1''+\mathbf{k}_2''-\mathbf{k}_1'-\mathbf{k}_2') \\ &\quad\times\langle \tfrac{1}{2}(\mathbf{k}_1''-\mathbf{k}_2''),\zeta''|t_c(2)|\tfrac{1}{2}(\mathbf{k}_1'-\mathbf{k}_2'),\zeta'\rangle, \end{aligned}\tag{3.6}$$

where $t_c(2)$ is the transition operator defined in the c.m. system of particles (1) and (2). Equation (3.4) becomes, with the aid of (3.6),

$$\begin{aligned}&\langle \mathbf{k}_f,\zeta_f|t(2)|\mathbf{k}_i,\zeta_i\rangle \\ &\quad=\delta(\mathbf{K}_f-\mathbf{K}_i)\sum_{\nu_f,\nu_i}A(\zeta_f,\nu_f;\zeta_i,\nu_i)\int d\mathbf{q}\,\phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}) \\ &\quad\times\langle \tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa}-\tfrac{1}{2}\mathbf{q},\nu_f|t_c(2)|\tfrac{3}{4}\mathbf{k}_i'-\tfrac{1}{2}\mathbf{q},\nu_i\rangle. \end{aligned}\tag{3.7}$$

We have introduced the vectors

$$\begin{aligned}\mathbf{K}&=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3, \\ \mathbf{k}'&=\tfrac{2}{3}[\mathbf{k}_1-\tfrac{1}{2}(\mathbf{k}_2+\mathbf{k}_3)], \\ \mathbf{q}&=\tfrac{1}{2}(\mathbf{k}_2-\mathbf{k}_3); \end{aligned}\tag{3.8}$$

also

$$\boldsymbol{\kappa}=\tfrac{1}{2}(\mathbf{k}_f'-\mathbf{k}_i'). \tag{3.9}$$

The Fourier transform, $\phi(\mathbf{q})$, of the (S-state) deuteron wave function, $\psi(\mathbf{r})$, expressed in relative coordinates, and normalized to unity, is given by

$$\phi(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}\psi(\mathbf{r}). \tag{3.10}$$

Finally,

$$A(\zeta_f,\nu_f;\zeta_i,\nu_i)=\sum_s b(\zeta_f,\nu_f,s)b(\zeta_i,\nu_i,s). \tag{3.11}$$

[37] The coefficients $b(\zeta,\nu,s)$ in Eq. (3.3) can be calculated quite easily. See, for example, reference 13.

The quantity on the right-hand side of Eq. (3.7), exclusive of the momentum-conserving delta function, is just the matrix element of $t(2)$ in the three-body c.m. system. The corresponding matrix element for $t(3)$ is identical in form to (3.7).

The quantities $A(\zeta_f,\nu_f;\zeta_i,\nu_i)$ are known constants. Hence, we need consider only the integrals

$$\begin{aligned}\langle t_2(\boldsymbol{\kappa})\rangle_{\nu_f,\nu_i}&\equiv\int d\mathbf{q}\,\phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}) \\ &\times\langle \tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa}-\tfrac{1}{2}\mathbf{q},\nu_f|t_c(2)|\tfrac{3}{4}\mathbf{k}_i'-\tfrac{1}{2}\mathbf{q},\nu_i\rangle. \end{aligned}\tag{3.12}$$

These integrals will be called the *average two-body amplitudes*, and their evaluation constitutes the principal difficulty in the calculation of the matrix elements of $t(2)$.

Henceforth, we restrict ourselves to the particular case when

$$|\mathbf{k}_f'|=|\mathbf{k}_i'|\equiv k'. \tag{3.13}$$

Also, we assume that $\mathbf{k}_i$ ($\mathbf{k}_f$) are the initial (final) wave vectors of the incident particle in the laboratory system. Thus, we are concerned only with the matrix elements of the two-body transition operators which correspond to the single scattering of the incident particle from each of the target nucleons. Actually, a good portion of our subsequent work is valid even when (3.13) is not satisfied.

Under the preceding assumptions, it is easily found that the integral equation satisfied by $t_0(2)$ appropriate to the matrix element contained under the integral sign in (3.12) is

$$\begin{aligned}t_0(2)&=v(2)+v(2)(E_{12}-K_{12}+i\epsilon)^{-1}t_0(2) \\ &=v(2)+t_0(2)(E_{12}-K_{12}+i\epsilon)^{-1}v(2), \end{aligned}\tag{3.14}$$

where

$$E_{12}=E_i'-\tfrac{1}{2}(\hbar^2/2M)\mathbf{q}^2, \tag{3.15}$$

$$E_i'=(\hbar^2/2M)(\tfrac{3}{4}\mathbf{k}_i'-\tfrac{1}{2}\mathbf{q})^2, \tag{3.16}$$

and K_{12} is the kinetic-energy operator for the relative motion of particles (1) and (2). The reduced two-nucleon mass is denoted by M. Also, we have omitted the subscript c in Eq. (3.14) for the sake of clarity. The corresponding equations for $t^{(\pm)}(2)$ are

$$t_i^{(+)}(2)=v(2)+v(2)(E_i'-K_{12}+i\epsilon)^{-1}t_i^{(+)}(2), \tag{3.17a}$$

$$t_f^{(-)}(2)=v(2)+t_f^{(-)}(2)(E_f'-K_{12}+i\epsilon)^{-1}v(2), \tag{3.17b}$$

where

$$E_f'=(\hbar^2/2M)(\tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa}-\tfrac{1}{2}\mathbf{q})^2. \tag{3.18}$$

Once again, c has been suppressed in these relations. Equation (3.17a) [(3.17b)] is defined only when operating to the left (right) of states of energy E_i' (E_f'). The domain of definition of (3.14) is arbitrary.

IV. AVERAGE TWO-BODY AMPLITUDES

The quantities $\phi(\mathbf{q})$ and $\phi(\mathbf{q}-\boldsymbol{\kappa})$ appearing in (3.12) define the momentum distribution of the struck target

nucleon before and after the collision, respectively. Therefore, the average amplitude (3.12) represents the sum, over the initial and final momentum distributions of the target particle, of all kinematically possible two-body collisions of the incident particle with a given target nucleon. However, the kinematics of these two-body scatterings are such that E_f' and E_i' are, in general, unequal. Thus, in order to evaluate (3.12) exactly, one needs to know all the matrix elements of the two-nucleon transition operators both off and on the two-body energy shell.[38] For N-N scattering these matrix elements are not known except for a limited range of energies and then only approximately.

In addition to the uncertainties in the description of N-N scattering, the deuteron wave function $\phi(\mathbf{q})$ is not accurately known for large values of $|\mathbf{q}|$.[39] It is thus not necessary to insist on an extremely accurate evaluation of the average amplitudes. We, therefore, seek approximate methods for evaluating the integral (3.12) which make allowance for our lack of knowledge of the integrand and yet which do not involve a prohibitive amount of calculation. We now describe several methods which satisfy these criteria to varying degrees.

If the matrix element of t_c, $\langle t_c\rangle$, could be taken outside the integral sign, the evaluation of (3.12) would be simplified considerably. That is, if we write (suppressing the spin indices)

$$\langle t(\boldsymbol{\kappa})\rangle=\langle \mathbf{p}_f|t_c|\mathbf{p}_i\rangle\int d\mathbf{q}\,\phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}), \tag{4.1}$$

where $\mathbf{p}_i$ and $\mathbf{p}_f$ are appropriate wave vectors (to be considered presently), the integration over $\mathbf{q}$ can be carried out to yield

$$\langle t(\boldsymbol{\kappa})\rangle=\langle \mathbf{p}_f|t_c|\mathbf{p}_i\rangle F_0(\boldsymbol{\kappa}), \tag{4.2}$$

where

$$F_0(\boldsymbol{\kappa})\equiv\int d\mathbf{r}\,[\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})]|\psi(\mathbf{r})|^2; \tag{4.3}$$

in our case the wave function $\psi(\mathbf{r})$ is real. All the $\langle t_c\rangle$ in (3.12) represent scatterings with the same momentum transfer $\hbar(\mathbf{k}_f'-\mathbf{k}_i')$. Therefore, for consistency, we should require that

$$|\mathbf{p}_f-\mathbf{p}_i|=|\mathbf{k}_f'-\mathbf{k}_i'|. \tag{4.4}$$

Equation (4.2) is approximate, except when t_c is a local operator. However, the approximation is not defined unless the relations between $\mathbf{k}_i'$, $\mathbf{k}_f'$ and $\mathbf{p}_i$, $\mathbf{p}_f$ are known. Equation (4.4) represents a necessary condition for these relations but does not, by itself, specify $\mathbf{p}_i$, $\mathbf{p}_f$ in terms of $\mathbf{k}_i'$, $\mathbf{k}_f'$.

[38] We refer to the energy-conserving and energy-nonconserving matrix elements of the two-body transition operator as being on-the-energy-shell (OES) and off-the-energy-shell (FES), respectively. The OES matrix elements of $t^{(\pm)}$ which occur in (3.12) can be identified with the two-body scattering amplitudes. This is not the case for the operator t_0 unless one also has $E_f'=E_i'=E_{12}$.

[39] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 1.

Suppose we simply set $\mathbf{q}=0$ in $\langle t_c\rangle$ in the integral (3.12). In this case one obtains the form (4.2) with

$$\mathbf{p}_i=\tfrac{3}{4}\mathbf{k}_i', \tag{4.5a}$$

and

$$\mathbf{p}_f=\tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa}. \tag{4.5b}$$

This approximation can only really be justified for small $\kappa=|\boldsymbol{\kappa}|$. Since $\phi(\mathbf{q})$ is peaked at $\mathbf{q}=0$, one may expect that for small κ the major contribution to the integral (3.12) will arise from a fairly small neighborhood of $\mathbf{q}=0$. It is implicitly assumed here that $\phi(\mathbf{q})$ has a much stronger dependence on $\mathbf{q}$ than does $\langle t_c\rangle$; this seems to be the actual state of affairs when k' corresponds to energies consistent with the validity of the IA.[12]

On the other hand, $\phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q})$ has peaks of equal magnitude at $\mathbf{q}=0$ and $\mathbf{q}=\boldsymbol{\kappa}$. So for large κ it is highly questionable whether the $\mathbf{q}$ dependence of the matrix element can be ignored. Also, it is evident that for nonzero κ the choice (4.5a, b) for $\mathbf{p}_i$ and $\mathbf{p}_f$ leads to an FES matrix element of t_c in Eq. (4.2). Therefore, to be consistent, as well as to avoid FES matrix elements, it has become customary[26] to define the magnitude of $\mathbf{p}_f$ by

$$|\mathbf{p}_f|=\tfrac{3}{4}k'; \tag{4.5c}$$

its direction is taken to be in the scattering plane and is determined by Eq. (4.4) with $\mathbf{p}_i$ given by (4.5a). We refer to the approximation (4.2), (4.4), (4.5a), and (4.5c) as the *Chew* approximation (CA).[4,40]

We next discuss an alternative approach to the problem of relating $\mathbf{p}_i$, $\mathbf{p}_f$ and $\mathbf{k}_i'$, $\mathbf{k}_f'$. Let us define an average value of $\mathbf{q}$, namely, $\langle\mathbf{q}\rangle$, by

$$F_0(\boldsymbol{\kappa})\langle\mathbf{q}\rangle=\int d\mathbf{q}\,\phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q})\mathbf{q}. \tag{4.6}$$

It is then easy to show that

$$\langle\mathbf{q}\rangle=\tfrac{1}{2}\boldsymbol{\kappa}. \tag{4.7}$$

Now, suppose that $\langle t_c\rangle$ in (3.12) is expanded in a power series in $\mathbf{q}$ about the point $\mathbf{q}=\langle\mathbf{q}\rangle$. If we retain only the zeroth-order and linear terms in this expansion, it then follows from Eqs. (4.6) and (4.7) that the average amplitude (3.12) is given by Eq. (4.2) with

$$\mathbf{p}_i=\tfrac{3}{4}\mathbf{k}_i'-\tfrac{1}{4}\boldsymbol{\kappa}, \tag{4.8a}$$

and

$$\mathbf{p}_f=\tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{4}\boldsymbol{\kappa}. \tag{4.8b}$$

The approximation of (3.12) as given by (4.2) and (4.8) has been discussed previously by several authors from a somewhat different point of view.[3,18,19] Since it is evident that all that is involved here is the assumption of a linear dependence of $\langle t_c\rangle$ on $\mathbf{q}$, we refer to this approximation as the *linear* approximation (LA).

The LA represents a simple and convenient extension of the CA to large momentum transfers. It is clear from

[40] G. F. Chew, Phys. Rev. 84, 1057 (1951).

Eqs. (4.8) that the LA is equivalent to evaluating $\langle t_c \rangle$ in (3.12) at a point midway between the peaks of $\phi(\mathbf{q})$ and $\phi(\mathbf{q}-\boldsymbol{\kappa})$, and then taking it outside the integral sign. For small momentum transfers ($\kappa \approx 0$), it follows from Eqs. (4.8) that the Chew and linear approximations become identical. Also, we note from (4.8a, b) that

$$|\mathbf{p}_f| = |\mathbf{p}_i|. \tag{4.8c}$$

Thus, the average two-body amplitude calculated in the LA involves only OES matrix elements of t_c.

The CA and the LA have been used extensively to calculate N-d scattering in the single-scattering approximation over a wide range of energies.[7,11–14,26,41] In general, the results of these calculations indicate that for small momentum transfers, not only do the single-scattering terms seem to account for the dominant features of N-d scattering, but also these terms appear to be well represented in either the CA or the LA.

On the other hand, for large momentum transfers the agreement with experiment is comparatively poor, particularly for the polarization. This may be due either to the failure of the single-scattering approximation or to an inaccurate evaluation of the average amplitudes. It is of interest, therefore, to examine how much of an improvement with experiment is obtained if one evaluates the average amplitudes in a somewhat more accurate manaer than is the case in either the CA or LA.

Consider the product function

$$P(\mathbf{q},\boldsymbol{\kappa}) \equiv \phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}). \tag{4.9}$$

We have observed previously that $P(\mathbf{q},\boldsymbol{\kappa})$ has peaks of equal magnitude at $\mathbf{q}=0$ and $\mathbf{q}=\boldsymbol{\kappa}$. Equation (3.10) suggests, and it can be verified by numerical calculation with typical deuteron wave functions, that: (1) For a given value of q, $P(\mathbf{q},\boldsymbol{\kappa})$ attains its maximum value for $\mathbf{q}$ in the direction of $\boldsymbol{\kappa}$, and (2) for $q>\kappa$, $P(\mathbf{q},\boldsymbol{\kappa})$ decreases quite rapidly. These two effects are more pronounced the larger the value of κ.

On the basis of the preceding argument, let us assume that most of the contribution to the integral (3.12) arises from those vectors $\mathbf{q}$ which satisfy

$$\mathbf{q} = q(\boldsymbol{\kappa}/\kappa). \tag{4.10}$$

Then, with the aid of Eq. (4.10), we have

$$\langle t_2(\boldsymbol{\kappa}) \rangle \approx \int d\mathbf{q}\, \phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}) \times \langle \tfrac{3}{4}\mathbf{k}_f' + \tfrac{1}{2}\boldsymbol{\kappa}[1-(q/\kappa)] | t_c(2) | \tfrac{3}{4}\mathbf{k}_i' - \tfrac{1}{2}\boldsymbol{\kappa}(q/\kappa) \rangle. \tag{4.11}$$

The integration over the angular variables of $\mathbf{q}$ results in the function

$$F(q,\boldsymbol{\kappa}) \equiv \int d\Omega_q\, \phi^*(\mathbf{q}-\boldsymbol{\kappa})\phi(\mathbf{q}). \tag{4.12}$$

For typical deuteron wave functions, $F(q,\boldsymbol{\kappa})$ can be evaluated in closed form.

[41] J. Sawicki and S. Watanabe, Nucl. Phys. **10**, 151 (1959).

In view of our discussion of $P(\mathbf{q},\boldsymbol{\kappa})$, we may expect that $F(q,\boldsymbol{\kappa})$ will decrease rapidly for $q>\kappa$. If we assume that this decrease is much more rapid than any possible variation of the matrix element in (4.11), then the integration over q in (4.11) for $q>\kappa$ can be approximated by setting $\langle t_c \rangle$ equal to its value at $q=\kappa$. When this is done, we obtain

$$\langle t_2(\boldsymbol{\kappa}) \rangle \approx \int_0^\kappa dq\, q^2 F(q,\boldsymbol{\kappa}) \times \langle \tfrac{3}{4}\mathbf{k}_f' + \tfrac{1}{2}\boldsymbol{\kappa}[1-(q/\kappa)] | t_c(2) | \tfrac{3}{4}\mathbf{k}_i' - \tfrac{1}{2}\boldsymbol{\kappa}(q/\kappa) \rangle + \langle \tfrac{3}{4}\mathbf{k}_f' | t_c(2) | \tfrac{3}{4}\mathbf{k}_i' - \tfrac{1}{2}\boldsymbol{\kappa} \rangle F_\kappa(\boldsymbol{\kappa}), \tag{4.13}$$

where

$$F_\kappa(\boldsymbol{\kappa}) \equiv \int_\kappa^\infty dq\, q^2 F(q,\boldsymbol{\kappa}). \tag{4.14}$$

Since (4.13) involves energy-nonconserving matrix elements of t_c, in contrast to the CA and the LA, we will call (4.13) the off-the-energy-shell approximation (FA) for the average two-body amplitude. We note that for small κ the first term on the right-hand side can be neglected. Thus, in the limit of small momentum transfers, all three approximations become indistinguishable from one another.

V. OFF-THE-ENERGY-SHELL APPROXIMATION

In this section we discuss some of the details involved in the calculation of the average amplitude as given by (4.13).

The two-body wave vectors associated with $\langle t_c \rangle$ will be denoted by

$$\begin{aligned} \mathbf{p}_i &= \tfrac{3}{4}\mathbf{k}_i' - \tfrac{1}{2}\boldsymbol{\kappa} z, \\ \mathbf{p}_f &= \tfrac{3}{4}\mathbf{k}_f' + \tfrac{1}{2}\boldsymbol{\kappa}(1-z), \end{aligned} \tag{5.1}$$

where

$$z \equiv q/\kappa. \tag{5.2}$$

When $z=\frac{1}{2}$, Eqs. (5.1) reduce to the relations (4.8) which hold in the LA; no such corresponding reduction can be obtained for the CA (except in the case of small κ).

The scattering angle, θ, in the three-body c.m. system is defined by

$$\mathbf{k}_i' \cdot \mathbf{k}_f' = (k')^2 \cos\theta; \tag{5.3}$$

it will then prove convenient to write

$$\kappa = k'x, \quad x = \sin(\tfrac{1}{2}\theta). \tag{5.4}$$

Finally, the magnitudes of $\mathbf{p}_i$ and $\mathbf{p}_f$ can be expressed in the form

$$\begin{aligned} p_i &= \tfrac{3}{4}k' f(x,z), \\ p_f &= \tfrac{3}{4}k' f(x,1-z), \end{aligned} \tag{5.5}$$

respectively, where

$$f(x,z) = [1+(4/9)x^2 z(z+3)]^{1/2}. \tag{5.6}$$

It is clear that we will be interested in values of x, z such that $0\leqslant(x,z)\leqslant 1$.

Let us next define the three scattering angles in the two-body c.m. system, θ_{fi}, θ_f, and θ_i, by

$$\begin{aligned}\mathbf{p}_f\cdot\mathbf{p}_i&=p_fp_i\cos\theta_{fi},\\ \mathbf{p}_f\cdot\mathbf{p}_L&=p_fp_L\cos\theta_f,\\ \mathbf{p}_i\cdot\mathbf{p}_L&=p_ip_L\cos\theta_i,\end{aligned}\tag{5.7}$$

where

$$\mathbf{p}_L=\tfrac{3}{4}\mathbf{p}_i-\tfrac{1}{4}\boldsymbol{\kappa}.\tag{5.8}$$

It can now be verified that in the domain $0\leqslant(x,z)\leqslant 1$, $\cos\theta_{fi}$ is virtually independent of z for a given value of x, the maximum variation in θ_{fi} being about 2.5°. Similarly, one can show that $\cos\theta_i$ is close to unity in the same domain of x and z, which corresponds to an upper limit on θ_i of about 8.0°. At this point we have sufficient kinematical information available so that we can proceed to simplify $\langle t_c\rangle$ in (4.13).

Let us study the partial-wave expansion[42,43] of $\langle t\rangle$ (it is convenient in this detailed discussion to omit the subscript c on t). In the singlet case the antisymmetrized matrix element of t can be written as

$$\langle\mathbf{p}_f,\nu_f|\,t^{(0,I)}\,|\,\mathbf{p}_i,\nu_i\rangle=-\frac{2}{\pi}\sum_l\Delta(l,0,I)\left(\frac{2l+1}{4\pi}\right)^{1/2}t_l(p_f|\,p_i)Y_l^0(\mathbf{p}_f,\mathbf{p}_i),\tag{5.9}$$

where

$$\Delta(l,S,I)=1-(-1)^{l+S+I},\tag{5.10}$$

and $S(=0,1)$, $I(=0,1)$ refer to the total ordinary- and total isotopic-spin quantum numbers, respectively. $Y_l^m(\mathbf{p}_f,\mathbf{p}_i)$ denotes the normalized spherical harmonic which is a function of the angles of $\mathbf{p}_f$ with respect to a set of coordinate axes whose polar axis is in the direction of $\mathbf{p}_i$.

We know that θ_{fi} is, to a good approximation, independent of z. Therefore, we evaluate all of the Y_l^0 in (5.9) at $z=\frac{1}{2}$. With this choice θ_{fi} is just the two-body scattering angle which occurs in the LA.

In the triplet case the t matrix element has the form

$$\langle\mathbf{p}_f,\nu_f|\,t^{(1,I)}\,|\,\mathbf{p}_i,\nu_i\rangle=-\frac{2}{\pi}\sum_{l,l',m}\Delta(l,1,I)Y_l^{m*}(\mathbf{p}_s,\mathbf{p}_i)Y_{l'}^{m+\nu_i-\nu_f}(\mathbf{p}_f,\mathbf{p}_s)\times\delta_{l,l'}\sum_{J=|l-1|}^{l+1}\bar{C}_{l'l}(J,m;\nu_f,\nu_i)t_{l',l}^J(p_f|\,p_i),\tag{5.11}$$

where

$$\bar{\delta}_{l,l'}=\delta_{l,l'}-\delta_{|l'-l|,2},\tag{5.12}$$

$$\bar{C}_{l'l}(J,m;\nu_f,\nu_i)=C_{l'1}(J,m+\nu_i;m+\nu_i-\nu_f,\nu_f)\times C_{l1}(J,m+\nu_i;m,\nu_i),\tag{5.13}$$

and $C_{j'j}(J,m;\nu_f,\nu_i)$ is the Clebsch-Gordan coefficient.[44] In Eq. (5.11) $\mathbf{p}_s$ refers to a vector along the axis of spin quantization (all angular momenta are to be quantized with respect to this direction). The vector $\mathbf{p}_s$ must be independent of the coordinates of $\mathbf{q}$ when the expansion (5.11) is used in Eq. (4.13).

The principal complication which appears in this case arises from the fact that the orientation of the wave vector $\mathbf{p}_i$ with respect to the axis of spin quantization is not a constant. This results in the appearance in (5.11) of two independent sets of angular variables. We examine to what extent the expression (5.11) can be simplified for convenient use in the calculation of the average amplitudes.

In all of our subsequent work we identify $\mathbf{p}_s$ with $\mathbf{p}_L$. Then (5.11) becomes

$$\langle\mathbf{p}_f,\nu_f|\,t^{(1,I)}\,|\,\mathbf{p}_i,\nu_i\rangle=-\frac{2}{\pi}\sum_{l,l',m}\Delta(l,1,I)Y_l^{m*}(\theta_i)Y_{l'}^{m+\nu_i-\nu_f}(\theta_f)\delta_{l,l'}\times\sum_{J=|l-1|}^{l+1}\bar{C}_{l'l}(J,m;\nu_f,\nu_i)t_{l',l}^J(p_f|\,p_i),\tag{5.14}$$

where, since $\mathbf{p}_i$, $\mathbf{p}_f$, and $\mathbf{p}_L$ are coplanar, we have set the various azimuthal angles in (5.11) equal to zero.

The expression (5.14) is still quite cumbersome due to the sum over m. If θ_i were equal to zero, we would have

$$[Y_l^m(\theta_i)]_{\theta_i=0}=[(2l+1)/4\pi]^{1/2}\delta_{0,m},\tag{5.15}$$

and so the summation over m would be trivial. We have observed previously that θ_i is rather small in the relevant domains of x and z. However, some of the Legendre functions are rapidly varying for small angles and sometimes attain their maximum values at angles only a few degrees away from zero.[45] Therefore, we expect that simply to set θ_i equal to zero in (5.14) would be a poor approximation.

Now let $|\,\bar{\nu}\rangle$ denote a two-body spin function defined with respect to an axis of quantization in the direction of $\mathbf{p}_i$. Then

$$\langle\nu_f|\,t\,|\,\nu_i\rangle=\sum_{\bar{\nu}',\bar{\nu}''}\langle\nu_f|\,\bar{\nu}'\rangle\langle\bar{\nu}'|\,t\,|\,\bar{\nu}''\rangle\langle\bar{\nu}''|\,\nu_i\rangle.\tag{5.16}$$

The quantities $\langle\nu|\,\bar{\nu}\rangle$ describe a rotation of a coordinate system with a polar axis defined by the direction of $\mathbf{p}_i$ into a coordinate system with a polar axis defined by $\mathbf{p}_L$.

[42] The partial-wave expansions to be employed here are developed, for example, in reference 43. It is supposed that t conserves total angular momentum, total ordinary spin, parity, is independent of the orientation of the total angular momentum, and is charge independent. The spin index ν takes on the values ±1, 0, and 0′, the index 0′ referring to singlet states.

[43] K. L. Kowalski and D. Feldman, J. Math. Phys. 2, 499 (1961).

[44] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Appendix A.

[45] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

From the well-known representation[46] of the rotation operator for this case, it follows that for θ_i less than about 8.0°,

$$\langle \nu | \bar{\nu} \rangle \approx \delta_{\nu,\bar{\nu}}. \tag{5.17}$$

Assuming (5.17) to be valid, we have then

$$\langle \mathbf{p}_f,\nu_f | t^{(1,I)} | \mathbf{p}_i,\nu_i \rangle = \pi^{-3/2} \sum_{l,l'} \Delta(l,1,I)(2l+1)^{1/2} Y_{l'}^{\nu_i-\nu_f}(\theta_{fi}) \delta_{l,l'} \times \sum_{J=|l-1|}^{l+1} \bar{C}_{l'l}(J,0;\nu_f,\nu_i) t_{l',l}^{J}(p_f|p_i). \tag{5.18}$$

It should be emphasized that (5.18) is entirely distinct from what one would obtain by setting θ_i equal to zero in (5.11), since θ_f and θ_{fi} are, in general, quite different. As in the singlet case, we make the additional approximation of setting θ_{fi} in (5.18) equal to its value at $z=\frac{1}{2}$.

It will prove convenient in future discussions to write

$$\langle \mathbf{p}_f,\nu_f | t | \mathbf{p}_i,\nu_i \rangle = (1/p_i) M_{\nu_f,\nu_i}(\mathbf{p}_f | \mathbf{p}_i). \tag{5.19}$$

The form of the M's is evident from Eqs. (5.9) and (5.18).

We next examine some special complications which arise if one chooses to employ t_0 as the transition operator, t_c, in Eq. (4.13). The calculation of the matrix elements of t_0 is known to be considerably more difficult than that of the operators $t^{(+)}$ or $t^{(-)}$.[17] Therefore, it is of considerable interest to investigate whether or not it is possible to approximate t_0 in (4.13) by either $t^{(+)}$ or $t^{(-)}$. It is apparent from Sec. III that this question reduces to whether or not E_{12} is nearly equal to E_i' or E_f' in the range of $\mathbf{q}$ which is of interest in the FA.

Let us define E_δ by

$$E_\delta = E_i' - E_{12}. \tag{5.20}$$

In the range of $\mathbf{q}$ which is involved in the FA one can show, with the aid of Eqs. (5.5) and (5.6), that

$$\begin{aligned} (E_\delta/E_i') &= (8/9)[zx/f(x,z)]^2, \\ (E_\delta/E_f') &= (8/9)[zx/f(x,1-z)]^2, \\ (E_i'/E_f') &= [f(x,z)/f(x,1-z)]^2. \end{aligned} \tag{5.21}$$

It is interesting to note that for $z=\frac{1}{2}$ we have

$$(E_\delta/E_i') = (E_\delta/E_f') = 2x^2(9+7x^2)^{-1}; \tag{5.22}$$

this is small even for backward scattering ($x=1$) and so in the LA one can to a good approximation identify $\langle t_c \rangle$ with the N-N scattering amplitudes, independently of which of the three two-body transition operators is used to represent t_c.[38]

It can be shown by direct calculation that, in the domain of interest, (E_{12}/E_i') is close to unity except when $(x,z) \approx 1$. On the other hand, (E_{12}/E_f') differs quite appreciably from unity except for very small x or when $z \approx \frac{1}{2}$. We, therefore, conclude that E_i' is a better approximation to E_{12} than is E_f', and in the FA we may identify $t^{(+)}$ with t_0. Most of the calculations to follow will be concerned primarily with the $t^{(+)}$ operator; however, we will see that the partial-wave amplitudes of $t^{(-)}$ are easily determined from the corresponding quantities for $t^{(+)}$.

VI. PARTIAL-WAVE AMPLITUDES OF THE TWO-NUCLEON TRANSITION OPERATORS

We proceed with the calculation of the partial-wave amplitudes of the operator $t_i^{(+)}$ [cf. Eq. (3.17a)]. Once this is done, we can, in view of the work of the preceding section, complete the evaluation of the integrals (4.13).

Now $t_i^{(+)}$ is related to the reactance operator, R_i, by

$$t_i^{(+)} = R_i - i\pi R_i \delta(E_i' - K_{12}) t_i^{(+)}. \tag{6.1}$$

R_i satisfies an integral equation identical to (3.17a) except for the replacement of $(E_i'-K_{12}+i\epsilon)^{-1}$ by $P(E_i'-K_{12})^{-1}$, where P denotes the Cauchy principal value. The use of R_i has the advantage that its partial-wave amplitudes are real if those of the potential are also real.

Most phenomenological N-N potentials contain a hard-core (h.c.) term.[47] Then, instead of R_i, it is more convenient to consider the operator[48]

$$R_i' = R_i - \bar{\Gamma}_i, \tag{6.2}$$

where $\bar{\Gamma}_i$ is the reactance operator for the case of pure h.c. scattering.[49] It can be shown that

$$R_i' = \bar{U}_i' + \bar{U}_i' P(E_i'-K_{12})^{-1} R_i', \tag{6.3}$$

where

$$\bar{U}_i' = (1+\bar{\omega}_i) v (1+\bar{\omega}_i^\dagger). \tag{6.4}$$

Here, v is the two-nucleon interaction exclusive of the h.c.; it is assumed that the coordinate representatives of v vanish for distances less than the h.c. radius.

With the use of expansions of the form (5.9) and (5.11), one can obtain from Eq. (6.3) an integral equation for the partial-wave amplitudes, $R'(p_f|p_i)$, of R_i', viz.,

$$R'(p_f|p_i) = \bar{U}'(p_f|p_i) + cP\int_0^\infty \frac{dp\, p^2}{p_i^2 - p^2} \bar{U}'(p_f|p) R'(p|p_i), \tag{6.5}$$

where

$$c = (4M/\pi\hbar^2). \tag{6.6}$$

We use the symbol $R'(p_f|p_i)$ in a generic sense. Thus, in the singlet and uncoupled triplet cases $R'(p_f|p_i)$ is equal to $R_l'(p_f|p_i)$ and $R_{J,J'}^{J}(p_f|p_i)$, respectively. In

[46] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), Chap. 4.

[47] T. Hamada, Progr. Theoret. Phys. (Kyoto) **24**, 1033 (1960); **25**, 247 (1961), and references cited therein.

[48] The operator $\bar{\Gamma}_i$, as well as the operator $\bar{\omega}_i$ which is introduced below, is defined in the Appendix.

[49] The following treatment of the scattering operators is elaborated upon more fully in references 17 and 43.

the coupled triplet case $R'(p_f|p_i)$ is a 2×2 matrix with diagonal (nondiagonal) components $R_{J\pm1,J\pm1}{}'^{J}(p_f|p_i)$ $[R_{J\pm1,J\mp1}{}'^{J}(p_f|p_i)]$. A similar notation is used for the partial-wave amplitudes of $\bar{U}_i'$.

The solution of Eq. (6.5) can be written as[17,43]

$$R'(p_f|p_i)=\varphi(p_f|p_i)R'(p_i|p_i), \tag{6.7}$$

where

$$\varphi(p_f|p_i)=\tau(p_f|p_i)+\int_0^\infty dp\,\Lambda(p_f|p)\varphi(p|p_i), \tag{6.8}$$

$$\tau(p_f|p)=\bar{U}'(p_f|p)[\bar{U}'(p_i|p)]^{-1}, \tag{6.9}$$

and

$$\Lambda(p_f|p)=c[p^2/(p_i^2-p^2)][\tau(p_f|p)-\tau(p_f|p_i)]\bar{U}'(p_i|p). \tag{6.10}$$

It follows from Eq. (6.2), with the aid of the expression (6.7) for $R'(p_f|p_i)$, that

$$R(p_f|p_i)=\varphi(p_f|p_i)R(p_i|p_i)+\theta(p_f|p_i), \tag{6.11}$$

where

$$\theta(p_f|p_i)=\bar{\Gamma}(p_f|p_i)-\varphi(p_f|p_i)\bar{\Gamma}(p_i|p_i). \tag{6.12}$$

In terms of partial-wave amplitudes, Eq. (6.1) becomes

$$t^{(+)}(p_f|p_i)=R(p_f|p_i)-i\tilde{c}R(p_f|p_i)t^{(+)}(p_i|p_i), \tag{6.13}$$

where

$$\tilde{c}=\tfrac{1}{2}\pi p_i c. \tag{6.14}$$

A relation for $t^{(+)}(p_f|p_i)$ similar to (6.11) can be obtained by combining Eqs. (6.11) and (6.13), viz.,

$$t^{(+)}(p_f|p_i)=\Phi(p_f|p_i)t^{(+)}(p_i|p_i)+\theta(p_f|p_i), \tag{6.15}$$

where

$$\Phi(p_f|p_i)=\varphi(p_f|p_i)-i\tilde{c}\theta(p_f|p_i). \tag{6.16}$$

Equation (6.15) permits a calculation of the FES partial-wave amplitudes of $t_i^{(+)}$ in terms of the OES amplitudes. The latter quantities can, in turn, be expressed in terms of phase shifts in the standard manner.[50] An expression for the partial-wave amplitudes of the operator $t_i^{(-)}$ which has nearly the same form as Eq. (6.15) can be obtained by use of the identity

$$t^{(-)}(p_i|p_f)=\tilde{t}^{(+)}(p_f|p_i). \tag{6.17}$$

In the present work both the FES and the OES partial-wave amplitudes of $t_i^{(+)}$ were calculated for the Hamada potential.[47] The OES amplitudes corresponding to two-body laboratory energies up to 300 MeV were determined exactly using the phase shifts listed in reference 47. For energies between 300 and 350 MeV, the OES amplitudes were obtained by extrapolating the quantity $p_i t^{(+)}(p_i|p_i)$ which varies rather smoothly with energy. For energies greater than 350 MeV, $p_i t^{(+)}(p_i|p_i)$ was set equal to its value at 350 MeV.

[50] See, for example, H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

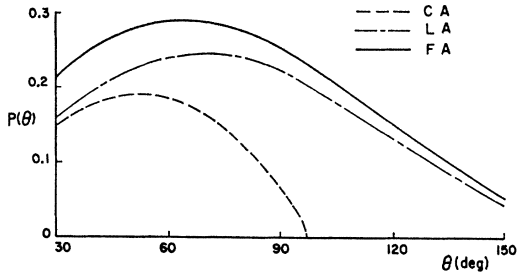

FIG. 1. Nucleon-deuteron polarizations at 40 MeV; θ is the c.m. scattering angle.

In order to calculate the FES amplitudes, one needs to know the function $\varphi(p_f|p_i)$. (The h.c. amplitudes are regarded as known.) It is evident from (6.8) that the simplest approximation to φ is just

$$\varphi(p_f|p_i)\approx\tau(p_f|p_i). \tag{6.18}$$

This approximation has been discussed elsewhere,[17,43] and it is expected to be valid when p_i and p_f are not too different, which is the circumstance in the FA. We have confined ourselves only to the consideration of the representation (6.18) for φ.

The partial-wave amplitudes of $\bar{U}_i'$ are required in order to calculate $\tau(p_f|p_i)$. These consist of a sum of integrals which have the form (in the triplet case, for example)

$$\bar{U}_{l',l}{}'^{J}(p_f|p_i)_\lambda=B(J)\int_a^\infty dr\,r^2F_{l'}(p_f,p_i,r)\times v_\lambda(r)F_l(p_i,p_i,r), \tag{6.19}$$

where

$$F_l(p_f,p_i,r)=j_l(p_f r)-j_l(p_f a)[n_l(p_i r)/n_l(p_i a)], \tag{6.20}$$

and a is the h.c. radius. The index λ is used to distinguish the central, tensor, spin-orbit, or quadratic spin-orbit parts of the potential, v. The integrals (6.19) were evaluated numerically with the upper limit of integration cut off at a distance of three pion Compton wavelengths beyond the core radius. The error resulting from this cutoff is estimated to be no more than a few percent, at least for the values of p_i and p_f employed.

Once the partial-wave amplitudes of $t_i^{(+)}$ are known, the integration (4.13) can be carried out in a relatively straightforward manner. It was found that M_{ν_f,ν_i} [cf., Eq. (5.19)] can be represented, at least to an accuracy consistent with all the other approximations which have been made, as a function of z (for $0\leqslant z\leqslant1$) by two linear segments which join at $z=\frac{1}{2}$. The slopes of these segments were deduced from the values of M_{ν_f,ν_i} at $z=0.1, \frac{1}{2}, 1.0$.

Finally, the deuteron wave function was taken to be

$$\psi(r)=(N/r)(e^{-\alpha r}-e^{-\beta r}), \tag{6.21}$$

where $\alpha=0.232$ f^{-1} and $\beta=1.332$ f^{-1}. The constant N is defined by the requirement that $\psi(r)$ be normalized to unity.

VII. RESULTS AND DISCUSSION

The average amplitudes (4.13) were computed in the FA in the manner discussed in the two preceding sections for incident nucleon (lab) energies (E_1) of 40, 95, and 150 MeV and for c.m. scattering angles (θ) ranging from 30° to 150°. The operator t_c was taken as $t_i^{(+)}$ except at 150 MeV where the calculations were also carried out with $\hat{t}_{fi}=\frac{1}{2}[t_f^{(+)}+t_i^{(-)}]$. The resultant matrix elements (3.7) were then employed to deduce the N-d cross sections $[\sigma(\theta)]$ and polarizations $[P(\theta)]$ in the single-scattering approximation.[51] The results are presented in Figs. 1–6. For purposes of comparison, the curves for the CA and the LA, for which cases $t_i^{(+)}$ and $t_f^{(-)}$ are identical, are also shown. The specific choice of energies and angles was motivated by the following considerations.

For $E_1<40$ MeV one may expect that the IA will cease to be a good approximation and, in addition, multiple-scattering effects will become increasingly important.[8] Therefore, a comparison of various methods for calculating the average two-body amplitudes which is based upon an identification of the single-scattering terms with the complete N-d transition matrix cannot be too meaningful for such low energies.

On the other hand, even for $E_1\approx 150$ MeV, it turns out that for large κ, two-nucleon scatterings corresponding to (two-body laboratory) energies in excess of 300 MeV contribute significantly to the average amplitudes. Thus, for $E_1>150$ MeV, information is required about the N-N transition matrix which is not available at the present time in any consistent form.

When θ is less than about 30°, the CA, the LA, and the FA all tend to become virtually identical and so nothing new can be learned. When $\theta>150°$, the pickup scattering is dominant and so a comparison of the direct-scattering terms with experiment is of little value except, perhaps, as part of a study of the nature of the interference between the direct and pickup scattering.

Now, in the CA or the LA, $\sigma(\theta)$ is proportional to the square of the form factor, $F_0(\kappa)$, and $P(\theta)$ is entirely independent of the deuteron wave function. Therefore, in the FA it seems reasonable to expect that $P(\theta)$ will

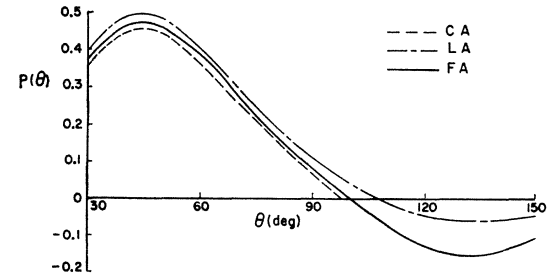

FIG. 2. Nucleon-deuteron polarizations at 95 MeV; θ is the c.m. scattering angle.

[51] The spin sums which need to be performed in order to relate the matrix elements (3.7) to observable quantities have been carried out many times in the literature (cf., reference 41).

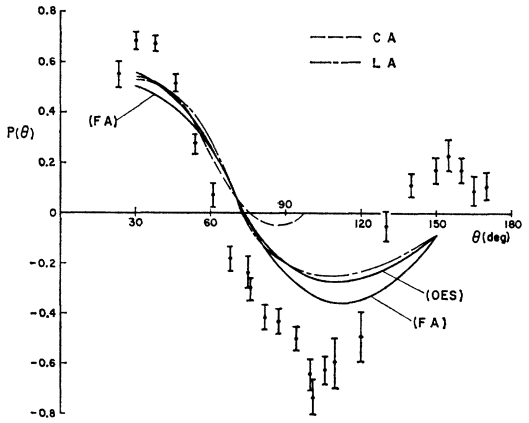

FIG. 3. Nucleon-deuteron polarizations at 150 MeV; θ is the c.m. scattering angle. (The experimental points are taken from reference 7.)

be relatively insensitive to any uncertainties in the deuteron wave function, in contrast to $\sigma(\theta)$. So, in order to ascertain better the relative merits of the various methods for evaluating the average amplitudes, we will devote most of the subsequent discussion to the polarization calculations.

In Figs. 1 and 2 some differences can be noted between the polarizations computed in the FA [with $t^{(+)}$] and those in either the CA or the LA. Unfortunately, there exists no experimental information on $P(\theta)$ at either 40 or 95 MeV, and so the significance of these differences is not apparent.

However, at 150 MeV, it is evident from Fig. 3 that a fairly large negative $P(\theta)$ is obtained at large angles in the FA. The improvement over either the CA or the LA is significant. The failure to obtain a better fit to the peaks in $P(\theta)$ (except in the pickup region) may be, in part, due to the Hamada potential itself; it is, however, clear that the neglect of the multiple scattering must also, in part, be responsible for the discrepancy, expecially at the larger angles.

In order to estimate the FES effects of $\langle \mathbf{p}_f|t_i^{(+)}|\mathbf{p}_i\rangle$ on the value of (4.13), we also calculated the average amplitudes, within the context of the FA, setting p_f equal to p_i. This we call the OES approximation. It is apparent from Fig. 3 that FES effects are needed to obtain an improved fit to $P(\theta)$.

At 150 MeV, the polarization computed in the (FA)′ [i.e., in the FA with $\hat{t}_{fi}$] is virtually identical to that obtained in the LA. Since the latter differs from the result derived with $t_i^{(+)}$, there seems to be some doubt as to whether the $t^{(+)}$ and $t^{(-)}$ operators can be regarded as equivalent, even in the IA [cf., Eq. (1.33) and reference 9]. The cross sections (Fig. 6) also show important differences between the $t_i^{(+)}$ and $\hat{t}_{fi}$ cases. On the basis of the present calculations, one may conjecture that Watson's form of the IA appears to be the most consistent in its application since the $t^{(\pm)}$ ambiguity does not arise; also, the best fit to the polarization is obtained with the $t_i^{(+)}$ operator which in the FA corresponds to t_0.

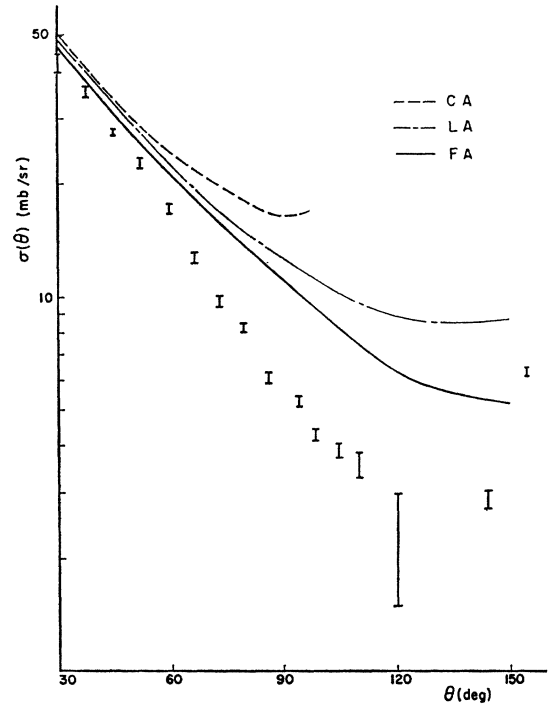

FIG. 4. Nucleon-deuteron scattering cross sections at 40 MeV; θ is the c.m. scattering angle. [The experimental points are taken from J. H. Williams and M. K. Brussel, Phys. Rev. **110**, 136 (1958).]

The cross sections for the several energies under consideration are presented in Figs. 4–6. A rather remarkable agreement is obtained with the large-angle data at 150 MeV using the various forms of the FA. However, this may be somewhat fortuitous. A comparison of the calculation of Postma and Wilson[7] in the CA, using the experimental deuteron form factor, with a second computation,[26] identical to the first except for the use of a form factor determined by a Hulthén wave function, leads us to suspect that with an improved deuteron wave function all the cross-section curves will be dropped down somewhat. Irrespective of the inadequacies of the deuteron wave function, Figs. 4–6 indicate that it is necessary to account for the internal motion of the target nucleons as well as the FES behavior of $\langle t_c \rangle$ in order to obtain a fit to the cross sections for large-angle scatterings.

VIII. CONCLUDING REMARKS

We have seen in the preceding section that it is possible to obtain a fair agreement with experiment, in the single-scattering approximation, if the dominant effects of the internal target nucleon momenta are taken into account. Nevertheless, it is clear that there is considerable room for improving the detailed fit to the data.

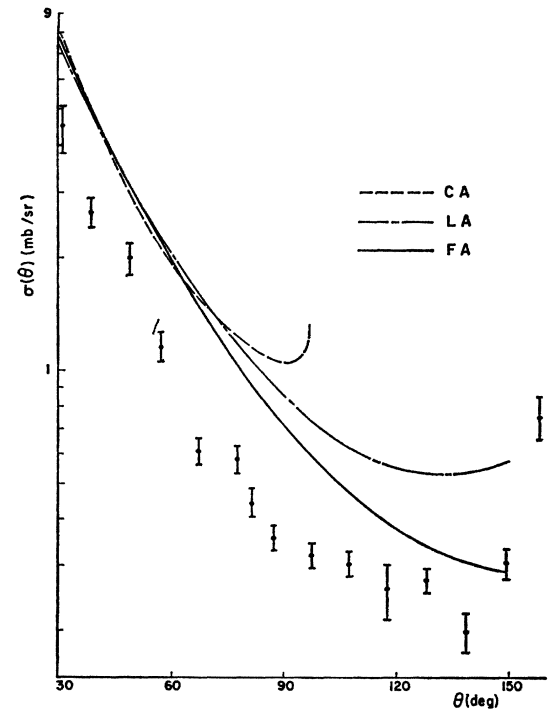

FIG. 5. Nucleon-deuteron scattering cross sections at 95 MeV; θ is the c.m. scattering angle. (The experimental points are taken from reference 19.)

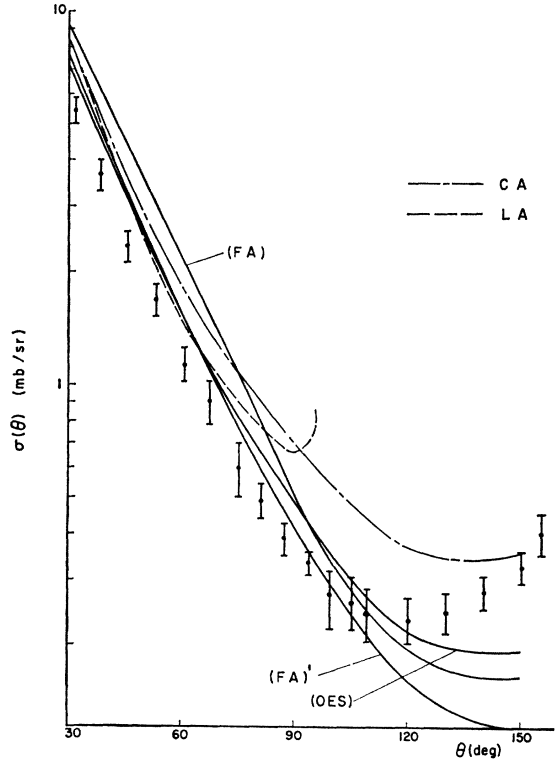

FIG. 6. Nucleon-deuteron scattering cross sections at 150 MeV; θ is the c.m. scattering angle. (The experimental points are taken from reference 7.)

Within the context of our calculation several possible improvements are obvious. For example, the FES behavior of $\langle t_c \rangle$ can be determined more accurately; also, the D state of the deuteron can be taken into account. However, such refinements do not appear to be called for at the present time in the light of current uncertainties in our knowledge of the two-nucleon system, particularly the deuteron.

In view of the difficulties associated with the evaluation of the single-scattering terms it would appear somewhat academic, at this stage, to consider the multiple scattering. However, there exists a considerable angular range (beyond about 75° at 150 MeV), exclusive of the pickup region, where the agreement with experiment obtained with the single-scattering terms still leaves something to be desired. It may very well be that a more accurate evaluation of the single-scattering terms is in order, but this does not seem too probable. It is more likely that any discrepancy is due to the onset of multiple scattering. In any case, some estimate of the magnitude of the multiple scattering would be of considerable interest.

The computation of multiple-scattering corrections is, of course, an exceedingly tedious task. Moreover, Everett[52] has suggested that the standard iteration techniques may be unreliable in the case of N-d scattering. However, it will be noticed that, with the use of an equation such as (2.31) [as applied to Eqs. (2.16)], only the terms corresponding to double scattering need be calculated. Equation (2.31) then effectively includes all orders of multiple scattering, although only the double-scattering terms are represented exactly.

[52] A. Everett, Phys. Rev. **126**, 831 (1962).

Finally, we remark that for N-d scattering the assumption of ground-state scattering is probably sufficient. In fact, due to the circumstance that for the deuteron the binding force acting on a target nucleon is actually the binding force of the entire nucleus, it would seem inconsistent with the IA to have too great a contribution to the scattering resulting from excited deuteron intermediate states.

ACKNOWLEDGMENTS

We should like to express our thanks to the members of the Brown University Computing Laboratory for advice and assistance.

APPENDIX

The hard-core operators $\bar{\Gamma}_i$ and $\bar{\omega}_i$ are defined in terms of their coordinate representatives as

$$\langle \mathbf{r} | \bar{\omega}_i | \mathbf{r}' \rangle = -\frac{\delta(r-a)}{a^2} \sum_l \left(\frac{2l+1}{4\pi}\right) \frac{\bar{g}_l(a|r')}{\bar{g}_l(a|a)} P_l(\mathbf{r}\cdot\mathbf{r}'), \quad \text{(A1)}$$

$$\langle \mathbf{r} | \bar{\Gamma}_i | \mathbf{r}' \rangle = -\frac{\delta(r-a)\delta(r'-a)}{a^4} \sum_l \left(\frac{2l+1}{4\pi}\right) \frac{P_l(\mathbf{r}\cdot\mathbf{r}')}{\bar{g}_l(a|a)}, \quad \text{(A2)}$$

where $\mathbf{r}$ is the relative position vector of two nucleons and a is the h.c. radius. The radial Green's function is given by

$$\begin{aligned} \bar{g}_l(r|r') &= (2M/\hbar^2) p_i j_l(p_i r') n_l(p_i r), \quad r > r', \\ &= (2M/\hbar^2) p_i j_l(p_i r) n_l(p_i r'), \quad r < r', \end{aligned} \quad \text{(A3)}$$

where j_l and n_l are the spherical Bessel and Neumann functions, respectively. Finally, P_l is the Legendre polynomial.