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Anomalous Solutions to the Dirac and Schrödinger Equations for the Coulomb Potential*

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Certain anomalous solutions to the Dirac and relativistic Schrödinger equations for the Coulomb potential, some of which have not been investigated before, are analyzed. These are solutions for orbital angular momentum $l=0$ and $l=\frac{1}{2}$ in the Schrödinger case and total angular momentum $j=0$ in the Dirac case that are quadratically integrable for all energies $E<0$. The purpose is to determine if there exist fundamental reasons for discarding them, or if they describe meaningful physical systems. Reasons are given why they may be discarded for real two-body systems, such as the hydrogen atom. These solutions can only be meaningful for systems which have an exact one-body, pure Coulomb relativistic Hamiltonian. They are valid for repulsive as well as attractive potentials. It is suggested that the $l=0$ relativistic Schrödinger solution be identified as the wave function of a nonrigid charged spherical shell. In addition, it is suggested that a particle fluctuating about its own center of mass in zitterbewegung-type motion may experience a repulsive Coulomb self-potential and have an exact one-particle-type Hamiltonian. The charge distributions obtained in this interpretation for two of the solutions have mean radii $\sim e^2/m_0c^2$ and rms radii $\sim \hbar/m_0c$.

I. INTRODUCTION

IN a previous paper,[1] the condition of finiteness of the wave function at the origin, usually imposed in nonrelativistic quantum mechanics, was discussed. The "anomalous" $l=0$ solution, treated by Tietz,[2] Kramers,[3] and others,[4] which is usually excluded by such a condition (see, e.g., Bethe and Salpeter,[5] or Schiff[6]), was shown not to obey the nonrelativistic (NR) Schrödinger equation and, hence, not to require exclusion from a NR theory. It does obey the relativistic Schrödinger (RS) equation, as was shown in reference 1, and as we show presently, there is another solution for $l=\frac{1}{2}$, as well as an analogous solution to the Dirac equation. None of these solutions has a satisfactory NR limit.

We now wish to consider the relativistic equations with Coulomb potential in more detail with regard to such "anomalous" solutions. We define these as quadratically integrable solutions to one of the equations of quantum mechanics which are superfluous as far as present theory is concerned. We wish to investigate their mathematical and physical properties in order to ascertain whether there exists some fundamental basis for discarding them, or whether they may describe meaningful physical systems. Particularly pertinent is the fact that they can describe repulsive, as well as attractive, Coulomb systems.

Our conclusion is that they may be discarded as far as real two-particle systems (e.g., the hydrogen atom) are concerned. This is based on the observation that the simple Coulomb Hamiltonian that generates these solutions is not physically valid for a two-body system if the system is in a state described by these solutions; viz., the equation and these solutions do not form a physically self-consistent scheme. We suggest that a pseudo one-body model exists, that of a charged spherical shell, for which one of the anomalous solutions, the one for $l=0$, is valid. Although we cannot yet *prove* that this model and the anomalous solution wave function we associate with it are physically meaningful, the results are very plausible. They are also important, since the charged spherical shell is such a tantalizing classical model for the electron. It has been discussed extensively in the literature from the classical standpoint,[7,8] it has been shown to be consistent with quantum electrodynamics and relativity without the necessity of nonelectromagnetic (Poincaré) forces,[7] and some promising but inconclusive results have been obtained through attempts to quantize this model.[9,10] In addition, we can now show that such a model is stable according to quantum mechanics, and will not blow up from its internal Coulomb stress.

We go on to suggest the possibility that the half-integral solutions may pertain to another pseudo one-body model. The suggestion is that a particle can move, to a limited extent, in its own field ("zitterbewegung"), but that in so doing it experiences a repulsive Coulomb self-potential. The anomalous solutions may be interpretable as bound states of this repulsive system. No attempt is made, in this paper, to set up a complete theory of this nature, although considerable work has been carried out and the indications obtained are

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[1] Baxter H. Armstrong and Edwin A. Power, Am. J. Phys. **31**, 262 (1963).
[2] T. Tietz, Soviet Phys.—JETP **3**, 777 (1956).
[3] H. A. Kramers, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1958).
[4] See, e.g., A. Sommerfeld, *Atombau and Spectrallinien* (Frederick Unger Publishing Co., New York, 1953), 2. Auf. 2. Bd.
[5] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1957).
[6] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949).
[7] F. Rohrlich, Am. J. Phys. **28**, 639 (1960).
[8] T. Erber, Fortschr. Physik **9**, 343 (1961).
[9] D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948).
[10] P. A. M. Dirac, Proc. Roy. Soc. (London) **A268**, 57 (1962).

favorable. Such an attempt is obviously a difficult problem with far-reaching consequences. The scope of the problem makes it desirable to merely present the solutions and their simpler properties with a provisional interpretation, based somewhat on conjecture, rather than await a comprehensive denouement. Whether the interpretation proposed is correct or not, the bearing of these solutions and the ideas presented therewith on the interpretation of currently accepted equations and theory is sufficiently important to warrant their general consideration.

Therefore, we first present the solutions mathematically, without regard to their possible physical interpretation, in Sec. II. A discussion *vis à vis* current theory is carried out in Sec. III, and Sec. IV is devoted to the theory of the charged spherical shell. The remainder of the paper (Sec. V) concerns the possible interpretation of the half-integral solutions, and a computation of their normalization and the mean radii of their "charge distributions."

II. THE ANOMALOUS SOLUTION TO THE DIRAC EQUATION FOR THE COULOMB POTENTIAL

A. The Radial Dependence

The general solution to the radial part of the Dirac equation for a Coulomb potential is given in reference 1 in terms of functions that vanish at infinity for all values of the angular momentum parameter κ and energy E. The two functions which constitute this solution are

$$f(\rho)=N_1e^{-\rho/2}\rho^{p-1}[(\kappa+\Lambda/\epsilon)\psi(1+p-\Lambda,\,2p+1;\rho)-\psi(p-\Lambda,\,2p+1;\rho)], \quad (2.1)$$
$$g(\rho)=N_2e^{-\rho/2}\rho^{p-1}[(\kappa+\Lambda/\epsilon)\psi(1+p-\Lambda,\,2p+1;\rho)+\psi(p-\Lambda,\,2p+1;\rho)].$$

In these formulas,

$$N_1=[(1-\epsilon)^{1/2}/(\kappa+\Lambda/\epsilon)]N;$$
$$N_2=[(1+\epsilon)^{1/2}/(\kappa+\Lambda/\epsilon)]N;$$

N is the normalization constant, κ is an eigenvalue of the operator[6] $\hbar^{-1}\beta(\boldsymbol{\sigma}'\cdot\mathbf{L}+\hbar)$ (the notation is that of reference 6), and ψ is a confluent hypergeometric function. It is defined in terms of the Whittaker function[11] $W_{k,\mu}(x)$, where $k=(c/2)-a$ and $\mu=(c/2)-\frac{1}{2}$, by

$$\psi(a,c;x)=e^{x/2}x^{-\frac{1}{2}-\mu}W_{k\mu}. \quad (2.2)$$

Its properties are adequately defined and discussed in the Bateman Manuscript Project, Higher Transcendental Functions.[12] The symbols ρ, Λ, ϵ, and p are defined in terms of standard quantities by the relations:

$$\Lambda\equiv\gamma\epsilon(1-\epsilon^2)^{-1/2},$$
$$\rho\equiv2\lambda_0r,$$
$$\epsilon=W/m_0c^2,$$
$$\lambda_0=(m_0c/\hbar)(1-\epsilon^2)^{1/2},$$
$$\gamma=Ze^2/\hbar c,$$
$$p^2=\kappa^2-\gamma^2.$$

W is the total energy, including the rest mass. We define E as $W-m_0c^2$. Whether the functions f and g of Eq. (2.1) are quadratically integrable depends on their behavior as $\rho\rightarrow0$. For $\psi(a,c;\rho)$ this is given by[1,13] $1/\Gamma(a)\rho^{c-1}$, so that

$$g(\rho),\,f(\rho)\sim1/\{\Gamma[1+(\kappa^2-\gamma^2)^{1/2}-\Lambda]\rho^{(\kappa^2-\gamma^2)^{1/2}+1}\}. \quad (2.4)$$

Therefore, as noted in reference 1, since the exponent of ρ is $1+(1-\gamma^2)^{1/2}$ for $l=0$, f and g are not integrable in that case. (The "large component" g for $l=0$ corresponds in the NR limit to the anomalous solution of the Schrödinger equation discussed in reference 1, which turns out to be an unsatisfactory solution at the origin.) The poles of the Γ function in Eq. (2.4), viz., negative integral values of $1-p-\Lambda$, lead to the normal hydrogen energy spectrum and eigenfunctions. In terms of κ, poles of the Γ function occur when the relationship[14]

$$1+(\kappa^2-\gamma^2)^{1/2}-\Lambda=1-n',\quad n'=1,2,3,\cdots, \quad (2.5)$$

is satisfied.

If now κ can take on the value $\frac{1}{2}$, then we have that

$$|p|=(1/4-\gamma^2)^{1/2}\cong\tfrac{1}{2}-\gamma^2,\quad(\gamma^2\ll1).$$

In this case, $|p|+1\cong\frac{3}{2}-\gamma^2$, so that f^2 and g^2 behave at the origin according to

$$f^2,\,g^2\sim1/[\Gamma(3/2-\gamma^2-\Lambda)\rho^{3-2\gamma^2}],\quad\rho\rightarrow0. \quad (2.6)$$

This is clearly quadratically integrable as long as $\gamma^2\neq0$, *for any energy $E<0$* [for which solutions of the form Eq. (2.1) exist]. In passing to the NR limit, we set $\gamma^2=0$ and, therefore, the integrability ceases. It is replaced by a logarithmic divergence as $\rho\rightarrow0$.

This integrability of the Dirac function for the lowest κ value (the value $\kappa=0$ does not occur in Dirac theory) off the hydrogen spectrum is analogous to the situation in the RS case where the $l=0$ solution is integrable off the hydrogen-like spectrum of the RS equation. (The

[11] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis* (Cambridge University Press, New York, 1950), 4th ed.

[12] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953).

[13] The solution, Eq. (2.1), is the same for both roots of the equation $p^2=\kappa^2-\gamma^2$. This can be shown by a method analogous to the proof indicated in Eqs. (13)–(15) of reference 1.

[14] An eigenvalue also occurs for $n'=0$ but only for negative κ. As pointed out in reference 5, this value requires special consideration; viz., in the present context, one must note that although the leading divergent term in $\psi(1+p-\Lambda,\,2p+1;\rho)$ which has the factor $[\Gamma(1+p-\Lambda)]^{-1}$ does not vanish, the coefficient $\kappa+\Lambda/\epsilon$ of this Γ function does vanish for negative κ, and the leading divergent term in $\psi(p-\Lambda,\,2p+1;\rho)$ contains $[\Gamma(p-\Lambda)]^{-1}$ which also vanishes; cf., reference 5, p. 67.

hydrogen-like spectrum of well-behaved eigenfunctions for $\kappa=\frac{1}{2}, \frac{3}{2}$, etc. in Dirac theory as well as the spectrum for $l=\frac{1}{2}, \frac{3}{2}$, etc. in Schrödinger theory is a separate problem from that which we are considering here; we discuss the general half-odd integral case briefly in Sec. V.)

For the repulsive Coulomb potential, one has the same equations as for the attractive case; the difference in the equation can be accounted for by selecting the appropriate sign of Z in the potential: $V(r)=Ze^2/r$. However, quadratically integrable solutions as terminating polynomial eigenfunctions will no longer be obtainable. The selection of the ψ- or Whittaker-type solutions [cf., Eq. (2.2)] makes it readily apparent that *the integrability of the anomalous solutions is independent of the sign of Z*.[15] The decisive term in the series expansion of the radial wave functions for the repulsive case can be written as

$$g(\rho), f(\rho)\sim 1/\{\Gamma[1+(1/4-\gamma^2)^{1/2}+|\Lambda|]\rho^{1+(1/4-\gamma^2)^{1/2}}\}, \quad \rho\rightarrow 0, \quad (2.7)$$

for the Dirac equation with $\kappa=\pm\frac{1}{2}$. In the repulsive case, the Γ functions have positive arguments and, therefore, have no poles and produce no spectrum. The anomalous solutions do not, of course, depend on this for their integrability, so we must conclude that these must describe repulsive as well as attractive systems, if they describe any systems at all. We must verify that acceptable angular solutions to the Dirac equation exist for the $\kappa=\pm\frac{1}{2}$ eigenvalues. We note from Eq. (2.4) that in the repulsive case there will be no integrable solutions for $|\kappa|>\frac{1}{2}$.

B. The Angular Dependence of the Dirac Equation for the Cases $\kappa^2=\frac{1}{4}, j=0$

The solution for the angular dependence of the Dirac equation is given by Bethe and Salpeter.[5] The theory presented in reference 5 is valid for half-integral spherical harmonics, as well as integral, as long as $l>\frac{1}{2}$ (we return to this point in Sec. V). In the case $l=\frac{1}{2}$, solutions are still obtainable, but with a modified formulation which avoids the divergent denominator $(2l-1)^{-1/2}$, which would appear in the formulation according to reference 5 for the case $j=l-\frac{1}{2}=0$. The method is indicated as follows. We require the Legendre functions in the half-odd-integral case, and take them to be defined as in reference 12, Sec. 3.4, viz.,

$$P_\nu^\mu(x)=[\Gamma(1-\mu)]^{-1}\times\left(\frac{1+x}{1-x}\right)^{\mu/2}F(-\nu,\nu+1;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}x), \quad (2.8)$$

where F is the ordinary hypergeometric function. The spherical harmonic solutions to the angular part of the Schrödinger equation are then taken to be

$$Y_{l,m}(\theta,\phi)=NP_l^{-|m|}(\cos\theta)e^{im\phi}, \quad (2.9)$$

where N is the normalization constant. The second solution to the differential equation in the half-odd-integral case, i.e., the solution which would be denoted $Q_\nu^{-|\mu|}$ on the basis of the above choice of P, is not, in general, well behaved (as is well known). The additional point which must be made here is that in the half-odd-integral case it turns out that we have [reference 12, Eq. (3.4) (17)]

$$Q_\nu^{-|\mu|}(x)=-\frac{\pi\Gamma(\nu-|\mu|+1)}{2\Gamma(\nu+|\mu|+1)}P_\nu^{|\mu|}(x). \quad (2.10)$$

This relation is the basis of the selection of $P_\nu^{-|\mu|}(\cos\theta)$ as the appropriate factor in the wave function. The sign of m has no physical significance in the θ factor of the wave function, so we are free to make this choice. (Changing the sign of m in the θ dependence would just, by Eq. (2.10), interchange the roles of the P and Q functions.) If now in Eq. (2.8) we take μ to be $-|\mu|$, and consider those functions for which $\nu=|\mu|$, we can obtain them quite easily by means of the relation[12]

$$F(a,b;b;z)=(1-z)^{-a}, \quad (2.11)$$

and the remaining functions required for smaller values of $|\mu|$ can be obtained from these by means of recurrence relations. From Eqs. (2.8) and (2.11) we obtain

$$P_\nu^{-\nu}(x)=(-\tfrac{1}{2})^\nu \sin^\nu\theta/\Gamma(1+\nu). \quad (2.12)$$

Thus, for example, for $\nu=\frac{3}{2}$, we have

$$P_{3/2}^{-3/2}(\cos\theta)=\frac{4(-\frac{1}{2})^{3/2}}{3\pi^{1/2}}\sin^{3/2}\theta \quad (2.13)$$

and by means of the recurrence relation [reference 12, Eqs. (3.8) (17) and (3.8) (19)]

$$P_\nu^{\mu+1}(x)=-(1-x^2)\frac{dP_\nu^\mu(x)}{dx}-\mu x P_\nu^\mu(x), \quad (2.14)$$

we obtain from Eq. (2.13)

$$P_{3/2}^{-1/2}(\cos\theta)=\frac{-2^{1/2}i}{\pi^{1/2}}\cos\theta\sin^{1/2}\theta. \quad (2.15)$$

The behavior of the function $P_\nu^{-|\mu|}(x)$ at $\theta=0$ is given by reference 12, [Eq. (3.9) 2(8)] to be

$$2^{-|\mu|/2}(1-x)^{|\mu|/2}\Gamma(1+|\mu|). \quad (2.16)$$

At $\theta=\pi$, Eq. [(3.4) (14)] of the same reference, which becomes

$$P_\nu^{-|\mu|}(-x)=P_\nu^{-|\mu|}(x)\cos\pi(\nu-|\mu|), \quad (2.17)$$

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[15] Except for $W=m_0c^2$; in this case the attractive solution is not integrable, cf., reference 26 and Eq. (3.32) of reference 5.

for half-odd-integral ν and μ, shows $P_\nu^{-|\mu|}(x)$ to have the same behavior as at $\theta=0$ to within a factor ± 1. Therefore, these functions are not singular and are quadratically integrable. Furthermore, the probability (and current) density in Dirac theory which consists of bilinear products of these functions will also be well behaved. In the NR limit, where derivatives of these functions appear, this good behavior may cease (this occurs in the $\kappa=\pm\frac{1}{2}$ cases we are now considering). However, our present interest is a case where the radial NR limit is also inadequate, and we assume that the model does not have a well-defined NR limit, so this feature should not hamper us. For $l=\frac{1}{2}$, we obtain from Eq. (2.12)

$$P_{1/2}^{-1/2}(\cos\theta)=i\left(\frac{2}{\pi}\right)^{1/2}\sin^{1/2}\theta. \tag{2.18}$$

For the Dirac equation, we shall also need the Q function for this value of l as well as the P function for $l=-\frac{1}{2}$. All of these functions turn out to be quadratically integrable, and are given by[16]

$$Q_{1/2}^{-1/2}(\cos\theta)=\frac{\pi^{1/2}}{2}\cos\theta\sin^{-1/2}\theta, \tag{2.19}$$

$$P_{-1/2}^{-1/2}(\cos\theta)=\frac{1}{i}\left(\frac{2}{\pi}\right)^{1/2}\sin^{-1/2}\theta. \tag{2.20}$$

Although the 4-component Dirac wave function can be constructed from well-behaved P-functions alone for $l>\frac{1}{2}$ (and is, therefore, well behaved), for the $l=\frac{1}{2}$ case this is not true. For this l value one requires two P functions for $Y_{l,m}$ and $Y_{l,m-1}$. But for $m=\frac{1}{2}$, we have that $m-1=-m$, so that we need $P_{1/2}^{-1/2}$ and $P_{1/2}^{1/2}$. But by Eq. (2.10), one of these must be a Q function. The existence of a well-behaved Dirac function when $\kappa=\frac{1}{2}$ is, therefore, dependent on the (fortuitous?) circumstance that both the P and Q functions are quadratically integrable for $l=\frac{1}{2}$, and that there is required only the one well-behaved P-function for $l=-\frac{1}{2}$. The normalized spherical harmonics corresponding to the Legendre functions of Eqs. (2.18) and (2.19) are

$$Y_{1/2,1/2}(\theta,\varphi)=\sin^{1/2}\theta e^{(i\phi/2)}/\pi,$$
$$Y_{1/2,-1/2}(\theta,\varphi)=(\cos\theta/\pi\sin^{1/2}\theta)e^{-i\phi/2}.$$

(The Legendre function for $l=-\frac{1}{2}$ appears automatically in the derivation of the Dirac 4-component wave function; no spherical harmonic need explicitly be assigned to it, in keeping with its unphysical nature.) The phase of the normalization constant has been chosen to remove the imaginary factor appearing in Eqs. (2.18) and (2.20). By use of these functions the angular solution to the Dirac equation can be obtained in complete analogy to the method of reference 5. The result obtained for $\kappa=\frac{1}{2}$ is

$$\begin{aligned} u_1&=(\sqrt{2}\pi)^{-1}\cos\theta\sin^{-1/2}\theta e^{-i\phi/2}g_+(r),\\ u_2&=(\sqrt{2}\pi)^{-1}\sin^{1/2}\theta e^{i\phi/2}g_+(r),\\ u_3&=(i\sqrt{2}\pi)^{-1}\sin^{-1/2}\theta e^{-i\phi/2}f_+(r),\\ u_4&=0,\end{aligned} \tag{2.21}$$

or

$$\begin{aligned} u_1^\dagger&=(\sqrt{2}\pi)^{-1}\sin^{1/2}\theta e^{-i\phi/2}g_+(r),\\ u_2^\dagger&=(-\sqrt{2}\pi)^{-1}\cos\theta\sin^{-1/2}\theta e^{i\phi/2}g_+(r),\\ u_3^\dagger&=0,\\ u_4^\dagger&=(i\sqrt{2}\pi)^{-1}\sin^{-1/2}\theta e^{i\phi/2}f_+(r).\end{aligned} \tag{2.22}$$

Equation (2.22), the second solution above, is the spin conjugate[3] of the first, Eq. (2.21). For the Coulomb potential the normalized radial functions $f_+(r)$ and $g_+(r)$ are given by Eq. (2.1) with κ set equal to $+\frac{1}{2}$. For κ of opposite sign, i.e., $-\frac{1}{2}$, the result is

$$\begin{aligned} v_1&=(i/\sqrt{2}\pi)\sin^{-1/2}\theta e^{-i\phi/2}g_-(r),\\ v_2&=0,\\ v_3&=(1/\sqrt{2}\pi)\cos\theta\sin^{-1/2}\theta e^{-i\phi/2}f_-(r),\\ v_4&=(i/\sqrt{2}\pi)\sin^{1/2}\theta e^{i\phi/2}f_-(r),\end{aligned} \tag{2.23}$$

with a corresponding spin-conjugate solution. The components of the probability density $\Psi^*\Psi$ and the current density (reference 6, p. 316) $\mathbf{S}=-c\Psi^*\boldsymbol{\alpha}\Psi$ for the solution, Eq. (2.21), are

$$\begin{aligned} \Psi^*\Psi&=(f_+^2+g_+^2)/(2\pi^2\sin\theta),\\ S_x&=2c\sin\phi g_+f_+,\\ S_y&=-2c\cos\phi g_+f_+,\\ S_z&=0.\end{aligned} \tag{2.24}$$

The trigometric radicals cancel each other and $\mathbf{S}$ is well behaved with (in polar coordinates) only a circulating ϕ component.

It can be easily verified that these solutions correspond to a total angular momentum eigenvalue $j=\pm\kappa-\frac{1}{2}$ equal to zero. That is to say, they are eigenfunctions of the operators

$$(\mathbf{L}+\tfrac{1}{2}\hbar\boldsymbol{\sigma}')^2,$$

and

$$(L_z+\tfrac{1}{2}\hbar\sigma_z'),$$

with eigenvalue zero (in the notation of reference 6).

C. The Relativistic Schrödinger Anomalous Solutions

We obtain the behavior at the origin of the solutions to the RS equation for the Coulomb potential that

[16] The definition of Q_ν^μ employed is that of reference 12, Eq. (3.4) (10). For half-odd-integral μ and ν this definition becomes

$$Q_\nu^{-|\mu|}(x)=\frac{\Gamma(1+\nu-|\mu|)\Gamma(|\mu|)}{2\Gamma(1+\nu+|\mu|)}\times\left(\frac{1-x}{1+x}\right)^{-|\mu|/2}F(-\nu,\nu+1;1-|\mu|;\tfrac{1}{2}-\tfrac{1}{2}x),$$

which leads to the result quoted in Eq. (2.19).

vanish at infinity again from reference 1. This behavior is

$$R(\rho)\sim 1/\Gamma(S'+1-\lambda')\rho^{S'+1}, \quad (2.25)$$

where

$$S'=-\tfrac{1}{2}+[(l+1/2)^2-\gamma^2]^{1/2},$$

and

$$\lambda'=Ze^2W/\hbar c(m_0{}^2c^4-W^2)^{1/2}.$$

This is quadratically integrable for $l=0$ and $l=\frac{1}{2}$ for all values of λ'. In the attractive case $(Z>0)$ there is, of course, the well-known hydrogen-like spectrum at the poles of the Γ function. In the repulsive case $(Z<1)$ the Γ-function has no poles so the *only* integrable solutions are for $l=0$ and $\frac{1}{2}$. The mathematics of the $l=0$ case has already been discussed in reference 1; consequently, at this point, we merely display the $l=\frac{1}{2}$ solution. It is given by

$$\Psi_{\pm}(\mathbf{r})=Ne^{-\rho/2}\rho^{-1/2+(1-\gamma^2)^{1/2}} \times\psi[\tfrac{1}{2}+(1-\gamma^2)^{1/2}-\lambda';1+2(1-\gamma^2)^{1/2};\rho] \times\frac{\sin^{1/2}\theta e^{\pm i\phi/2}}{\pi}, \quad (2.26)$$

where N is the radial normalization constant and $\rho=\{2(m_0{}^2c^4-W^2)^{1/2}/\hbar c\}r$. The θ derivative of this wave function (whose angular part is $P_{1/2}{}^{-1/2}$) is $Q_{1/2}{}^{-1/2}$, and as pointed out previously, this is also quadratically integrable [cf, Eq. (2.19)].

III. DISCUSSION: THE ANOMALOUS SOLUTIONS AND THE REAL TWO-BODY PROBLEM

We have now two types of anomalous solutions: one to the RS equation and one to the Dirac equation. Both fail in the NR limit; the first by ceasing to obey the proper equation, and the second by ceasing to be quadratically integrable.

Rather than attempting to exclude such anomalous relativistic solutions from quantum-mechanical theory, we wish to investigate the possibility of associating such solutions with different classical models than the point-charge model of the hydrogen atom (or analogous system such as a mesic atom). Such an attempt should be tantamount to finding another system, perhaps an extended one, whose Hamiltonian is mathematically similar to that of two point charges, and in proving that the association is reasonable. A charged spherical shell is such a system which is, furthermore, not now included in the domain of elementary quantum mechanics.

However, before attempting such a program, it seems essential to answer the question: if these solutions are to be physically meaningful, why do they not describe a state of the hydrogen atom (or similar attractive two-particle system) or a state of a repulsive two-particle system? It is reasonably certain on empirical grounds that such an association cannot be made. Present theory is adequate for the hydrogen atom and hydrogenic-type systems, and no bound states of doubly charged systems appear to exist. An interesting answer to this question which provides additional insight into the physical significance of eigenvalues can be given as follows.

If we give the anomalous solutions a physical interpretation, a primary feature is the fact that the charge density of the particle they would describe is very high at the origin. It is of the form $\rho(\mathbf{r})=f(\theta)/r^q$, where q lies between $2-2\gamma^2$ and $3-\gamma^2$ depending upon which solution is being considered. Thus, the functions are singular at this point. Now this means that the difference between the ordinary Dirac and RS equations and the "true" equations for the systems they represent will be very important. So important, in fact, that the ordinary equations *will be incorrect for energies off their hydrogen-like energy spectra.* "True" equations are those that correctly account for the finite extension of the nucleus and the relativistic two-particle character of the system. For example, in the hydrogen atom, the finite extension of the proton charge cloud could not be neglected in the presence of a large electron probability density at the origin; i.e., the Coulomb potential would not be correct for a real physical hydrogen atom whose electron probability density $\Psi^*\Psi$ diverges as $1/r^2$ at the origin. Secondly, some of the relativistic two-body correction terms behave like r^{-3} and $\delta^3(\mathbf{r})$ near the origin,[5] and, thus, clearly should be included before obtaining a solution for which they are not small. On the other hand, for energies lying on the hydrogen-like spectrum, the electron density at the origin is very small, and the finite-nucleus-extension and two-body effects are, therefore, small. If the real two-body systems have states whose energies lie off the hydrogen spectrum, the equation describing these states will have to contain much more structure than the ordinary Dirac and RS equations and one can expect its solution will be quite different than the anomalous solutions we have obtained. This correct equation then (on the basis of empirical knowledge) would have stable eigenstates in the attractive case only if this turns out to be a legitimate way to describe neutral elementary particles, such as the neutron, Λ^0, etc., and we would expect no stable eigenstates for two repelling particles. Thus, we see that the simple classical model $p^2/2m-Ze^2/r$ for the Hamiltonian of a hydrogen atom, when carried over into quantum mechanics, produces an equation whose solutions are physically consistent with (or insensitive to) the approximations inherent in it only on an eigenvalue spectrum. For system energies off this spectrum, the equation itself is no longer useful. The equation and its eigenfunctions form a physically self-consistent scheme. A condition that the wave function be finite at the origin (or "small" if we include the relativistic case) will be tantamount to a condition that a wave equation has a state or set of states physically consistent with the approximate formulation of the equation.

For example, from the Fourier transform $\phi(\mathbf{p})$ of a wave function $\psi(\mathbf{r})$ we obtain a relation between $\Psi(\mathbf{r})$ for small r and $\phi(\mathbf{p})$ for large p; explicitly, we can state

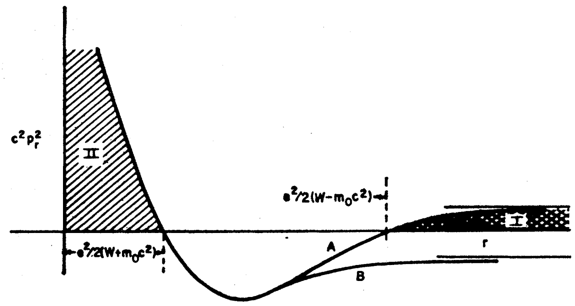

FIG. 1. The square of the radial momentum of a charged relativistic sphere with angular momentum $L=0$. Region I: Ordinary classical region, $W>m_0c$ (Curve A); Curve B for $W<m_0c^2$ has no allowed ordinary classical region. Region II: Relativistically allowed region (dependent upon negative energy root).

that

$$\psi(0)=(2\pi)^{-3/2}\int d^3p\phi(\mathbf{p}), \tag{3.1}$$

where the major contribution comes from large p [the Fourier transforms of hydrogenic functions, for example, fall off as inverse powers of p in contrast to the exponential decrease of $\Psi(r)$ for large r[17]]. If the contribution from large p is sufficiently great $\Psi(r)$ will diverge as $r\to 0$ and a relativistic formulation is required to obtain Ψ. If it is sufficiently great and $\Psi(r)$, therefore, sufficiently divergent, the formulation will need to exceed the generality of Dirac or RS theory and contain a more exact two-body relativistic description.

We conclude, therefore, that off the hydrogen-like spectrum, the anomalous solutions are solutions only to an "artificial" pure Coulomb potential; viz., a model which has an exact relativistic one-body type equation. Such a model exists, that of a charged, nonrigid sphere, and we undertake a classical analysis of this model in the next section.

IV. THE CHARGED SPHERICAL SHELL

A. Classical Considerations

In the case of a uniform, spherically symmetric charge distribution, the potential at a distance R from the center is e/R (where e is the total charge) and the field energy is exactly $e^2/2r$, where r is the radius of the sphere. By symmetry, the angular momentum of this distribution must vanish. For a distribution with angular momentum $L>0$, the notion of rigidity would have to be invoked, or a nonuniform distribution allowed. In either event, the energy would cease to be simply $e^2/2r$; there would be additional magnetic and multipole terms. It is also unlikely that a meaningful collective description could be obtained with three or less degrees of freedom, and this would be necessary in order to employ a one-body formulation.

In the case of a spherically symmetric charge distribution with vanishing angular momentum, we can write the NR Hamiltonian as

$$H=\tfrac{1}{2}(p_r^2/2m_0+e^2/r)=E, \tag{4.1}$$

where $p_r=m_0v_r$ is the total outward radial momentum, v_r the common radial velocity of all the elements of the system, m, the total mass, and E the energy. In addition, in the rest frame of the system, we can also obtain a simple form for the relativistic total energy of the system.[18]

$$(W-e^2/2r)=(m_0^2c^4+c^2p_r^2)^{1/2}, \tag{4.2}$$

or

$$(W-e^2/2r)^2=m_0^2c^4+c^2p_r^2. \tag{4.3}$$

This again has the form of a one-particle expression. Writing this equation in the form

$$W^2=m_0^2c^4+c^2p_r^2+We^2/r-e^4/4r^2, \tag{4.4}$$

reveals the interesting result that W can be less than m_0c^2 for sufficiently small values of r. The presence of the $-e^4/4r^2$ term in the relativistic energy expression, which the classical one does not have, endows the system with an effectively attractive potential (at small distances). If it should have made sense to include an angular momentum term in Eq. (4.4), it would have been

$$W^2=m_0^2c^4+c^2p_r^2+We^2/r+(4L^2c^2-e^4)/4r^2. \tag{4.5}$$

An angular momentum $L>e^2/2c$, or $L/\hbar>e^2/2\hbar c$ would nullify this binding term. This very small value provides an interesting consistency with the previously stated limitation to $L=0$. According to Eq. (4.4), real values of p_r exist at arbitrarily small r which satisfy the equation for a constant value of W.

Figure 1 shows $c^2p_r^2=W^2-m_0^2c^4-(We^2/r^2)+(e^4/4r^2)$ plotted against r. The asymptotic value $W^2-m_0^2c^4$ is positive for curve A and negative for curve B. As long as $W^2>m_0^2c^4$, as for curve A, an ordinary classically accessible region exists (the hatched region I under the curve A).

There is another region, designated II, for which real values of the momentum exist; viz., $p_r^2\geq 0$ for

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[17] In addition to noting that $\phi(\mathbf{p})$ falls off slowly as $p\to\infty$, perhaps, we should also note that $\phi(0)$ is finite. This can be seen directly from the Schrödinger equation in momentum space for $p=0$:

$$\phi(0)=(2\pi)^{-3/2}\int V(r)\Psi(\mathbf{r})d^3r/E.$$

As long as $V\Psi$ is no more divergent than r^{-c}, $c<3$, $\phi(0)$ is finite. Even for the anomalous solutions for the Coulomb potential, this condition is satisfied.

[18] This follows by carrying out the sum in the expression

$$\frac{1}{8\pi}\int(E^2+H^2)dv+\sum_i(\Delta_i m_0c^2)/(1-v^2/c^2)^{1/2},$$

over all the elements $\Delta_i m_0c^2$ in the spherically symmetric rest system; cf., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 79. Although this basic expression is invariant to a Lorentz transformation, the reduction to a simple one-particle-like description can only be carried out in the rest frame.

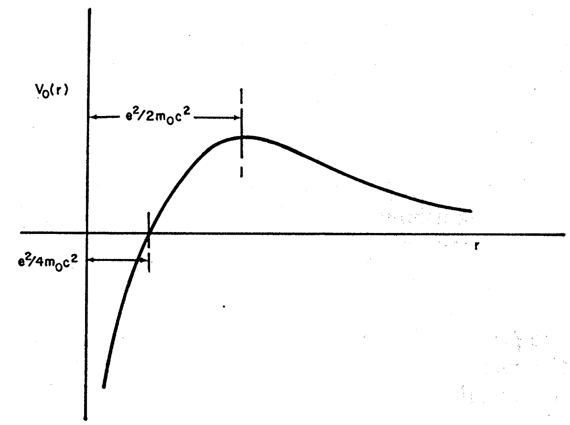

FIG. 2. Approximate classical potential for a relativistic charged sphere with angular momentum $L=0$.

$r \leq e^2/2(W+m_0c^2)$. However, its existence depends on the negative root of the mechanical energy expression, which is normally excluded in classical mechanics. One can see this by writing Eq. (4.4) in the form

$$W=(e^2/2r)\pm(m_0{}^2c^4+c^2p_r{}^2)^{1/2}. \qquad (4.6)$$

Clearly, for constant W at arbitrarily small r we must choose the negative root (which our original purely classical expression did not have). This region will be accessible in quantum mechanics since negative energy states quite naturally enter it and are allowed. The mechanical energy term whose absolute value is $(m_0{}^2c^4+c^2p_r{}^2)^{1/2}$ must be large and negative in order to cancel the large positive value of the field energy $e^2/2r$, to within the constant energy difference W. This should be the precise realization of the model referred to by Dirac,[19] but whose description he avoided since his intent was the construction of electrodynamics rather than a particle model. Classically, the system in this state would be dynamically unstable. The outward repulsive Coulomb force acting on a negative-mechanical-mass-sphere should collapse the sphere into the origin. However, in quantum mechanics, one might hope to avoid this collapse, as in the usual transition from a classical to a quantum system. There is no simple NR analog to this behavior near the origin. In the complete NR limit, the term depending on e^4 in Eq. (4.4) vanishes and this equation yields back Eq. (4.1). This latter equation cannot have $E<0$ for real values of p. If rather than taking the complete NR limit of Eq. (4.4), we neglect $W-m_0c^2$ relative to m_0c^2 [i.e., set $W^2-m_0c^2=(W-m_0c^2)\times 2m_0c^2$], we obtain

$$W\cong m_0c^2+p_r{}^2/2m_0+We^2/2m_0c^2r-e^4/8m_0c^2r^2, \qquad (4.7)$$

with the condition that

$$p_r{}^2/2m_0+We^2/2m_0c^2r-e^4/8m_0c^2r^2\ll m_0c^2. \qquad (4.8)$$

[19] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).

If we set $W-m_0c^2=E$, then we can write Eq. (4.7) as

$$E=p_r{}^2/2m+V_0(r), \qquad (4.9)$$

with

$$V_0(r)=(W/m_0c^2)e^2/2r-e^4/8m_0c^2r^2. \qquad (4.10)$$

In this approximation our model relates to classical NR behavior in the potential well $V_0(r)$, which is shown in Fig. 2 for $W=m_0c^2$. The subscript 0 is placed on V_0 to emphasize the restriction to $l=0$. As long as $r>e^2/2m_0c^2$ the customary "blowing up" of the sphere occurs from the repulsive Coulomb force. However, if $r<e^2/2m_0c^2$, this force is inward, and classically the sphere tends to collapse. The dynamics of the potential of Eq. (4.10) is not straightforward, however, because of the energy dependence it exhibits. An explicit solution for the energy E can be readily obtained; it is

$$E=\{p_r{}^2/2m_0+e^2/2r-e^4/8m_0c^2r^2\}/(1-e^2/2m_0c^2r).$$

B. Suggested Quantum Mechanics of a Charged Spherical Shell

Since the anomalous solutions, to be interpreted as wave functions, must describe bound states and, furthermore, are satisfactory in relativistic theory only, we interpret this to mean that if they describe a physical system, this system must cease to exist, or cease to be bound, in the NR limit. As we have shown, this is the case in classical theory for repulsive pure-Coulomb systems for which $E\leq 0$ and the angular momentum vanishes. Such a system can be bound in the classical, relativistic theory (if we include negative mechanical energies); however, in the NR limit it reverts to the well-known situation in which only values of $E>0$ are allowed (unless, of course, an additional attractive potential is supplied at the origin). This is due to the existence of an effectively attractive potential term in the relativistic Hamiltonian, which dominates in the region of $r\sim e^2/mc^2$. However, the NR limit of the wave function for this system cannot behave properly at the origin since, in this limit, no attractive potential exists there. It is reasonable, therefore, that the NR wave function obey a Schrödinger-like equation, but with an unsatisfactory behavior at the origin. As we have already pointed out, two real particles, either repulsive or attractive, cannot form a meaningful pure Coulomb system (viz., a system for which the Coulomb potential Ze^2/r remains valid as $r\rightarrow 0$). A simple charged sphere is such a system (a charged sphere with a point charge rather than a real charge at the center would also be such a system, but probably not a meaningful one, since the whole "point" of a particle model is to avoid point particles). In a sense, it is an artificial system, but because of its importance as a particle model in classical electrodynamics it is worth investigating.

Since we have not yet been able to perform the required normalization integrations [of the form $\int e^{-\rho}\rho^q\psi(a,c;\rho)^2d\rho$] in the general case for any energy, we select the analytically simplest situation as an exploratory example. This example for which the integrations can be readily performed is the case $W=m_0c^2$, viz., the total energy is equal to the rest mass of the sphere.

The RS equation for the Coulomb potential is well known.[1,6] If we set $W=m_0c^2$ in this equation with the repulsive Coulomb potential $eV(r)=Ze^2/r$, we obtain

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}-\left\{1+\frac{4l(l+1)+1-4\gamma^2}{t^2}\right\}y=0, \quad (4.12)$$

where use has been made of the definitions

$$\begin{aligned} a_0&=\hbar^2/me^2,\\ t&=(8Zr/a_0)^{1/2},\\ \gamma&=Ze^2/\hbar c,\\ R(r)&=Ny(t)/t. \end{aligned} \quad (4.13)$$

The solutions to Eq. (4.12) are Hankel functions of order $\nu=[(2l+1)^2-4\gamma^2]^{1/2}$ so that the radial wave function becomes

$$R(r)=NH_\nu^{(1)}(it)/t. \quad (4.14)$$

If we could normalize the ψ function in the general case, we could, of course, make use of the connection between $H_\nu^{(1)}$ and limit of $\psi(-a, c; \rho/a)$ as $a\rightarrow 0$. From the behavior of Hankel functions at the origin[20]

$$iH_\nu^{(1)}(it)\cong\frac{(\nu-1)}{i^\nu\pi}\left(\frac{2}{t}\right)^\nu, \quad t\rightarrow 0 \quad (4.15)$$

we obtain the behavior of $R(r)$ to be

$$R(r)\sim 1/t^{\nu+1}, \quad (4.16)$$

and find that $R(r)$ is integrable for both $l=0$ and $l=\frac{1}{2}$. The latter case is integrable in spite of the fact that there is no "classical" interpretation of such a bound state in terms of an effectively attractive potential at the origin [cf, (Eq. (4.5)]. Since it requires two angular coordinates in addition to the radial coordinate it is unlikely that we can associate this $l=\frac{1}{2}$ solution with a sphere. Therefore, we defer a discussion of this solution to Sec. V and consider only the $l=0$ case at this point. We interpret $R^*(r)R(r)r^2dr$ as the probability that the shell lies between r and $r+dr$, and normalize $R(r)$ according to

$$\int R^2(r)r^2dr=1, \quad (4.17)$$

the normalization constant N is given by

$$N^{-2}=2\left(\frac{a_0}{8Z}\right)^3\int[H_{(1-4\gamma^2)^{1/2}}^{(1)}(it)]^2t^3dt. \quad (4.18)$$

The integral can be computed to an adequate approximation for our present purpose by neglecting $4\gamma^2$. The result is, for the radial wave function,

$$R(r)=(96\pi^2Z^3/a_0^3)^{1/2}H_{(1-4\gamma^2)^{1/2}}^{(1)}(it)/t. \quad (4.19)$$

If we interpret the square of this wave function as a radial charge distribution, its mean radius, defined as

$$\bar{r}\equiv\int R^2(r)r^3dr, \quad (4.20)$$

turns out to have the value $(3\pi^2a_0/64Z)\int[H_\nu^{(1)}(it)]^2t^5dt$. This can be evaluated and yields

$$\bar{r}\equiv(3/10)(a_0/Z) \quad (4.21)$$

to the lowest order in γ^2. The result is large in terms of the classical electron radius and the region where the "attractive potential" term dominates. However, we have set $E=m_0c^2$ so that there is no binding energy.

V. THE HALF-INTEGRAL, ZERO-ENERGY ANOMALOUS SOLUTIONS AS INTRINSIC WAVE FUNCTIONS

A. Suggested Physical Interpretation

It seems unlikely that the half-integral anomalous solutions that we have found, involving as they do, the three coordinates of a point, can be associated with a symmetric spherical shell as their classical model. Difficulties arise in defining the meaning of $\Psi^*(\mathbf{r})\Psi(\mathbf{r})$ as there is no point to single out in this model with which to associate a probability or charge density. In the $l=0$ RS case, the one coordinate involved appears compatible with the one coordinate needed to define a sphere, and we suggested that $\Psi^*(r)\Psi(r)r^2dr$ can be interpreted as the probability that the shell lies between r and $r+dr$. There is one other "classical model," if it can be called that, that may be describable by an exact Coulomb Hamiltonian, and the conjecture that it is provides a basis for the physical interpretation of the half-integral anomalous solutions. The conjecture is as follows. It is well known that an electron exhibits a fluctuating behavior about its mean position, known as "zitterbewegung."[21] We propose that a particle experiences a repulsive Coulomb self-potential in the process of fluctuating or oscillating about its mean center-of-mass position. This self-potential would arise from the emission and absorption of virtual photons, the process which leads to the self-energy. We expect to associate a probability (and, hence, a charge) distribution with this fluctuating motion, and consequently need a wave equation to generate it. Analytically, what is proposed is as follows. The relativistic wave equations for the motion of a particle are obtained by adding the external potential $-A_\mu(\mathbf{r})$ to the energy-momentum 4-vector p_μ, either squaring or dotting into γ_μ (the Dirac matrix

[20] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945), 4th ed.

[21] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1961).

4-vector) to obtain an invariant expression, inserting the operators for p_μ and generating a wave function Ψ; e.g.,

$$(cp_\mu - eA_\mu)\gamma_\mu\Psi = m_0c^2\Psi. \tag{5.1}$$

However, the particle itself generates a potential 4-vector

$$\left(\frac{e\mathbf{v}}{c(r-\mathbf{v}\cdot\mathbf{r}/c)}, \frac{e}{r-\mathbf{v}\cdot\mathbf{r}/c}\right).$$

Therefore, we postulate that if we insert this potential (as a self-potential) into Eq. (5.1) and solve it in the *rest frame of the particle*, that we will obtain the motion of the particle relative to its own center-of-mass (which remains at rest). Since the vector potential vanishes in the rest frame, we assume a simple static Coulomb self-potential

$$V(r) = Ze^2/r, \tag{5.2}$$

should be inserted in the wave equation and the total energy set equal to m_0c^2. The factor Z is retained throughout to indicate scaling on the value of the charge, but it is expected to have the value unity. We would expect to interpret the motion as either a rotation or a fluctuating motion caused by the emission and self-absorption of photons, or both. We thereby postulate a concrete dynamical model to the notion of zitterbewegung. The solution to the resulting "intrinsic" wave equation yields quadratically integrable, bound-state type wave functions some of whose average spatial extensions are of the order of e^2/m_0c^2 and rms extensions are $\sim\hbar/m_0c$, and which are, therefore, consistent with the initial postulate. It is well known that the Coulomb interaction between two particles can be interpreted as arising from the emission and absorption of virtual photons. Our postulate should, therefore, be equivalent to assuming that the emission and self-absorption of photons by a particle results in the particle experiencing a repulsive Coulomb self-potential.

The solutions we obtain in this fashion are only for the one energy $W = m_0c^2$; the repulsive anomalous solutions are integrable in general for all $E \leq m_0c^2$. We assume that this freedom will ultimately allow the inclusion of higher order radiative effects than a simple Coulomb self-potential, and a difference between the mechanical mass m_0c^2 that should be inserted in the intrinsic equation, and the total energy W that would be the actual observed external rest mass.

We note that obtaining a radius in this fashion (which is $\sim e^2/m_0c^2$) would not imply a localization of a particle to within this distance, since the coordinate r can only be an internal one; viz., the equation we solve is obviously not the equation for the center of mass of the particle. We assume that a theory could be formulated such that the c.m. obeyed the customary equations for a point particle, and can be bound or free according to the well-known circumstances and models. The expressions for the mean radii of the order of e^2/m_0c^2 are of the form

$$\bar{r} \propto \gamma^2 a_0.$$

Hence, in the NR limit, $\bar{r} \to 0$ and we return to a point particle. The solutions pass over, in the NR limit to solutions of the Schrödinger equation

$$H\psi = 0,$$

where

$$H = -\frac{\hbar^2}{m}\nabla^2 + \frac{Ze^2}{r}.$$

This is the origin of our terminology "zero-energy" solutions. We can alternatively exhibit the consistency of our conjecture as follows. Consider a charge distribution of the form

$$\rho(\mathbf{r}) = C/r^{3-\gamma^2}, \tag{5.3}$$

where C is a constant and $\gamma^2 \ll 1$. A test charge in this distribution will experience a radial electric field E_r which can be obtained from Gauss' theorem:

$$E_r = \frac{1}{4\pi}\int_0^r \rho dv/r^2 = \frac{C}{r^{2-\gamma^2}} \tag{5.4}$$

Therefore, a particle in this distribution *will experience a Coulomb field* to lowest order in γ^2. If we interpret the anomalous $l = \frac{1}{2}$ solutions to the exact Coulomb problems as wave functions, the charge distributions they yield (averaged over angles) have the form given in Eq. (5.3) for small r. Thus, to terms of order γ^2, this charge distribution obtained from a Coulomb potential, yields back a Coulomb potential. Outside the range $r \sim e^2/mc^2$ the distribution falls off exponentially, so that the potential becomes accurately Coulombic outside this range. [Actually, the appearance of γ^2 is a quantum-relativistic effect which probably should be ignored in a classical consistency argument such as is constituted by Eqs. (5.3) and (5.4) and the argument phrased as follows: the charge distribution Eq. (5.3) can be made *arbitrarily* closely consistent with the Coulomb field of Eq. (5.4) for sufficiently small γ^2.]

Making use of this interpretation in quantum theory will ultimately require the obtaining of a relativistically invariant separation of the motion of a particle into a motion of its center of mass and the "zitterbewegung" motion relative to the center of mass (c.m.). If this is obtainable, then in the absence of any interaction between the motion relative to the c.m. and the motion of the c.m., the wave equation for the total motion will separate into an equation for the motion of the c.m. (which would be the currently accepted point-particle equations) and an equation (the one we are dealing with here) for the motion relative to the c.m. The total wave function will then be a product $\Psi(\mathbf{r}_{\mathrm{c.m.}})\Psi(\mathbf{r}_i)$ of the usual wave function interpreted as $\Psi(\mathbf{r}_{\mathrm{c.m.}})$ and our internal wave function $\Psi(\mathbf{r}_i)$. In this fashion, the results of present ordinary quantum mechanics are reproduced as long as the internal state does not change, and by orthonormality $\Psi(\mathbf{r}_i)$ drops out of the computation of experimental predictions. The internal Dirac wave function, being an eigenfunction for $j=0$, would not affect the usual angular momentum and spin of a Dirac particle described by $\Psi(\mathbf{r}_{\text{c.m.}})$. Thus, to first and possibly higher order there would be no corrections in Dirac theory. The first-order corrections to the energy that would arise from the finite extension of the charge cloud vanish due to the angular behavior of the charge clouds of the anomalous solutions; viz.,

$$V(\mathbf{R})=\int\frac{\rho(\mathbf{r})dv}{|\mathbf{R}-\mathbf{r}|}=\frac{e}{R}+O(R^{-3}).$$

In the RS case, it is not readily apparent which, if any, real particles might be described by the $l=\frac{1}{2}$ solution [Eq. (2.26)]. It appears a possibility, however, that the RS theory might now yield an alternate way of describing an electron. The effects of spin would be added by first-order corrections via the half-integral internal solution, and treated, therefore, in ordinary configuration space.

The general problem of half-integral angular momenta has been considered before.[22–25] Pauli[22] showed that NR transition probabilities between half-odd-integral states are not, in general, well behaved. His arguments do not, however, apply to the relativistic case, where there have been previous indications[23] that they may be needed to describe extended systems. (In Dirac theory, the probability current does not involve the derivatives of the wave function; hence, the requirements of good behavior are not as stringent as in NR theory.) One can readily verify[25] that *both* half-odd-integral and integral angular eigenfunctions cannot occur in the description of a given system. Because of this, the half-integral hydrogen-like spectrum of the Dirac and NR Schrödinger equations for the Coulomb potential can be disallowed on the basis of the experimental verification of the validity of the integral spectrum. [Cf., e.g., Eq. (2.5); this is satisfied by a spectrum of energy values for $\kappa=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots$. The complete set of angular momentum eigenfunctions generated by this spectrum are the half-odd-integral spherical harmonics. These lead to single-valued well-behaved probability and current densities.] The problem discussed by Pauli[22] crops up in a theory which requires a set of half-integral spherical harmonics, corresponding to different values of the total angular momentum. The models to which we suggest in this paper that these solutions belong require only a single total angular momentum eigenvalue for a given system, therefore, this problem does not arise here.

It appears likely, on the basis of Pauli's arguments, the indications from studies of extended relativistic systems,[23,24] and the results presented herein, that the half-integral angular momentum eigenfunctions may only pertain to relativistic cases that have no adequate NR limit.

B. Computation of Normalization Constants and Mean Radii

1. $l=\frac{1}{2}$ *RS Case*

On the basis of the preceding postulates we consider the rest frame of the particle only, and assume that the total energy is to be identified with the rest mass of the particle. The half-integral solution to the zero-energy repulsive Coulomb RS equation, Eq. (4.12) is then

$$\Psi_{\pm}(\mathbf{r})=N[H_{2(1-\gamma^2)^{1/2}}^{(1)}(it)/t]\frac{\sin^{1/2}\theta e^{\pm i\phi/2}}{\pi}. \quad (5.5)$$

Normalizing to unity, as before, [Eq. (4.17)], we obtain

$$N^{-2}=2\left(\frac{a_0}{8Z}\right)\int[H_{(1-\gamma^2)^{1/2}}^{(1)}(it)]^2t^3dt. \quad (5.6)$$

Evaluating this integral [see Appendix A, Eqs. (A1) through (A3)] we obtain, to lowest order in γ^2,

$$N=\frac{\pi\gamma}{4}\left(\frac{8Z}{a_0}\right)^{3/2}.$$

The mean radius, defined as

$$\bar{r}=\int\Psi^*\Psi r^3drd\Omega, \quad (5.7)$$

is given by

$$\bar{r}=N^2\left(\frac{a_0}{8Z}\right)^4\int[H_{(1-\gamma^2)^{1/2}}^{(1)}(it)]^2t^5dt.$$

The integral has the approximate value $128/5\pi^2$ so that $\bar{r}$ becomes

$$\bar{r}=\gamma^2\left(\frac{a_0}{5Z}\right)=\left(\frac{Z}{5}\right)e^2/m_0c^2. \quad (5.8)$$

The quasidivergent character of the wave function, Eq. (5.5), reduces the value of $\bar{r}$ by a factor γ^2 below the scale parameter of the equation (a_0) so that $\bar{r}$ is of the order of magnitude of the "classical electron radius" e^2/m_0c^2. By use of Eq. (4.15) we can write the behavior of Ψ as given by Eq. (5.5), at the origin, as

$$|\Psi|^2\sim\left(\frac{8Z}{a_0}\right)^{\gamma^2}\frac{\gamma^2\sin\theta}{r^{3-\gamma^2}}, \quad r\rightarrow 0, \quad (5.9)$$

which is the basis of our previously stated Eq. (5.3).

[22] W. Pauli, in *Handbuch der Physik*, edited by H. Geiger and Karl Scheel (Springer-Verlag, Berlin, 1933), 2nd ed. See especially p. 126.
[23] D. Bohm, P. Hillion, and J. P. Vigier, Progr. Theoret. Phys. (Kyoto), **24**, 761 (1960).
[24] G. Lochak, Cahiers Phys. **102**, 1 (1959).
[25] David Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1951).

2. $j=0$ Dirac Case

We assume a radial wave function of the form

$$\Psi=\begin{pmatrix} G(r)/r \\ F(r)/r \end{pmatrix}$$

and set $W=m_0c^2$ in the Dirac equation for the repulsive Coulomb potential Ze^2/r. This yields the coupled, first-order equations

$$\begin{aligned} &\frac{dG}{dx}+\frac{\kappa}{x}G-\left(2-\frac{\gamma}{x}\right)F=0, \\ &\frac{dF}{dx}-\frac{\kappa}{x}F-\frac{\gamma}{x}G=0, \end{aligned} \tag{5.10}$$

where $\gamma=Ze^2/\hbar c$, $x=\beta r$, and $\beta=m_0c/\hbar$. κ is an eigenvalue of the operator whose square is defined by the equation

$$(\mathbf{L}+\tfrac{1}{2}\hbar\boldsymbol{\sigma}')^2+\tfrac{1}{4}\hbar^2=\hbar^2\kappa^2$$

(cf., reference 6, Sec. 44 for notation). Elimination of G between the two linear equations, Eq. (5.10), leads to the second-order equation for F:

$$\frac{d^2F}{dt^2}+\frac{1}{t}\frac{dF}{dt}-\left[1+\frac{4(\kappa^2-\gamma^2)}{t^2}\right]F=0, \tag{5.11}$$

which is independent of the sign of κ. In this equation, we have made use of the substitutions

$$t^2\equiv 8rZ/a_0 \quad\text{and}\quad a_0=\hbar^2/m_0e^2,$$

as in Eq. (4.13). This again [cf., Eq. (4.12)] is a Bessel equation whose solution that vanishes at infinity is

$$F=\text{const}\times H_\nu^{(1)}(it), \tag{5.12}$$

with $\nu=2(\kappa^2-\gamma^2)^{1/2}$ where $H_\nu^{(1)}(it)$ is again the well-known Hankel function.

The solution for $\kappa=+\frac{1}{2}$. The solution to Eqs. (5.10) when $\kappa=\frac{1}{2}$ can be written, by use of the Hankel function recurrence relations, in the form

$$G_+=N_+\{itH_{(1-4\gamma^2)^{1/2}+1}{}^{(1)}(it) -[(1-4\gamma^2)^{1/2}-1]H_{(1-4\gamma^2)^{1/2}}{}^{(1)}(it)\}, \tag{5.13}$$

$$F_+=-2N_+\gamma[H_{(1-4\gamma^2)^{1/2}}{}^{(1)}(it)], \tag{5.14}$$

where N_+ is a normalization constant. We choose as normalization condition

$$\int[G(r)^2+F(r)^2]dr=1, \tag{5.15}$$

and neglect terms of order γ^2 in order to obtain a value of N_+ accurate to this order. The normalization condition, Eq. (5.15) becomes, to this accuracy,

$$\left(\frac{a_0}{4Z}\right)N_+^2\int[itH_{2-2\gamma^2}{}^{(1)}(it)]^2t\,dt=1. \tag{5.16}$$

This integral can be performed [see Eqs. (A1) through (A3)] with the result that N_+ is given by

$$N_+=\frac{\pi\gamma}{(a_0/Z)^{1/2}}, \tag{5.17}$$

to terms of order γ^2 in comparison. With this result, the average radius,

$$\bar{r}\equiv\int r[G^2(r)+F^2(r)]dr \tag{5.18}$$

can now be computed for the $\kappa=\frac{1}{2}$ solution. Neglecting the small function compared to the large leads to the result

$$\bar{r}_+=\frac{a_0^2}{32Z^2}N_+^2\int[itH_{2-2\gamma^2}(it)]^2t^3dt. \tag{5.19}$$

The integral has the value $128/5\pi^2$ again to terms of order γ^2. Inserting N_+ from Eq. (5.17) we obtain

$$\bar{r}_+=\tfrac{4}{5}\gamma^2\left(\frac{a_0}{Z}\right)=\tfrac{4}{5}(e^2/m_0c^2)Z, \tag{5.20}$$

to the accuracy already stated. The computation of $\langle r^2\rangle_{\text{av}}$ can be carried out similarly. The result is

$$\langle r^2\rangle_{\text{av}}=\frac{a_0^3}{2^8Z^3}N_+^2\int[itH_{2-2\gamma^2}(it)]^2t^5dt. \tag{5.21}$$

In this case, the integral has the (approximate) value $3\times2^8/7\pi^2$, which yields

$$\begin{aligned}\langle r^2\rangle_{\text{av}}&=(3/7)\gamma^2(a_0/Z)^2,\\ &=(3/7)(\hbar/m_0c)^2,\end{aligned} \tag{5.22}$$

or

$$\langle r^2\rangle_{\text{av}}{}^{1/2}=0.655(\hbar/m_0c). \tag{5.23}$$

The Solution for $\kappa=-\frac{1}{2}$. For $\kappa=-\frac{1}{2}$, the solution to Eqs. (5.10) can be written

$$G_-=N_-\{itH_{(1-4\gamma^2)^{1/2}-1}(it) -[(1-4\gamma^2)^{1/2}-1]H_{(1-4\gamma^2)^{1/2}-1}(it)\}, \tag{5.24}$$

$$F_-=2\gamma N_-[H_{(1-4\gamma^2)^{1/2}-1}(it)].$$

It should be observed that in this case the large component has a term which vanishes as $t\to0$ in addition to a quasi-divergent term of order γ^2. In some computations with this function, the term of order γ^2 cannot now be neglected as it will dominate the "large" term as $t\to0$ in spite of the γ^2 coefficient. As before the normalization condition will be given by Eq. (5.15). By use of $\nu+2\kappa\cong-2\gamma^2$ and by neglect of small terms, this yields

$$\frac{a_0}{4Z}N_-^2\int\{4\gamma^2[H_\nu^{(1)}(it)]^2+4\gamma^2[itH_{\nu-1}{}^{(1)}(it)H_\nu^{(1)}(it)] +[itH_{\nu-1}{}^{(1)}(it)]^2\}t\,dt=1. \tag{5.25}$$

The integrals can be worked out from the Appendix. The result is (again neglecting terms of order γ^2)

$$N_- = (3\pi^2 Z/a_0)^{1/2}. \qquad (5.26)$$

It should be noted that N_- does not contain a factor γ as was the case for N_+ in Eq. (5.17). This is one indication of the great difference between the two solutions. The $\kappa=-\frac{1}{2}$ solution can be seen to have a "structure" to it in distinction to the simple monotonic behavior of the $\kappa=+\frac{1}{2}$ solution.

The solution, Eq. (5.24), reduces in the limit to the nonrelativistic solution for $l=-\frac{1}{2}$. As previously mentioned, the angular solution for this value of l is needed (both for $\kappa=+\frac{1}{2}$ and $\kappa=-\frac{1}{2}$) so it is not surprising that the radial solution is needed also. Proceeding as before, to compute the mean radius, we find it is given approximately by

$$\bar{r}_- = \frac{a_0^2 N_-^2}{32Z^2} \int t^5 [iH_0^{(1)}(it)]^2 dt. \qquad (5.27)$$

The value of the integral is $64/15\pi^2$. Inserting Eq. (5.26) leads to the result,

$$\bar{r}_- = (1/10)a_0/Z.$$

In this case, the rms radius turns out to be

$$\langle r_-^2 \rangle_{av}^{1/2} = (27/280)^{1/2}(a_0/Z).$$

VI. CONCLUSION

We have shown that the anomalous solutions can be discarded, at least on physical grounds, for the attractive two-body systems which are already known to be described adequately without them. It is proposed that they indicate that quantum mechanically, a repulsive *pure* Coulomb system need not blow up from its internal electrostatic stress. The author's attention was drawn to these anomalous solutions by the initial conjecture, presented in Sec. V, that the solution of the wave equations for $E=0$ or $W=m_0c^2$ should yield an intrinsic description of a particle via its instantaneous motion relative to its c.m. As stated previously, extensive calculations have been carried out beyond the simple mean radii ones presented herein that appear to indicate physical consistency for this idea. However, as is obvious, the scope of establishing an adequately consistent theory of elementary particles along these, as along any, lines is formidable. In view of this and the unorthodox nature of the approach, it seems desirable to present the basic ideas and simple descriptive results only, as we have done, and defer more elaborate computations to another paper. Most, if not all, the parameters describing elementary particles require the inclusion of radiative interactions for their description. Our preliminary results indicate that these can be included in our intrinsic wave equations to permit particle decays, and to cause a coupling to arise between the internal and external degrees of freedom. We hope to report on these results in a subsequent article. One of the solutions we consider here was obtained before by Temple.[26] Our current speculations are not totally unrelated to his, but the problems in setting up a theory such as he attempted are now known to be vastly more difficult than his attempt would imply.

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VII. APPENDIX. THE EVALUATION OF INTEGRALS INVOLVING PRODUCTS OF TWO HANKEL FUNCTIONS

The normalization and mean radius integrals of the "zero-energy" equations Eq. (4.12) and (5.10) are of the form $\int_0^\infty H_\nu^{(1)}(it) H_\mu^{(1)}(it) t^q dt$. Some of these can be obtained, upon neglect of terms of order γ^2, directly from the reduction formulas given by Watson.[27] Those which are of a quasidivergent nature; viz., those whose integrands behave like $t^{-1+\gamma^2}$ at the origin can be integrated by an intuitive procedure, wherein one replaces the Hankel functions by their behavior at the origin [Eq. (4.15)] and integrates from zero to some small but finite value of t. To terms of order γ^2, the divergence of the integrand at the origin yields the full value of the integral from zero to infinity. That is to say,

$$\int_0^\infty [itH_{2-2\gamma^2}^{(1)}(it)]^2 t dt \cong \frac{2^4}{\pi^2}\int_0^\delta \frac{dt}{t^{1-4\gamma^2}} = \frac{2^4}{\pi^2}\frac{\delta^{4\gamma^2}}{4\gamma^2} \cong \frac{4}{\pi^2\gamma^2}. \qquad (A1)$$

An exact result is obtainable, although the expression is somewhat more cumbersome[12]:

$$2^{\rho+2}\Gamma(1-\rho)\int_0^\infty K_\mu(\alpha t)K_\nu(\beta t)t^{-\rho}dt = \alpha^{\rho-\nu-1}\beta^\nu \left\{ {}_2F_1\left(\frac{1+\nu+\mu-\rho}{2}, \frac{1+\nu-\mu-\rho}{2}; 1-\rho; 1-\beta^2/\alpha^2\right)\right\} \Gamma\left(\frac{1+\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1+\nu-\mu-\rho}{2}\right) \times\Gamma\left(\frac{1-\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1-\nu-\mu-\rho}{2}\right), \qquad (A2)$$

[26] G. Temple, Proc. Roy. Soc. (London) **A145**, 344 (1934).
[27] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1952), 2nd ed., Secs. 5.11–5.14.

where $K_\nu(t)=(\pi i/2)e^{i\pi\nu/2}H_\nu^{(1)}(it)$. The conditions for the validity of this expression are given in reference 12 as $\mathrm{Re}(\alpha+\beta)>0$ and $\mathrm{Re}(\rho\pm\mu\pm\nu+1)>0$. However, the sign of ρ in the second condition is incorrect. The basic requirement is that the integral converge,[28] and the expression, Eq. (4.15), for $H_\nu^{(1)}(t)$ for small t can be used to establish the correct limitation on ρ, μ, and ν. Convergence at infinity is assured by the exponential behavior of the K functions at large distances. For the quasidivergent integrals needed in the text, $\alpha=\beta$, and $\mu=\nu$. In these cases, Eq. (A2) simplifies to

$$\int K_\nu^2(t)t^n dt=\frac{2^{n-2}}{\Gamma(1+n)}\Gamma\left(\tfrac{1}{2}+\nu+\frac{n}{2}\right)\times\Gamma^2\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1+n}{2}-\nu\right). \quad \text{(A3)}$$

[28] E. C. Titchmarsh, Proc. London Math. Soc. 2, 97 (1927).

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Simplified Approach to the Ground-State Energy of an Imperfect Bose Gas

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The well-known first two terms in the asymptotic density series for the ground-state energy of a Bose gas, $E_0=2\pi N\rho a[1+(128/15\sqrt{\pi})(\rho a^3)^{1/2}]$, where a is the scattering length of the pair potential, is ordinarily obtained by summing an infinite set of graphs in perturbation theory. We show here how this same series may be obtained by elementary methods. Our method offers the advantages of simplicity and directness. Another advantage is that the hard-core case can be handled on the same basis as a finite potential, no pseudopotential being required. In fact, the analysis of the hard-core potential turns out to be simpler than for a finite potential, as is the case in elementary quantum mechanics. In an Appendix we discuss the high-density situation and show that for a certain class of potentials Bogoliubov's theory is correct in this limit. Thus, Bogoliubov's theory, which is never correct at low density unless a pseudopotential is introduced, is really a high-density theory.

INTRODUCTION

IMPORTANT and often brilliant theoretical investigations by many workers in the past few years have given us considerable insight into the nature of the ground state and low-lying excited states of a many-particle Bose gas at low density with repulsive pairwise forces. While the intermediate density problem is still unsolved, we at least know now how to begin a consistent expansion (possibly divergent) in the density. In appropriate units[1] we have the following well-known formulas for the ground-state energy, E_0, and the energies of the elementary excitations of long wavelength, $\epsilon(\mathbf{k})$:

$$E_0=2\pi N\rho a\{1+(128/15\sqrt{\pi})(\rho a^3)^{1/2}+\cdots\}, \quad (1.1)$$

$$\epsilon(\mathbf{k})=2(\pi\rho a)^{1/2}k+\cdots, \quad (1.2)$$

where N is the number of particles, $\rho=N/V$ is the density, and $a>0$ is the scattering length of the two-body potential. The omitted higher terms in Eq. (1.1) depend upon the shape of the potential as well as the scattering length; Eq. (1.2) is justified if $k\ll(\rho a)^{1/2}$.

While Eqs. (1.1) and (1.2) may now be regarded as well established and, therefore, elementary, it was not always so. The first attempt to find E_0 was based on perturbation theory. Aside from the fact that perturbation theory cannot be justified in this case (E_0 is enormously greater than the spacing between the unperturbed ground and first excited states), it is easily seen that all terms in the perturbation series beyond the second are divergent for any potential. By this is meant that although the terms are not actually infinite, they are proportional to a higher power of N than the first.

Nevertheless, it was held for a long time that the first term in the perturbation series, viz.,

$$E_0'=2\pi N\rho a', \quad (1.3a)$$

where

$$a'=\frac{1}{4\pi}\int v(\mathbf{x})d^3x, \quad (1.3b)$$

was exact,[2] $v(\mathbf{x})$ being the two-body potential. Equation

[1] $\hbar=1$, $m=1$.

[2] We must be careful to define the meaning of exact and approximate as used in this paper. We are interested in calculating E_0 as a function of density for a given fixed potential; we are, therefore, concerned with an asymptotic series in the density whose coefficients, and, indeed, whose entire form are functionals of the potential. As such, Eq. (1.1) is exact in that it gives correctly the first two terms in an asymptotic series. Equations (1.3) and (1.6) are only approximations to that series. If, on the other hand, we regard the potential as being proportional to some parameter, λ, and if we were interested in a double expansion in ρ *and* λ, then Eqs. (1.3) and (1.6) would be exact. We are not interested in this latter type of series because for a large potential, such as a hard core, it is relatively useless. Indeed, there is no need for such a double series, because it is the burden of this paper, as well as of a good deal of previous work, that it is just as easy to generate the former type of single series as the latter double series.