Coupled Channel Approach to $J=\frac{3}{2}^+$ Resonances in the Unitary Symmetry Model*

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An analysis is made of the $p_{3/2}$ pseudoscalar meson-baryon scattering amplitudes in all isotopic spin and strangeness states. In analogy with pion-nucleon scattering, the basic assumption that the single baryon exchange contribution provides the dominant force in these amplitudes is made, but all the coupled two-particle channels in each isotopic spin and strangeness state are included. A relativistic scattering matrix for each state is constructed by a matrix formulation of the N/D method so that it satisfies the requirements of unitarity and symmetry. Being fully relativistic, unlike the static model, the theory does not contain adjustable cutoff parameters. The known masses of the mesons and baryons and the Yukawa-type meson-baryon coupling constants are the only parameters that enter into the calculation. The octet model of Gell-Mann and Ne'eman is used to define the coupling constants in terms of a single parameter f and the known pion-nucleon coupling constant. Then the requirement that $N^*(T=\frac{3}{2};S=0)$, $Y_1^*(T=1;S=-1)$, the recently observed $\Xi^*(T=\frac{1}{2};S=-2)$, and a yet-to-be-discovered $Z^-(T=0,S=-3)$ exist as a tenfold representation of the unitary symmetry model restricts f to a rather narrow range. There are no other resonances in the $p_{3/2}$ state for f in this range, except possibly one in the $T=0$, $S=-1$ system. The positions and widths of these resonances are discussed.

I. INTRODUCTION

RECENTLY, a resonance in the cascade-pion system, in the isotopic spin $T=\frac{1}{2}$ state, has been observed.[1] Although the spin and parity of this resonance have not been measured, the indications are that the spin is greater than $\frac{1}{2}$. If it is assumed that the resonance is in the $p_{3/2}$ state, being the analog of the well-known $p_{3/2}$ resonance in the pion-nucleon system, then one encounters an interesting problem. Experimentally the cascade particle appears to have the same spin,[2] as well as the same isotopic spin, as the nucleon. The Chew-Low static model[3] then, which successfully explained the pion-nucleon 3, 3 resonance, should be even more appropriate in the cascade-pion system because of the increased cascade mass. This reasoning, however, leads us to expect a resonance in the $T=\frac{3}{2}$ cascade-pion state and not in the $T=\frac{1}{2}$ state.

On the other hand, the observed $T=\frac{1}{2}$ resonance was predicted by, and neatly fits into, the group of baryon-meson $p_{3/2}$ resonances known as the tenfold representation of the unitary symmetry group $SU(3)$.[4] The main purpose of this paper is to investigate whether predictions based on dynamical considerations, to be specified below, can be reconciled with those based on symmetry arguments.

The basic assumption in our dynamical treatment is, as is generally believed from analyses of pion-nucleon scattering, the importance of the Born approximation term arising from the single baryon exchange diagram. However, in treating the Born approximation term we depart from the static model[5] with its reliance on an arbitrary cutoff parameter. Our calculation is, in general, closer to the fully relativistic treatments of Baker,[6] and Frautschi and Walecka,[7] as will be discussed in more detail in the following sections. But this does not alter the fact that one cannot obtain a resonance in the $T=\frac{1}{2}$ cascade-pion system since the Born approximation term is repulsive in that state.

One possible approach, suggested immediately by the treatments of the Y^* resonances in the strangeness -1 channels,[8] is based on the idea that the coupled two-particle inelastic channels are important. This notion is reinforced when one notes that the $\Sigma\bar{K}$ and $\Lambda\bar{K}$ thresholds are much closer to that of the $\Xi\pi$ system than are their analogs, ΣK and ΛK, in the pion-nucleon problem. In this way we are led to a matrix formulation for our dynamical considerations. When one turns to a coupled-channel calculation, however, one immediately encounters the problem of an overabundance of coupling constants which, especially in this system, are so poorly defined by experiment that they assume the roles of undetermined parameters. Since we prefer to work with as few arbitrary constants as possible, we turn to the group-theoretical considerations for a model of the coupling constants.

The octet model of the strongly interacting par-

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[1] G. M. Pjerrou, D. J. Prowse, P. Schlein, W. E. Slater, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **9**, 114 (1962); L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra, G. C. Moneti, R. R. Rau, N. P. Samios, I. O. Skillicorn, S. S. Yamamoto, M. Goldberg, L. Gray, J. Leitner, S. Lichtman, and J. Westgard, *ibid.* **9**, 180 (1962).

[2] L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra, G. C. Moneti, R. R. Rau, N. P. Samios, I. O. Skillicorn, S. S. Yamamoto, M. Goldberg, L. Gray, J. Leitner, S. Lichtman, and J. Westgard, Phys. Rev. Letters **9**, 229 (1962).

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[4] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962); S. L. Glashow and J. J. Sakurai, Nuovo Cimento **25**, 337 (1962); R. Cutkosky, J. Kalckar, and P. Tarjanne, Phys. Letters **1**, 93 (1962).

[5] While this work was in progress, work by R. H. Capps [Nuovo Cimento **27**, 1208 (1963)] was brought to our attention. In it he carries out a similar analysis based on the static model.

[6] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958).

[7] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[8] D. Amati, A. Stanghellini, and B. Vitale, Nuovo Cimento **13**, 1143 (1959); K. C. Wali, T. Fulton, and G. Feldman, Phys. Rev. Letters **6**, 644 (1961).

ticles proposed by Gell-Mann, and independently by Ne'eman,[9] leads to a two-parameter set of expressions for the baryon-pseudoscalar meson coupling constants. These are conveniently summarized by Sakurai.[10] Furthermore, the requirement that the pion-nucleon coupling constant be $g_{N\pi}{}^2/4\pi=15$ reduces the dependence to a single arbitrary parameter. We adopt the octet-model set of coupling constants as a working hypothesis.

As mentioned in the opening remarks, the octet model forms the basis for predictions of the interactions of elementary particles. These predictions go far beyond the simple coupling constant considerations outlined above. Among the more striking of these is the suggestion that a family of $p_{3/2}$ resonances can be identified as belonging to the tenfold representation in $SU(3)$. The members of this family[4] are taken to be the familiar pion-nucleon 3, 3 resonance, the strangeness -1 Y_1^*, the recently discovered Ξ^*, and a yet-to-be-discovered resonance or bound state in the strangeness -3 channel with isospin zero, tentatively named the Z^- by Glashow and Sakurai.[11] For the prediction to be fulfilled, all of these resonances must have the angular-momentum quantum numbers of the 3, 3 resonance.

If we take these suggestions seriously, then it is clear that an explanation of the cascade-pion resonance is too narrow in scope. Rather we should attempt to treat all of the baryons and pseudoscalar mesons on an equal footing, apart from their mass differences which will obviously affect dynamical considerations, and see whether an underlying structure suggestive of all four resonances in the representation 10 exists. In this approach it is, of course, necessary to make the usual assumptions about the spins and parities of the particles involved. The baryons N, Λ, Σ, Ξ are taken to be spin-$\frac{1}{2}$ particles with positive relative parities while the mesons π, K, $\bar{K}$, η are assumed to be pseudoscalar.

Therefore, in each state of isotopic spin and strangeness, we consider coupled systems of the allowed two-particle states, ignoring the inelastic channels consisting of three or more particles. Finally, we restrict our attention to the $p_{3/2}$ partial-wave state. We may summarize our dynamical model as follows. The relativistic scattering matrix for each state will be constructed by a matrix formulation of the N/D method so that it satisfies the requirements of unitarity and symmetry and contains the singularities arising from the single baryon exchange graphs in all two-particle channels. From the N/D formulation the condition for a resonance or bound state in a given channel is easily isolated and, in this approximate model, the dynamical structure underlying the unitary symmetry group is understood. Because the approach taken is relativistic and, therefore, requires no cutoff parameter, the calculation contains only the unknown coupling constant parameter which is then restricted to a small range by comparison with experiment.

In Sec. II we review the pertinent kinematical considerations and describe our choice of variable and scattering amplitude. Section III is devoted to the description of the many-channel N/D formalism and the comparison of our model with the calculations of pion-nucleon scattering mentioned above. In Sec. IV we turn to the relationship of our treatment to the predictions of the unitary symmetry group. Section V contains the analysis of the mass-difference corrections and a brief discussion of dynamic factors not included in our model. The octet-model single-parameter expressions for the coupling constants are given in Appendix I, while Appendix II contains the tabulation of the appropriate isotopic spin coefficients in the different channels.

II. KINEMATICS AND CHOICE OF AMPLITUDE

The kinematical considerations for baryon-meson scattering have been developed extensively in the literature.[7,12] The principal modifications required in the analysis to follow stem from the fact that we are concerned with inelastic two-body scattering processes as well as elastic ones. This necessitates a more careful treatment of the scattering amplitude in order to eliminate the kinematical singularities inherent in inelastic processes. The following equations, therefore, are intended mainly as a guide to our final choice of partial-wave amplitudes.

We begin with the general two-body scattering amplitude for a process involving a spin-$\frac{1}{2}$ baryon and a spin-zero meson in the initial and final states. Let the initial baryon and meson have masses M_1 and μ_1 and four-momenta p_1 and q_1, respectively; and let M_2 and μ_2, p_2 and q_2, represent the corresponding quantities for the final baryon and meson (Fig. 1). The Lorentz invariant T matrix for this process is defined by[13]

$$S_{fi}=\delta_{fi}-(2\pi)^4 i\delta^4(p_2+q_2-p_1-q_1)\times\left[\frac{M_1M_2}{4E_1E_2\omega_1\omega_2}\right]^{1/2}\bar{u}_2Tu_1, \quad (1)$$

where $E_{1,2}$ and $\omega_{1,2}$ represent the energies of the appropriate baryons and mesons, respectively, and the

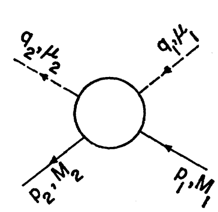

FIG. 1. General meson-baryon scattering diagram.

[9] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[10] J. J. Sakurai (to be published).

[11] S. L. Glashow and J. J. Sakurai, Nuovo Cimento **26**, 622 (1962).

[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[13] We use natural units $\hbar=c=1$ and the energies are measured in terms of the pion mass μ.

Dirac spinors are normalized so that $\bar{u}u=1$. For the two-body processes we are considering, T has the general structure

$$T=-A(s,t,u)+\frac{q_1+q_2}{2}B(s,t,u), \qquad (2)$$

where A and B are scalar functions of the familiar Mandelstam variables

$$\begin{aligned} s&=-(p_1+q_1)^2,\\ t&=-(q_1-q_2)^2,\\ u&=-(p_1-q_2)^2,\\ s+t+u&=M_1^2+M_2^2+\mu_1^2+\mu_2^2. \end{aligned}$$

The invariant amplitudes A and B are linear combinations of the amplitudes referring to the total isotopic spin states involved in the scattering.

In the center-of-mass system the differential cross section is given by

$$\frac{d\sigma}{d\Omega}=\frac{q_f}{q_i}\sum_{\text{spins}}\left|\left\langle f\left|f_1+\frac{(\boldsymbol{\sigma}\cdot\mathbf{q}_f)(\boldsymbol{\sigma}\cdot\mathbf{q}_i)}{q_fq_i}f_2\right|i\right\rangle\right|^2, \qquad (3)$$

where q_i and q_f are the magnitudes of the three-momenta $\mathbf{q}_i$ and $\mathbf{q}_f$ of the initial and final mesons, respectively. The scalar functions f_1 and f_2 in the two-component representation implied by Eq. (3) are related to the invariant amplitudes by

$$8\pi Wf_1=(E_1+M_1)^{1/2}(E_2+M_2)^{1/2}\times\left[A+\left(W-\frac{M_1+M_2}{2}\right)B\right], \qquad (4)$$

$$8\pi Wf_2=(E_1-M_1)^{1/2}(E_2-M_2)^{1/2}\times\left[-A+\left(W+\frac{M_1+M_2}{2}\right)B\right]. \qquad (4)$$

In the expressions above, W is the total energy in the c.m. system. For a given baryon-meson state with masses M and μ, the quantities E, ω, and q are related to W by the expressions

$$\begin{aligned} E&=(W^2+M^2-\mu^2)/2W,\\ \omega&=(W^2-M^2+\mu^2)/2W, \qquad (5)\\ q&=[W^4-2(M^2+\mu^2)W^2+(M^2-\mu^2)^2]^{1/2}/2W. \end{aligned}$$

As is well known,[7] the partial-wave scattering amplitude, $f_{l\pm}$, for a state with total angular momentum $J=l\pm\frac{1}{2}$ and parity $(-1)^{l+1}$ is given by the projection

$$f_{l\pm}(W)=\frac{1}{2}\int_{-1}^{1}dx[f_1(W,x)P_l(x)+f_2(W,x)P_{l\pm1}(x)], \qquad (6)$$

where $x=\cos\theta$, θ being the scattering angle. From Eqs. (4), (5), and (6)

$$\begin{aligned} f_{l\pm}=&\frac{1}{16\pi W^2}[(W+M_1)^2-\mu_1^2]^{1/2}[(W+M_2)^2-\mu_2^2]^{1/2}\\ &\times\left[A_l+\left(W-\frac{M_1+M_2}{2}\right)B_l\right]\\ &+\frac{1}{16\pi W^2}[(W-M_1)^2-\mu_1^2]^{1/2}[(W-M_2)^2-\mu_2^2]^{1/2}\\ &\times\left[-A_{l\pm1}+\left(W+\frac{M_1+M_2}{2}\right)B_{l\pm1}\right], \qquad (7) \end{aligned}$$

where

$$A_l=\frac{1}{2}\int_{-1}^{1}AP_l(x)dx,$$

$$B_l=\frac{1}{2}\int_{-1}^{1}BP_l(x)dx.$$

For our purposes the coupled-channel scattering matrix for given quantum numbers of strangeness, total isotopic spin, angular momentum, and parity is limited to two-particle states. In order to keep the notation as simple as possible, the partial-wave amplitude for the process $M_i+\mu_i\rightarrow M_j+\mu_j$ in the isotopic spin state T will be denoted by $f_{l\pm}{}^T(j,i)$. Time-reversal invariance requires the scattering matrix to be symmetric, i.e.,

$$f_{l\pm}{}^T(j,i)=f_{l\pm}{}^T(i,j).$$

And the unitarity condition in the physical region demands

$$\operatorname{Im}f_{l\pm}{}^T(j,i)=\sum_n f_{l\pm}{}^{T*}(j,n)q_nf_{l\pm}{}^T(n,i), \qquad (8)$$

where the summation extends over the allowed two-particle states in our "elastic" approximation. It should be noted that the phase space factor q_n appearing in the unitarity condition implicitly contains a step function which vanishes below the physical threshold in channel n and is unity above that threshold. In a single-channel case, Eq. (8) leads to the familiar form $q^{-1}\sin\delta_{l\pm}{}^T\exp(i\delta_{l\pm}{}^T)$ for the partial-wave amplitude in the physical region.

Now, in the choice of our amplitudes, we are guided by the considerations discussed by Frautschi and Walecka, namely, we wish to retain as many features of the correct relativistic theory as possible. It is evident from Eq. (7) that the amplitudes $f_{l\pm}{}^T(j,i)$ contain kinematic poles and branch cuts that are in no way related to the dynamic singularities contained in the invariant amplitudes A and B. Furthermore, these kinematic singularities cannot conveniently be removed if we chose to work in the W^2 plane. On the other hand, they are easily eliminated by taking the amplitudes to be functions of the complex variable W.

The appropriate partial-wave amplitudes, which are devoid of kinematic singularities in the W plane and ensure the correct threshold behavior of the $f_{l\pm}{}^T$, are obtained by defining

$$G_{l+}{}^T(j,i)=\frac{2W}{q_j{}^l[(W+M_j)^2-\mu_j{}^2]^{1/2}}\times f_{l+}{}^T(j,i)\frac{2W}{q_i{}^l[(W+M_i)^2-\mu_i{}^2]^{1/2}},$$
$$G_{l-}{}^T(j,i)=\frac{2W}{q_j{}^{l-1}[(W-M_j)^2-\mu_j{}^2]^{1/2}}\times f_{l-}{}^T(j,i)\frac{2W}{q_i{}^{l-1}[(W-M_i)^2-\mu_i{}^2]^{1/2}}. \tag{9}$$

These definitions are based on the assumption that in the vicinity of the thresholds the projections of the invariant amplitudes behave as

$$A_l(j,i)\propto q_j{}^l q_i{}^l,$$
$$B_l(j,i)\propto q_j{}^l q_i{}^l.$$

The unitarity condition satisfied by the $G_{l\pm}{}^T$, obtained from Eqs. (8) and (9), is given by

$$\operatorname{Im}G_{l\pm}{}^T(j,i)=\sum_n G_{l\pm}{}^{T*}(j,n)\rho_{l\pm}(n)G_{l\pm}{}^T(n,i), \tag{10}$$

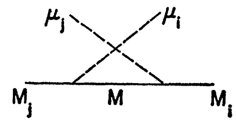

FIG. 2. Single baryon exchange diagram. M_i and μ_i (M_j and μ_j) refer to the initial (final) baryon and meson masses, respectively. M is the mass of the intermediate baryon.

where the phase space factor $\rho_{l\pm}(n)$ is

$$\rho_{l+}(n)=[(W+M_n)^2-\mu_n{}^2]q_n{}^{2l+1}/4W^2,$$
$$\rho_{l-}(n)=[(W-M_n)^2-\mu_n{}^2]q_n{}^{2l-1}/4W^2. \tag{11}$$

An additional property of the partial-wave amplitudes $f_{l\pm}{}^T(j,i)$ and $G_{l\pm}{}^T(j,i)$ in the W plane is the "crossing relation"

$$G_{l+}{}^T(-W)=-G_{l+1,-}{}^T(W),$$
$$\rho_{l+}(-W)=-\rho_{l+1,-}(W), \tag{12}$$

which implies a set of cuts on the negative real axis, whose discontinuities are bounded by unitarity.

Finally, as pointed out in the introduction and as is further discussed in the following section, in our analysis of scattering in the $p_{3/2}$ state we will approximate the dynamical singularities of $G_{1+}{}^T(j,i)$ by those arising from the single baryon exchange graph (Fig. 2). If the isotopic spin factors and coupling constants are suppressed, the general expression for the $p_{3/2}$ contribution of this diagram is[14]

$$G_{B,1+}(j,i)=\frac{(W+M_j+M_i-M)}{4q_j{}^2q_i{}^2}\Bigg[\left\{\frac{3\alpha_{ji}(W)}{[(W+M_j)^2-\mu_j{}^2][(W+M_i)^2-\mu_i{}^2]}-2\frac{(W-M_j-M_i+M)}{(W+M_j+M_i-M)}\right\}$$
$$-\frac{1}{2\beta_{ji}(W)}\left\{\frac{3\alpha_{ji}(W)}{[(W+M_j)^2-\mu_j{}^2][(W+M_i)^2-\mu_i{}^2]}-2\frac{(W-M_j-M_i+M)}{(W+M_j+M_i-M)}\right.$$
$$\left.-\frac{[(W-M_j)^2-\mu_j{}^2][(W-M_i)^2-\mu_i{}^2]}{\alpha_{ji}(W)}\right\}\ln\left(\frac{1+\beta_{ji}(W)}{1-\beta_{ji}(W)}\right)\Bigg], \tag{13}$$

where

$$\alpha_{ji}(W)=W^4-(M_j{}^2+M_i{}^2+\mu_j{}^2+\mu_i{}^2-2M^2)W^2-(M_j{}^2-\mu_j{}^2)(M_i{}^2-\mu_i{}^2),$$
$$\beta_{ji}(W)=4W^2q_jq_i/\alpha_{ji}(W).$$

The singularities of the Born approximation term are contained in the logarithmic factor and can be chosen to consist of two branch cuts on the real axis and a third branch cut running the length of the imaginary axis. The real-axis branch cuts are reflections of each other across the imaginary axis and, in the case of elastic scattering $(M_i,\mu_i)=(M_j,\mu_j)$, the branch points are given by the simple expressions

$$W_1=(M_j{}^2-\mu_j{}^2)/M,$$
$$W_2=(2M_j{}^2+2\mu_j{}^2-M^2)^{1/2}. \tag{14}$$

This choice of cuts is the simplest one consistent with the crossing relation and the reality condition on $G_{1+}(j,i)$. Equation (14) forms the basis of the pole approximation for processes such as pion-nucleon scattering, for which the real-axis cuts are roughly one-third of a pion mass in length and can reasonably be replaced by poles. However, in scattering processes involving the heavy mesons, the cuts may extend up to four pion masses in length and the pole approximation becomes increasingly difficult to justify.

It will be convenient later to have a more tractable expression for $G_{B,1+}(j,i)$ than the exact functional de-

[14] Later when we have occasion to refer to the Born approximation term [Eq. (13)] for particular particle states, we will employ the notation $G_B(BC,A,DE)$ where the letters A through E will be replaced by the appropriate particle symbols, with D and E (B and C) representing the initial (final) baryon and meson, respectively, while A indicates the intermediate baryon.

pendence (13). Such an expression is readily obtained for the low-energy region by expanding the logarithm in powers of q_jq_i, the result being

$$G_{B,1+}(j,i)\approx 8W^4(W-M_j-M_i+M)/3\alpha_{ji}^2(W). \quad (15)$$

A further simplification results from noting that $\alpha_{ji}(W)$ can be factored into the form

$$\alpha_{ji}(W)=(W-c)(W+c)(W^2+d^2),$$

where c and d are real quantities. In Eq. (15), the second-order pole at $W=c$ lies on the branch cut and marks the position of the dominant contribution arising from the cut.

III. PION-NUCLEON SCATTERING AND THE MATRIX N/D FORMATION

It is clear that the minimum requirement for any dynamical model to be used in the understanding of resonances in baryon-meson systems is that it reproduce the experimentally well-established features of pion-nucleon scattering in the $p_{3/2}$ state. As a guide, therefore, in our choice of a dynamical coupled-channel formalism, we turn to three of the well-known theoretical treatments of this N-π problem.

One of the earliest successful approaches to pion-nucleon scattering was the Chew-Low static model and its dispersion-theoretic refinements.[12] In this model the single nucleon exchange diagram and unitarity form the basis of the dynamics; but the approximation of taking the nucleon mass to be very large compared with the pion mass has the effect of keeping only one part of the Born approximation singularities given by Eqs. (13) and (14)—namely, the short branch cut nearest the physical region. This short branch cut becomes the pole at the origin in the static model. In addition, the static model treatment requires an arbitrary cutoff because the integrals involved are divergent.

The static model is easily generalized to the relativistic case by replacing the nearby cut by a pole whose residue (as pointed out by Frautschi and Walecka) is just that of the static model, and by ignoring the other cuts in the Born approximation. Unitarity is satisfied by using the N/D method[15] with a once-subtracted (at the pole) dispersion relation for D. This relativistic formulation requires no cutoff parameter and, in fact, contains no arbitrary constants. When this program is carried through, it is found that no resonance is predicted in the 3, 3 state for pion-nucleon scattering; that is, the short cut in the Born approximation is not sufficiently attractive to cause the phase shift to go through 90°. From this viewpoint it is evident that the primary factor in the static model's successful prediction of the 3, 3 resonance is the necessity for an arbitrary cutoff which allows the integrals to be made as large as required.

The results of the static model were placed on a sounder footing by Frautschi and Walecka, who showed that the single nucleon exchange term did indeed produce a resonant structure for the 3, 3 amplitude if the branch cut along the imaginary axis was included in the dynamic singularities along with the short branch cut. Their single channel calculation, fully relativistic and devoid of cutoff parameters, was based on a many-pole approximation N/D technique which required the solution of simultaneous algebraic equations. This point will be important when we consider the coupled-channel formalism. For our purposes the central result of their analysis is the fact that the Born approximation, treated relativistically, does reproduce the general features of pion-nucleon scattering in the $p_{3/2}$ state. It is also pertinent to note their comment to the effect that the single nucleon exchange singularities are too strong (i.e., the 3, 3 resonance occurs too close to the physical threshold). They remark that a reduction of the discontinuity across the imaginary axis cut by roughly a factor of 2 leads to closer agreement with experiment.

The third pion-nucleon treatment to be discussed is Baker's application of the determinantal method.[6] His formalism, based primarily on perturbation expansions, bears a remarkable similarity to the N/D method. The first-order determinantal result for the partial-wave amplitude may be considered to be an N/D representation, where N is the full Born approximation term [Eq. (13)] with appropriate isotopic spin factors and coupling constants, and D is expressed by a once subtracted dispersion integral over the physical cut. The integrand in the latter expression contains N in the standard way. The subtraction point is chosen to be the nucleon mass. This choice means that the full amplitude is normalized so that its discontinuity across the short cut is essentially equal to the Born approximation discontinuity. These points will be discussed in more detail in the application of our model to pion-nucleon scattering. The many-channel formalism we employ is, apart from small modifications arising from crossing, a matrix generalization of Baker's work. Before proceeding to the matrix N/D formulation, it is important to point out that the resonant structure of the 3, 3 amplitude was also obtained by Baker, with the position of the resonance about $1.2\,\mu$ higher than the experimentally observed position.

The success of these single-channel treatments makes it plausible that a relativistic coupled-channel formalism based on unitarity and the single baryon exchange term can be useful in understanding baryon-meson resonances in the $p_{3/2}$ state. We do not expect to be able to predict the exact locations of the resonances, since resonance positions are known to be sensitive to many dynamical effects we cannot include.

The most convenient manner in which to ensure the unitarity of the scattering matrix is by means of a matrix generalization of the N/D method. Such a generalization is not unique but depends, rather, on the approximations to be used. An extension of the single-pole approximation to the many-channel case has been

[15] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

given by Feldman, Matthews, and Salam,[16] while a more general N/D coupled integral equation formalism has been presented by Bjorken.[17] The single-pole approximation allows one to extend the static model to coupled channels without difficulty. However, the static model requires either the nonrelativistic treatment in which an arbitrary cutoff is introduced or the relativistic treatment in which one cannot obtain even the pion-nucleon resonance. In either case, there is difficulty involved in the replacement of very long branch cuts by single poles. For these reasons, the single-pole approximation is inadequate for our purposes.

A matrix generalization of the many-pole approximation used by Frautschi and Walecka would be quite acceptable as a dynamical model, apart from the extensive work involved in determining the correct residues for the poles. The limitation in this case, however, stems from the difficulty in going from the single-channel method to the matrix form as illustrated by Costa and Ferrari,[18] who based their analysis on Bjorken's matrix N/D formalism. An explicit representation for the many-pole approximation to a scattering matrix of arbitrary dimensionality exists,[19] but suffers from the fact that the integrals contained in the denominator matrix D must be subtracted n times, where n is the number of poles. For these reasons we will not employ the generalization of the many-pole approximation.

Fortunately, a straightforward generalization of Baker's single-channel first-order determinantal method is easily carried out. We take the numerator matrix N to be given by the full Born approximation terms [Eq. (13)] with appropriate isotopic spin factors and coupling constants, and construct D so that the two requirements of symmetry and unitarity of the T matrix are satisfied. Since the numerator matrix in this model is symmetric, Bjorken's matrix formalism, while ensuring unitarity, will not in general satisfy the symmetry requirement unless one proceeds to solve the coupled matrix integral equations.[20] For our approximation scheme such an exact treatment is unwarranted.

Instead, we will utilize the "symmetrized" many-channel N/D method[19] defined as follows:

$$T(W)=N(W)D^{-1}(W),$$

$$D(W)=1-\frac{1}{2\pi}\int\frac{\rho(W')N(W')}{W'-W}dW'-\frac{N^{-1}(W)}{2\pi}\times\int\frac{N(W')\rho(W')}{W'-W}dW'\cdot N(W), \tag{16}$$

where $\rho(W)$ is a diagonal matrix of phase space factors and the ranges of the integrals are over the physical region as determined by the step functions implied by the elements of $\rho(W)$.[21] It is easily verified that the above formulation satisfies unitarity in the physical region and that for symmetric $N(W)$ it leads to a symmetric T matrix. In specializing Eq. (16) to the $p_{3/2}$ state we introduce the following modifications. The crossing relation [Eq. (12)] indicates the existence of physical cuts (for the $d_{3/2}$ amplitudes in this case) running along the negative real axis in the W plane. The unitarity condition bounds the discontinuities across these cuts and it is proper, therefore, to construct the scattering matrix so that unitarity is satisfied on both sets of physical branch cuts. We also write once-subtracted dispersion relations for $D(W)$, subtracting at the point $W=S$, in order to define suitably renormalized coupling constants. In each amplitude, S will be chosen in the region of the right-hand dynamic branch cuts, the singularities nearest to the physical region in our model. The subtraction also ensures the convergence of the integrals so that, in contrast to the static model, we require no cutoff parameters. When these modifications are incorporated, the scattering matrix $G_{1+}(W)$, whose elements are defined by Eq. (9), is written

$$G_{1+}(W)=G_{B,1+}(W)D_{1+}^{-1}(W),$$

$$D_{1+}(W)=1-\frac{(W-S)}{2\pi}\int_{W_0}^{\infty}dW'\left[\frac{\rho_{1+}(W')G_{B,1+}(W')}{(W'-S)(W'-W)}+\frac{\rho_{2-}(W')G_{B,1+}(-W')}{(W'+S)(W'+W)}\right]$$
$$-G_{B,1+}^{-1}(W)\frac{(W-S)}{2\pi}\int_{W_0}^{\infty}dW'\left[\frac{G_{B,1+}(W')\rho_{1+}(W')}{(W'-S)(W'-W)}+\frac{G_{B,1+}(-W')\rho_{2-}(W')}{(W'+S)(W'+W)}\right]G_{B,1+}(W), \tag{17}$$

[16] G. Feldman, P. T. Matthews, and A. Salam, Nuovo Cimento **16**, 549 (1960).
[17] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[18] G. Costa and F. Ferrari, Nuovo Cimento **22**, 214 (1961).
[19] A. W. Martin, Ph.D. thesis, Stanford University, Stanford, California, 1962 (unpublished).
[20] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).
[21] See discussion following Eq. (8).

where the elements of $G_{B,1+}(W)$ are given by Eq. (13) with the appropriate isotopic spin factors and coupling constants, to be found in Appendixes I and II. The integrals in Eq. (17) run only over the positive real axis with W_0 representing the various physical thresholds.

The relationship of our approximation for $G_{1+}(W)$ to the exact solution of the coupled integral equation formalism, taking only the single baryon exchange graphs for the dynamical singularities, may be viewed in the following way. Both scattering matrices satisfy unitarity and are symmetric. Both have the dynamical branch cuts contained in $G_{B,1+}(W)$. However, the exact solution would have the precise discontinuities of G_B across the cuts, while our model provides only a reasonable approximation to these discontinuities for the distant cuts. The nearby branch cuts, on the other hand, are treated accurately in our formalism because of our choice of subtraction point. These statements will be made quantitative in the analysis of pion-nucleon scattering to follow.

Having defined our dynamical model for coupled-channel processes, we turn to the resonance (or bound state) condition in the many-channel case. This condition, which has been discussed by several authors,[22] can be conveniently stated in terms of the inverse K matrix defined by

$$G_{l\pm}^{-1}=K_{l\pm}^{-1}-i\rho_{l\pm}, \tag{18}$$

where $K_{l\pm}^{-1}$ is a real symmetric matrix both above and below thresholds. The vanishing of $\det K_{l\pm}^{-1}$ then gives the resonance location. In the matrix formulation of the N/D method this corresponds to

$$\det K^{-1}(W_r)=\det[\mathrm{Re}D(W_r)\cdot N^{-1}(W_r)]=0, \tag{19}$$

where W_r is the position of the resonance. It should be noted that only the vanishing of det ReD gives what one considers the dynamical resonances. In a single-channel case Eq. (19) reduces to the familiar condition $\rho(W_r)\cot\delta(W_r)=0$.

Finally, in the single-channel case one defines the width Γ of a resonance by noting that near the resonance position

$$\tan\delta=(\Gamma/2)/(W_r-W),$$

which leads to

$$\Gamma/2=\lim_{W\to W_r}(W_r-W)\rho(W)K(W). \tag{20}$$

The extension of these considerations to the many-channel case utilizes the definition of a partial width Γ_i for an elastic channel

$$\Gamma_i/2=\lim_{W\to W_r}(W_r-W)\rho_i(W)K_{ii}(W). \tag{21}$$

The sum of the partial widths, $\Gamma=\sum_i\Gamma_i$, is the full width of the resonance, while the relative branching ratios into the different channels can be expressed in terms of the partial widths and phase space factors.

We conclude this section with an application of our model to pion-nucleon scattering, considered as a single-channel process, in order to fully illustrate the nature of our approximation. Using the isotopic spin factors listed in Appendix II, the two isotopic spin amplitudes, $T=\frac{3}{2}$ and $T=\frac{1}{2}$, for N-π scattering in the $p_{3/2}$ partial-wave state are given by the single-channel form of Eq. (17) as

$$G^{3/2}=\frac{2g^2G_B(N\pi,N,N\pi)}{1-2g^2I(N\pi,N,N\pi)-2ig^2\rho(N\pi)G_B(N\pi,N,N\pi)},$$
$$G^{1/2}=\frac{-g^2G_B(N\pi,N,N\pi)}{1+g^2I(N\pi,N,N\pi)+ig^2\rho(N\pi)G_B(N\pi,N,N\pi)}, \tag{22}$$

where for simplicity the dependence on the total energy variable W has been suppressed and the angular momentum subscript has been dropped. $I(N\pi,N,N\pi)$ represents the real part of the integral in the denominator function,

$$I(N\pi,N,N\pi)=\mathrm{Re}\frac{(W-N)}{\pi}\int_{M+\mu}^{\infty}dW'\left[\frac{\rho_{1+}(W')G_B(W')}{(W'-N)(W'-W-i\epsilon)}+\frac{\rho_{2-}(W')G_B(-W')}{(W'+N)(W'+W+i\epsilon)}\right], \tag{23}$$

and is plotted in Fig. 4. The subtraction point is chosen to be the nucleon mass in keeping with our treatment of the nearby branch cuts. As discussed in Appendix I, it is convenient to absorb the usual factor of 4π into the definition of the coupling constant so that g^2, the square of the pion-nucleon coupling constant appearing in Eq. (22) has the value $g^2=15$.

From the resonance condition and Eq. (22), it is clear that a resonance in the low energy region is possible only in the $T=\frac{3}{2}$ state. The position of the 3, 3 resonance in this model is found to be 2.2 μ higher than the observed position. That it is also higher than the location determined by Baker is due to the inclusion of the integral over the left-hand physical cut. This integral has the effect of reducing the value of $I(N\pi,N,N\pi)$ by roughly 10% in the vicinity of the resonance. The source of the discrepancy with the result of Frautschi and Walecka, who obtain the resonance at too low an energy, is found in the treatment of the distant singularities. In the $T=\frac{3}{2}$ amplitude the discontinuity across the imaginary axis cut, in our model, is only 56% of the actual Born approximation discontinuity in the vicinity of the origin, while the actual discontinuity across the short left-hand branch cut is reduced by slightly more than a factor of 2. In brief, the attractive Born approximation term is made less attractive in this model.

[22] See, for example, R. H. Dalitz, Rev. Mod. Phys. 33, 471 (1961), and also reference 8.

TABLE I. Coupled two-particle pseudoscalar meson-baryon states for each strangeness and isotopic spin channel. The last column contains the factorized form for Det Re$D(W)$ in each case. The quantities S, P, Q, R, $\bar{R}$, and U are defined in Eq. (27).

Strangeness S	Isotopic spin T	Coupled two-particle states	Det Re$D(W)$
1	0	NK	$\bar{R}$
	1	NK	U
0	$\frac{1}{2}$	$N\pi, \Lambda K, \Sigma K, N\eta$	$PQ\bar{R}U$
	$\frac{3}{2}$	$N\pi, \Sigma K$	RU
	0	$N\bar{K}, \Sigma\pi, \Xi K, \Lambda\eta$	$SPQU$
−1	1	$N\bar{K}, \Sigma\pi, \Lambda\pi, \Xi K, \Sigma\eta$	$PQR\bar{R}U$
	2	$\Sigma\pi$	U
−2	$\frac{1}{2}$	$\Xi\pi, \Lambda\bar{K}, \Sigma\bar{K}, \Xi\eta$	$PQRU$
	$\frac{3}{2}$	$\Xi\pi, \Sigma\bar{K}$	$\bar{R}U$
−3	0	$\Xi\bar{K}$	R
	1	$\Xi\bar{K}$	U

In the $T=\frac{1}{2}$ amplitude, on the other hand, the effect of the distant singularities is enhanced, the imaginary axis cut by a factor of roughly $1\frac{1}{2}$ and the left-hand cut by a factor of $2\frac{1}{4}$. As pointed out by Frautschi and Walecka, the $T=\frac{1}{2}$ amplitude is not very sensitive to these distant singularities because of the strong repulsion of the nearby branch cut. Hence, it is not surprising that our values for the $T=\frac{1}{2}$ phase shift differ very little from their results (excluding the contribution due to the π-π singularities). In both approaches, then, qualitative agreement with experiment is obtained in each of the isotopic spin states.

In computing the resonance width, however, which from Eq. (20) is given by

$$\frac{\Gamma}{2}=\frac{\rho(W_r)G_B(W_r)}{(dI/dW)_{W=W_r}}, \tag{24}$$

we obtain too large a value. This is due primarily to the phase space factor in Eq. (24), proportional to q^3, to which the width is very sensitive in the low-energy region. If the phase space factor to be evaluated is taken arbitrarily at the observed resonance position, then Eq. (24) predicts the half-width at half-maximum to be about 100 MeV, as compared to the experimental value of roughly 70 MeV.

IV. SYMMETRY CONSIDERATIONS

As stated in the introduction, all of the Yukawa type coupling constants required for the single baryon exchange diagrams can be expressed in terms of a single parameter, denoted by f, once the octet-model expressions are normalized to the experimentally measured pion-nucleon coupling constant. The explicit single-parameter forms derived from the unitary symmetry model are given in Appendix I. Using these coupling constants and the formalism described in the previous sections, we wish to examine the resonance conditions in the various $p_{3/2}$ states corresponding to different isotopic spin and strangeness. However, to illustrate clearly the implications of adopting the octet-model coupling constant scheme in our calculations, we first consider a simplified model in which all baryons have the same mass B and all pseudoscalar mesons have the mass m.

In this extreme model all channels have the same threshold and phase space factor $\rho(W)$, and the matrices $G_B(W)$ and $D(W)$[Eq. (17)] assume the simple forms

$$\begin{aligned} G_B(W)&=h(W)\mathcal{B},\\ D(W)&=1-I(W)\mathcal{B}-i\rho(W)h(W)\mathcal{B}, \end{aligned} \tag{25}$$

where $h(W)$ represents the scalar Born approximation function $G_B(Bm,B,Bm)$ from Eq. (13), and $I(W)$ is the real part of the integral

$$I(W)=\text{Re}\frac{W-B}{\pi}\int_{B+m}^{\infty}dW'\left[\frac{\rho_{1+}(W')h(W')}{(W'-B)(W'-W-i\epsilon)}+\frac{\rho_{2-}(W')h(-W')}{(W'+B)(W'+W+i\epsilon)}\right]. \tag{26}$$

In Eq. (25), $\mathcal{B}$ is a constant symmetric matrix containing the isotopic spin factors and coupling constants. The relevant information needed to construct $\mathcal{B}$ for a given isotopic spin and strangeness state is contained in the appendices.

From Eq. (25) it is clear that $G_B(W)$ and $D(W)$ can be simultaneously diagonalized by the orthogonal transformation which diagonalizes $\mathcal{B}$. This means that, in this model, the full scattering matrix $G(W)$ for each isotopic spin and strangeness state is diagonalized (put into the eigenphase representation) by a transformation determined solely by coupling constants and isotopic spin factors. Now it is well known[4] that the two-particle baryon-pseudoscalar meson states can be expressed as linear combinations of states forming the basis of the representations 1, 8, 8, 10, $\overline{10}$, and 27 of the $SU(3)$ group. The consequence of choosing the octet model coupling constant scheme is that the elements of the diagonal forms of the various scattering matrices correspond precisely to these six representations. In other words, the orthogonal matrix relating the particle states to the unitary symmetry basis states is exactly the matrix which diagonalizes $\mathcal{B}$ in each strangeness and isotopic spin channel.

For our purposes the primary result following from these considerations is that the determinant of the real part of $D(W)$, being invariant under orthogonal transformations, can be expressed in terms of n factors, where n is the dimensionality of $D(W)$. Each such factor has the general structure of the denominator function in a single-channel case and can be identified with one of the six representations of $SU(3)$ listed above. Thus, the determination of the resonance condition amounts to determining the range of values for the coupling constant parameter f such that one or several of these factors can vanish for a given value of $I(W)$. Table I contains the factorized expressions for det Re$D(W)$ in each isotopic spin and strangeness state, where we have introduced the convenient notation:

$$\begin{aligned} S&=1-(4/3)(5-10f-4f^2)g^2I(W),\\ P&=1+(2/3)(5-10f-4f^2)g^2I(W),\\ Q&=1+2(1-2f+4f^2)g^2I(W),\\ R&=1-(8/3)(1+f-2f^2)g^2I(W),\\ \bar{R}&=1-(8/3)(1-5f+4f^2)g^2I(W),\\ U&=1-(4/3)(1-2f+4f^2)g^2I(W). \end{aligned} \tag{27}$$

It is clear from Table I that the factors S, P, Q, R, $\bar{R}$, and U are to be identified in that order with the 1-, 8-, 8-, 10-, $\overline{10}$-, and 27-fold representations of $SU(3)$ and that the vanishing of any one of them corresponds to the realization, in this oversimplified model, of a family of resonances belonging to that representation. In order to find the restrictions on the values of the parameter f such that resonances are possible in some states and not in others it is necessary to evaluate the integral $I(W)$ of Eq. (26). For this purpose we utilize the baryon and pseudoscalar meson mass formula[9,23] to determine the masses B and m:

$$B=\frac{N+\Xi}{2}=\frac{3\Lambda+\Sigma}{4}=8.2\,\mu,$$

$$m^2=K^2=\frac{3\eta^2+\pi^2}{4}=(3.5\,\mu)^2.$$

The real part of the integral, $I(W)=I(Bm,B,Bm)$, is plotted in Fig. 3, from which it follows that the quantity $g^2I(W)$ in Eq. (27) lies in the range 0 to roughly $\frac{2}{3}$ in the low-energy region. The range of values for the parameter f which will give resonances in each of the six representations are then found to be

$$\begin{array}{ll} S & -2.84<f<0.34,\\ P & f<-3.1,\quad 0.59<f,\\ Q & \text{no possibility},\\ R & -0.28<f<0.78,\\ \bar{R} & f<0.15,\quad 1.15<f,\\ U & f<-0.06,\quad 0.56<f. \end{array} \tag{28}$$

In applying this idealized model to the currently available experimental data, we use the apparent absence of $p_{3/2}$ resonances in the NK system (strangeness $+1$) to exclude the vanishing of $\bar{R}$ and U. The lack of a $p_{3/2}$ resonance in the $N\pi$ $T=\frac{1}{2}$ system then further excludes the vanishing of P. These arguments leave us with a relatively narrow range of allowed values for f, namely, $0.15<f<0.56$. It is amusing to note that this

[23] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

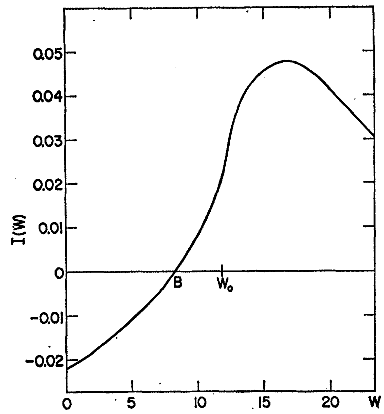

FIG. 3. $I(W)=I(Bm,B,Bm)$ defined in Eq. (26), plotted as a function of the total energy W in units of the pion mass μ. $B=8.2\,\mu$, $m=3.5\,\mu$. The threshold, W_0, is 11.7 μ and the subtraction point is at the baryon mass B.

range of values definitely predicts the vanishing of R, giving resonances in the tenfold representation of the baryon-meson states, while the vanishing of S, corresponding to a $p_{3/2}Y_0^*$ resonance, may or may not occur.

From Eq. (27) it is seen that the maximum attraction in the tenfold representation, given by the factor R, occurs at $f=\frac{1}{4}$. This value of f predicts the unitary singlet Y_0^* resonance at exactly the same location. These two cases differ, however, in that the resonance position given by R is relatively insensitive to variations of f about the value $\frac{1}{4}$, while the position determined by S is strongly dependent on these variations.

It remains to be verified that the results of this idealized model, particularly with regard to the values of f which allow a resonant structure in the states of the tenfold representation, hold as well when the actual particle masses are incorporated into the calculation. This question, as well as the related questions of resonance locations, widths, and branching ratios, is treated in the following section.

V. NUMERICAL RESULTS AND DISCUSSION

The symmetry model analysis described above has the virtue of explicitly showing the possibility, for certain ranges of f, that resonant structures can occur in different sets of scattering amplitudes. It is clear, however, that the mass differences between the elementary particles cannot reasonably be neglected in the low-energy physical region. Not only are the positions and widths of resonances and the relative branching ratios dependent on the actual particle masses, but also the location of the dynamical singularities[24] as is evident from Eq. (14). In this section we apply the relativistic formalism developed in Secs. II and III to the various

[24] From the considerations discussed in Sec. III, the subtraction points for the different strangeness states were chosen to be: $S=+1$, $5.25\,\mu$; $S=0$, $N=6.80\,\mu$; $S=-1$, $\Sigma=8.57\,\mu$; $S=-2$, $\Xi=9.57\,\mu$; $S=-3$, $11.1\,\mu$.

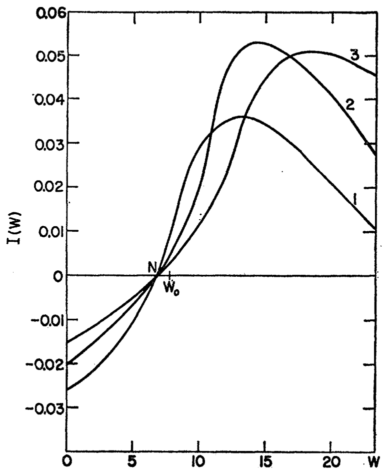

FIG. 4. The elastic integrals necessary in the coupled system with strangeness $S=0$, plotted as functions of the total energy W. Curves 1, 2, and 3 refer to $I(N\pi,N,N\pi)$, $I(N\eta,N,N\eta)$, and $I(\Sigma K,\Xi,\Sigma K)$, respectively. $I(\Lambda K,\Xi,\Lambda K)$ differs very little from curve 3, and has not been plotted. The subtraction point N is the nucleon mass, 6.80 μ, and W_0 is the $N\pi$ threshold, 7.80 μ.

scattering matrices composed of the two-particle states listed in Table I. The only input from the unitary symmetry model, as mentioned in the introduction, is the choice of coupling constants; and the parameter f arising from that choice is the only arbitrary constant appearing in our calculation.

The first point to be made is that the ranges for f that lead to resonant structures [Eq. (28)] in the appropriate amplitudes are not significantly modified by the introduction of the actual particle masses. This is most easily illustrated by the following treatment of the symmetrized N/D method [Eq. (16)]. We rewrite the matrix equation for D in the condensed notation

$$D=1-\tfrac{1}{2}I-\tfrac{1}{2}N^{-1}\tilde{I}N,$$

where I represents the matrix of integrals and $\tilde{I}$ is its transpose. Then defining the antisymmetric matrix R as $R=\tilde{I}N-NI$, we obtain the equivalent form

$$D=1-I-\tfrac{1}{2}N^{-1}R.$$

In the "equal mass" model of Sec. IV the matrix R is identically zero. When the correct particle masses are used in determining R, it is found that this term makes a negligible contribution to the determinant of D.

Thus, the primary modification in determining the resonance conditions is due to the variation of the integrals for different particle states and subtraction points. Several examples of the integrals involved in the calculation are plotted in Figs. 4 and 5. Figure 4, which contains three of the integrals required in the analysis of pion-nucleon scattering in the $p_{3/2}$ partial-wave state, is intended as an illustration of the dependence on the particle state when the subtraction point is held fixed. Figure 5, on the other hand, contains the three integrals necessary in the five single-channel calculations (Table I) and illustrates the dependence upon the choice of subtraction point as well as the particle state. The integrals, which are of the principal-value type above the physical threshold, were evaluated on a computer without approximating either the phase space factor [Eq. (11)] or the Born approximation term [Eq. (13)].

It is worthwhile to point out the following features of the integrals. As would be expected, there is little difference in the low energy region between integrals involving the Λ mass and those with the corresponding dependence on the Σ mass. Second, integrals with a higher physical threshold and the same subtraction point lie below those (in the low energy region) with a lower threshold. This conforms, in part, with expectations regarding the relative importance of inelastic channels with high thresholds. But at the same time it is clear from Fig. 4 that the ΣK channel cannot meaningfully be ignored in the calculation of the 3, 3 pion-nucleon resonance. Third, the integrals for particle states involving the heavier mesons attain a greater maximum value than do those for the states containing pions. Fourth, the integrals for inelastic amplitudes do not differ appreciably from the related "elastic" integrals and, in the low-energy region, their behavior is governed by the particle state associated with the phase space factor. Finally, the integrals tend logarithmically to $-\infty$ as W tends to ∞ along the positive real axis. For consistency, we have verified that no ghosts occur near the physical region.

Incorporating the computed integrals into the symmetrized N/D formalism and evaluating the determinant of the real part of D, we obtain resonant structures in the four scattering matrices corresponding to the tenfold representation. The calculated locations of the resonances and (in parentheses) the experimentally

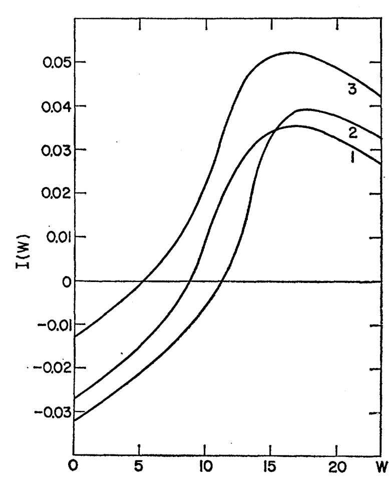

FIG. 5. The integrals necessary in the single channel cases plotted as functions of W. Curves 1, 2, and 3 refer to $I(\Sigma\pi,\Sigma,\Sigma\pi)$, $I(\Xi\bar{K},\Sigma,\Xi\bar{K})$, and $I(NK,\Sigma,NK)$, respectively. The corresponding integrals with the intermediate Λ particle differ very little from the above curves.

observed positions are

$$\begin{array}{lll} N_{3,3}^* & 10.1\,\mu & (9.0\,\mu) \\ Y_1^* & 11.6\,\mu & (10.0\,\mu) \\ \Xi^* & 12.6\,\mu & (11.1\,\mu) \\ Z^- & 13.6\,\mu & (?). \end{array} \quad (29)$$

In calculating these values we have taken $f=\frac{1}{4}$ which corresponds to the lowest possible resonance positions in our model. For f in the range $\frac{1}{6}$ to $\frac{1}{2}$, no resonances are predicted in the other strangeness and isotopic spin systems except the $S=-1$, $T=0(Y_0^*)$ state. As mentioned in the previous section, the position of this unitary singlet resonance is strongly dependent on the value of f, and the resonance vanishes for f greater than roughly $\frac{1}{3}$. With $f=\frac{1}{4}$, the position of the $p_{3/2}Y_0^*$ is predicted to be very close to that of the Y_1^* resonance, namely, $11.6\,\mu$.

Concentrating on the pion-nucleon system for the moment, we note that the inclusion of the ΣK channel lowers the position of the 3, 3 resonance by roughly a pion mass in this model. Furthermore, the width of the resonance is computed to be $\Gamma/2=73$ MeV if we again arbitrarily evaluate the phase space factor at the observed position. This value is to be compared with the 100 MeV half-width obtained in the single-channel model. Both of these effects strongly suggest that the inelastic channels will be of importance in more exact treatments of pion-nucleon scattering. In addition, the effect of the two-particle inelastic channels in the $T=\frac{1}{2}$ system is to reduce the magnitude of the small negative $N\pi$ phase shift and improve the agreement with experiment.[25]

In calculating widths and branching ratios for the remaining $p_{3/2}$ resonances, we consistently evaluate the sensitive phase space terms at the observed resonance locations. The $T=\frac{1}{2}$ cascade-pion resonance width is computed to be $\Gamma/2=5.8$ MeV in this model, in close agreement with experiment.[1] For the Y_1^*, with $f=\frac{1}{4}$, the branching ratios into the $\Lambda\pi$ and $\Sigma\pi$ channels were calculated both for the equal mass model of Sec. IV and the correct mass treatment of this section. In the first case, we obtain $\Lambda\pi$ 91%, $\Sigma\pi$ 9%, while introducing the actual particle masses modifies this to $\Lambda\pi$ 94%, $\Sigma\pi$ 6%, in relatively good agreement with experiment.[26] The width of the Z^- resonance was not evaluated because of the lack of experimental data, while the Y_0^* width was omitted because of its sensitive dependence on the parameter f.

It is, of course, difficult to gauge the effects of dynamical singularities not included in our approximate model. However, qualitative remarks concerning the influence of such diagrams as single ρ or ω vector meson exchange between the baryon and meson lines can be made if ρ and ω are assumed to couple, respectively, to the conserved isotopic spin current and the conserved hypercharge current. Using the simple rules[27] that ρ exchange is repulsive when the isotopic spins are parallel, and attractive otherwise, and that like hypercharges repel in the case of ω exchange while unlike hypercharges attract, we may draw the following rough conclusions. In the pion-nucleon system the ρ exchange is not sufficiently repulsive in the $T=\frac{3}{2}$ state to prevent the 3, 3 resonance from occurring, nor is it attractive enough in the $T=\frac{1}{2}$ state to make the $p_{3/2}$ phase shift positive in the low-energy region.

Although the vector meson exchange terms are not the dominant interactions in the $p_{3/2}$ amplitudes, they are likely to be important in determining resonance positions. In this regard we note that the vector meson terms are attractive in both the Y_1^* and Ξ^* systems, as contrasted with the repulsion in the $N_{3,3}^*$ system noted above. In the Z^- amplitude we may suppose the vector meson effects to be small because of a cancellation between the ρ and ω. On the basis of these extremely sketchy arguments it is interesting to speculate on the possibility of a dispersion theoretic verification of the "equal spacing" rule[28] for the resonances of the tenfold representation. The resonance locations obtained in this calculation [Eq. (29)] show a general equal spacing structure which might easily be improved by including the vector meson contributions.

Finally, we wish to point out two possible experiments relating to the coupling constant parameter f. The first concerns the obvious question of the existence of a Y_0^* resonance in the $p_{3/2}$ state. In particular, if the known Y_0^* at the total energy $10.2\,\mu$ turns out to have these quantum numbers, then the value $f=\frac{1}{4}$ would seem very reasonable. If, however, the predicted resonance is found much higher or does not exist at all, then f must be made larger in this model. As mentioned previously, the prediction of the tenfold family of resonances is relatively insensitive to such adjustments of the coupling constant parameter.

An independent experimental check on f follows from the observation that in this model, if the Σ-Λ mass difference is ignored, the effect of the single baryon exchange term in the $T=0$ NK system is very sensitive to f. For f greater than $\frac{1}{4}$ (and less than 1) the interaction is repulsive in the $p_{3/2}$ state, while values of f less than $\frac{1}{4}$ lead to an attractive interaction. On the basis of rough arguments, we expect the vector meson exchange diagrams effectively to cancel each other in this isotopic spin state. Hence, the sign of the $p_{3/2}$ phase shift in the low-energy region may indicate the appropriate value of f. Similarly, if these oversimplified considerations are valid, we would expect the $p_{1/2}$ phase shift for $T=0$ NK

[25] W. D. Walker, W. D. Shephard, and J. Davis, Phys. Rev. **118**, 1612 (1960).
[26] M. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, and S. G. Wojcicki, Phys. Rev. Letters **5**, 520 (1960).
[27] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).
[28] M. Gell-Mann, in *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962).

scattering to have the opposite sign to that of the $p_{3/2}$ state.

Of course, if the Ξ^* resonance is not in the $p_{3/2}$ state, or if the predicted Z^- is not found, then the basic assumption of this paper, namely, the octet-model expressions for the Yukawa-type coupling constants, must be re-examined. On the other hand, if these predictions are borne out experimentally, then one may feel reasonably confident that the coupling-constant expressions evaluated in the neighborhood of $f=\frac{1}{4}$ will remain meaningful in more refined dynamical treatments. When the usual factor of 4π is absorbed into the coupling constants and the definitions of Appendix I are used, the numerical values obtained with $f=\frac{1}{4}$ are

$$g_{N\pi}^2=h_{\Sigma K}^2=g^2\approx 15,$$
$$g_{\Lambda\pi}^2=g_{\Lambda K}^2=g_{\Lambda\eta}^2=g_{\Sigma\eta}^2=g_{\Xi\eta}^2=\tfrac{3}{4}g^2\approx 11,$$
$$g_{\Sigma\pi}^2=g_{\Xi\pi}^2=g_{\Sigma K}^2=\tfrac{1}{4}g^2\approx 4,$$
$$h_{\Lambda K}^2=g_{N\eta}^2=0.$$

These values appear to be consistent with the experimental data now available.

In conclusion we would like to thank Dr. J. Uretsky and Professor J. J. Sakurai for several helpful discussions.

APPENDIX I

Octet-Model Coupling Constants

In order to establish the notation for coupling constants used in this paper, we write the interaction Lagrangian density for baryons and pseudoscalar mesons in the form

$$\mathfrak{L}_{\text{int}}=(4\pi)^{1/2}[g_{N\pi}\boldsymbol{\pi}\cdot\bar{N}\boldsymbol{\tau}N+g_{\Lambda\pi}(\boldsymbol{\pi}\cdot\bar{\Lambda}\boldsymbol{\Sigma}+\text{H.c.})-ig_{\Sigma\pi}\boldsymbol{\pi}\cdot\bar{\boldsymbol{\Sigma}}\times\boldsymbol{\Sigma}+g_{\Xi\pi}\boldsymbol{\pi}\cdot\bar{\Xi}\boldsymbol{\tau}\Xi+g_{\Lambda K}(\bar{N}\Lambda K+\text{H.c.})+g_{\Sigma K}(\bar{N}\boldsymbol{\tau}\cdot\boldsymbol{\Sigma}K+\text{H.c.})$$
$$+h_{\Lambda K}(\bar{\Xi}\Lambda K^c+\text{H.c.})+h_{\Sigma K}(\bar{\Xi}\boldsymbol{\tau}\cdot\boldsymbol{\Sigma}K^c+\text{H.c.})+g_{N\eta}\eta\bar{N}N+g_{\Lambda\eta}\eta\bar{\Lambda}\Lambda+g_{\Sigma\eta}\eta\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\Sigma}+g_{\Xi\eta}\eta\bar{\Xi}\Xi],$$

where the space-time dependence of the interaction has been suppressed. The particle symbols refer to the field operators in the standard way and H.c. means Hermitian conjugate. The charge structure is contained in the conventional isotopic spin representation, where

$$K^c=-i\tau_2K^*=\begin{pmatrix}-\bar{K}^0\\ K^-\end{pmatrix}$$

is the isotopic spinor of the second kind for the $\bar{K}$ meson. It is convenient to include the factor of $(4\pi)^{1/2}$ in the definition of the coupling constants so that, for example, we have $g_{N\pi}^2\approx 15$ as the experimental value for the pion-nucleon coupling constant.

In the Gell-Mann–Ne'eman octet model for the strongly interacting particles, there are only two independent ways,[10] known as D- and F-type coupling, in which to write Yukawa couplings which are invariant under unitary-symmetry transformations. It follows that the most general form for these couplings is an arbitrary linear combination of the D- and F-type expressions. If we let dg be the coefficient of the D-type form, and fg that for the F-type form, then the resulting expression for the pion-nucleon coupling constant is $g_{N\pi}=(d+f)g$. Without loss of generality we may choose $g=(15)^{1/2}$, in which case experiment restricts the coefficients to $(d+f)=1$.

The combination, therefore, of the known value of $g_{N\pi}^2$ and the octet-model assumption limits the twelve Yukawa-type coupling constants to a one parameter set of expressions. Setting $d=1-f$,[29] we obtain the following representation for the baryon-pseudoscalar meson coupling constants:

$$\begin{array}{lll}
g_{N\pi}=g, & g_{\Lambda K}=-(1/\sqrt{3})(1+2f)g, & g_{N\eta}=-(1/\sqrt{3})(1-4f)g,\\
g_{\Lambda\pi}=(2/\sqrt{3})(1-f)g, & g_{\Sigma K}=(1-2f)g, & g_{\Lambda\eta}=-(2/\sqrt{3})(1-f)g,\\
g_{\Sigma\pi}=2fg, & h_{\Lambda K}=-(1/\sqrt{3})(1-4f)g, & g_{\Sigma\eta}=(2/\sqrt{3})(1-f)g,\\
g_{\Xi\pi}=-(1-2f)g, & h_{\Sigma K}=-g, & g_{\Xi\eta}=-(1/\sqrt{3})(1+2f)g.
\end{array}$$

APPENDIX II

Isotopic Spin Factors

The isotopic spin structure given in the interaction Lagrangian density of Appendix I allows one to determine easily the appropriate coefficients for the Born approximation terms arising from the single baryon exchange diagram (Fig. 2). We list the isotopic spin factors for each strangeness and total isotopic spin state, using the notation (BC,A,DE) to represent the particles involved in each diagram (cf., footnote 14). The symmetry of the matrices permits us to suppress half of the off-diagonal terms in each case. Finally, for the single channel cases, we use the notation (BC,BC) to represent the total isotopic spin amplitude for the Born approximation contribution.

[29] It should be noted that the parameter f is related to α in Gell-Mann's notation (reference 9) by $f=1-\alpha$.

The results are as follows:

$S=1,\ T=0$ $$(NK,NK)=3(NK,\Sigma,NK)-(NK,\Lambda,NK),$$

$S=1,\ T=1$ $$(NK,NK)=(NK,\Sigma,NK)+(NK,\Lambda,NK),$$

$S=0,\ T=\frac{1}{2}$
$$\begin{array}{c} \\ N\pi \\ \Lambda K \\ \Sigma K \\ N\eta \end{array}\begin{array}{cccc} N\pi & \Lambda K & \Sigma K & N\eta \\ \end{array}$$
$$\begin{array}{c|cccc} & N\pi & \Lambda K & \Sigma K & N\eta \\ \hline N\pi & -(N\pi,N,N\pi) & \sqrt{3}(N\pi,\Sigma,\Lambda K) & 2(N\pi,\Sigma,\Sigma K)+(N\pi,\Lambda,\Sigma K) & \sqrt{3}(N\pi,N,N\eta) \\ \Lambda K & & (\Lambda K,\Xi,\Lambda K) & -\sqrt{3}(\Lambda K,\Xi,\Sigma K) & (\Lambda K,\Lambda,N\eta) \\ \Sigma K & & & -(\Sigma K,\Xi,\Sigma K) & \sqrt{3}(\Sigma K,\Sigma,N\eta) \\ N\eta & & & & (N\eta,N,N\eta) \end{array},$$

$S=0,\ T=\frac{3}{2}$
$$\begin{array}{c|cc} & N\pi & \Sigma K \\ \hline N\pi & 2(N\pi,N,N\pi) & (N\pi,\Lambda,\Sigma K)-(N\pi,\Sigma,\Sigma K) \\ \Sigma K & & 2(\Sigma K,\Xi,\Sigma K) \end{array},$$

$S=-1,\ T=0$
$$\begin{array}{c|cccc} & N\bar{K} & \Sigma\pi & \Xi K & \Lambda\eta \\ \hline N\bar{K} & 0 & \sqrt{6}(N\bar{K},N,\Sigma\pi) & (N\bar{K},\Lambda,\Xi K)-3(N\bar{K},\Sigma,\Xi K) & \sqrt{2}(N\bar{K},N,\Lambda\eta) \\ \Sigma\pi & & (\Sigma\pi,\Lambda,\Sigma\pi)-2(\Sigma\pi,\Sigma,\Sigma\pi) & \sqrt{6}(\Sigma\pi,\Xi,\Xi K) & \sqrt{3}(\Sigma\pi,\Sigma,\Lambda\eta) \\ \Xi K & & & 0 & \sqrt{2}(\Xi K,\Xi,\Lambda\eta) \\ \Lambda\eta & & & & (\Lambda\eta,\Lambda,\Lambda\eta) \end{array},$$

$S=-1,\ T=1$
$$\begin{array}{c|ccccc} & N\bar{K} & \Sigma\pi & \Lambda\pi & \Xi K & \Sigma\eta \\ \hline N\bar{K} & 0 & 2(N\bar{K},N,\Sigma\pi) & \sqrt{2}(N\bar{K},N,\Lambda\pi) & -(N\bar{K},\Lambda,\Xi K)-(NK,\Sigma,\Xi K) & \sqrt{2}(N\bar{K},N,\Sigma\eta) \\ \Sigma\pi & & (\Sigma\pi,\Sigma,\Sigma\pi)-(\Sigma\pi,\Lambda,\Sigma\pi) & -\sqrt{2}(\Sigma\pi,\Sigma,\Lambda\pi) & 2(\Sigma\pi,\Xi,\Xi K) & \sqrt{2}(\Sigma\pi,\Sigma,\Sigma\eta) \\ \Lambda\pi & & & (\Lambda\pi,\Sigma,\Lambda\pi) & \sqrt{2}(\Lambda\pi,\Xi,\Xi K) & (\Lambda\pi,\Lambda,\Sigma\eta) \\ \Xi K & & & & 0 & \sqrt{2}(\Xi K,\Xi,\Sigma\eta) \\ \Sigma\eta & & & & & (\Sigma\eta,\Sigma,\Sigma\eta) \end{array},$$

$S=-1,\ T=2$ $$(\Sigma\pi,\Sigma\pi)=(\Sigma\pi,\Lambda,\Sigma\pi)+(\Sigma\pi,\Sigma,\Sigma\pi),$$

$S=-2,\ T=\frac{1}{2}$
$$\begin{array}{c|cccc} & \Xi\pi & \Lambda\bar{K} & \Sigma\bar{K} & \Xi\eta \\ \hline \Xi\pi & -(\Xi\pi,\Xi,\Xi\pi) & \sqrt{3}(\Xi\pi,\Sigma,\Lambda\bar{K}) & 2(\Xi\pi,\Sigma,\Sigma\bar{K})+(\Xi\pi,\Lambda,\Sigma\bar{K}) & \sqrt{3}(\Xi\pi,\Xi,\Xi\eta) \\ \Lambda\bar{K} & & (\Lambda\bar{K},N,\Lambda\bar{K}) & -\sqrt{3}(\Lambda\bar{K},N,\Sigma\bar{K}) & (\Lambda\bar{K},\Lambda,\Xi\eta) \\ \Sigma\bar{K} & & & -(\Sigma\bar{K},N,\Sigma\bar{K}) & \sqrt{3}(\Sigma\bar{K},\Sigma,\Xi\eta) \\ \Xi\eta & & & & (\Xi\eta,\Xi,\Xi\eta) \end{array},$$

$S=-2,\ T=\frac{3}{2}$
$$\begin{array}{c|cc} & \Xi\pi & \Sigma\bar{K} \\ \hline \Xi\pi & 2(\Xi\pi,\Xi,\Xi\pi) & (\Xi\pi,\Lambda,\Sigma\bar{K})-(\Xi\pi,\Sigma,\Sigma\bar{K}) \\ \Sigma\bar{K} & & 2(\Sigma\bar{K},N,\Sigma\bar{K}) \end{array},$$

$S=-3,\ T=0$ $$(\Xi\bar{K},\Xi\bar{K})=3(\Xi\bar{K},\Sigma,\Xi\bar{K})-(\Xi\bar{K},\Lambda,\Xi\bar{K}),$$

$S=-3,\ T=1$ $$(\Xi\bar{K},\Xi\bar{K})=(\Xi\bar{K},\Sigma,\Xi\bar{K})+(\Xi\bar{K},\Lambda,\Xi\bar{K}).$$