Spontaneous Breakdown of Octet Symmetry*

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A model of strong interactions with the octet symmetry of Gell-Mann and Ne'eman is considered, and a spontaneous breakdown of this symmetry leading to nondegenerate baryon masses is sought. The Gell-Mann mass formula is deduced for the physically relevant symmetry-breaking solutions.

INTRODUCTION

CONSIDER the possibility that octet symmetry[1] exactly characterizes strong interactions—that there are no medium-strong symmetry-breaking interactions. Observed departures from octet symmetry are attributed to the dynamic instability of fully symmetric solutions of the quantum field theory. Baker and Glashow[2] showed the possibility for such a spontaneous symmetry breakdown in the symmetric Sakata model.[3] They showed that solutions could exist which retain only the reduced symmetries of isospin and hypercharge conservation. Applying related considerations to a model with octet symmetry, we find that there may be solutions with only these reduced symmetries, but that a mass sum rule must be satisfied in the approximation where the mass splittings are small compared to the cutoff. If the solutions are required to violate R symmetry (thus, to break the $N-\Xi$ degeneracy), this sum rule is the Gell-Mann mass formula.

I. MEANING OF THE MASS FORMULA

Gell-Mann's formula relates the masses of the eight baryons,

$$\tfrac{1}{2}m_N+\tfrac{1}{2}m_\Xi=\tfrac{3}{4}m_\Lambda+\tfrac{1}{4}m_\Sigma, \qquad (1.1)$$

and the squared masses of the eight pseudoscalar mesons,

$$\mu_K{}^2=\tfrac{3}{4}\mu_\chi{}^2+\tfrac{1}{4}\mu_\pi{}^2. \qquad (1.2)$$

Both formulas are well satisfied—to 0.5% for the baryons, and to 2% in mass for the mesons. Okubo[4] generalized Gell-Mann's formula for any irreducible unitary multiplet, obtaining

$$m=a+bY+c[T(T+1)-\tfrac{1}{4}Y^2] \text{ for fermions,} \qquad (1.3)$$

$$\mu^2=\alpha+\beta[T(T+1)-\tfrac{1}{4}Y^2] \text{ for bosons.} \qquad (1.4)$$

These relations are equivalent to the physical mass Lagrangian (or, the inverse noninteracting renormalized Green's function) having transformation properties under SU_3 of the superposition of a unitary singlet and a $\mathbf{T}=0$, $Y=0$ member of a unitary octet. Because the reduction of a direct product of an irreducible representation of SU_3 with its adjoint always contains a singlet just once and an octet at most twice, it follows that the masses within such a multiplet may be expressed [as in (1.3)] in terms of no more than three parameters. For some multiplets like the 10, the decomposition yields but one octet and the mass formula simplifies to[5]

$$m=a+bY.$$

Gell-Mann got his mass formula by introducing simple symmetry-breaking interactions, e.g.,

$$g\phi \textstyle\sum_{ij} \bar{\psi}_i(\delta_{ij}+\epsilon O_{ij})\psi_j,$$

where ϕ is a scalar unitary singlet meson, and $\mathbf{O}$ transforms like the $\mathbf{T}=0$, $Y=0$ member of a unitary octet (i.e., like the χ meson) and ϵ is a small parameter. Departures from baryon degeneracy to order ϵ satisfy (1.3), but the mass formula does not persist to order ϵ^2.

Here, we try to obtain split masses satisfying the mass formula without breaking the symmetry of the dynamics.

II. THE MODEL

We discuss eight baryons with four-fermion interactions invariant under SU_3, the baryons behaving like a unitary octet. Our considerations are otherwise independent of the form of the interaction. We regard mass as entirely dynamical in origin; therefore, we take the bare baryon masses as zero. Solutions to the quantum field theory depend parametrically only upon the square of the cutoff momentum Λ^2, and upon the dimensionless couplings $g\Lambda^2$ (g standing for the various symmetric four-fermion coupling strengths). In the approximations of reference 2, the mass operator is a constant, the masses being determined by coupled algebraic equations

$$m_i=\textstyle\sum_j g_{ij}m_jh(\Lambda^{-2}m_j{}^2), \qquad (2.1)$$

which results from putting the expression for the bare mass equal to zero. The g_{ij} are linear combinations of

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[4] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).
[5] M. Gell-Mann, in *Proceedings of the International Conference on High-Energy Physics at Geneva, 1962* (CERN, Geneva, 1962); S. L. Glashow, in International Summer School on Group Theoretical Methods in Elementary Particle Physics at Istanbul, 1962 (Gordon and Breach, London, to be published).

the various $g\Lambda^2$ and the function $h(x)$ depends on how the cutoff is introduced. We do not confine ourselves to this approximate expression, for the *exact* solution to the model is expected to yield an analogous result,

$$m_i=\sum_j \mathfrak{F}_{ij}(g\Lambda^2,(m/\Lambda)^2)m_j, \qquad (2.2)$$

in which $\mathfrak{F}_{ij}$ depends upon all the baryon masses and is not a linear function of $g\Lambda^2$. Invariance of the dynamics under SU_3 is bound up in the allowed structure of the functions $\mathfrak{F}_{ij}$.

Only in a special basis of SU_3 are the masses diagonal (i.e., does the choice of fields correspond to particles of definite mass). What is required in order to study the group-theoretic properties of (2.2) is to re-express these equations in a general basis, where they take the form of 8×8 matrix equations,

$$M_{ij}=\sum_{kl} \mathfrak{F}_{ij;kl}(g\Lambda^2,(\mathbf{M}/\Lambda)^2)M_{kl}, \qquad (2.3)$$

in terms of a Hermitian mass matrix whose eigenvalues are m_i. Let the 8×8 matrix U_{ij} represent an element of SU_3. Under this transformation,

$$M_{ij}\rightarrow M_{ij}'=\sum_{ab} U_{ia}M_{ab}U_{bj}^\dagger, \qquad (2.4)$$

and symmetry of the dynamics under SU_3 requires that $\mathbf{M}'$ also satisfy (2.3).

Expand (2.3) in powers of Λ^{-2} about $\mathbf{M}=0$,

$$M_{ij}=\sum_{kl} F_{ij;kl}(g\Lambda^2)M_{kl}+O(\mathbf{MMM}\Lambda^{-2}). \qquad (2.5)$$

Invariance of (2.5) under (2.4) requires that $F_{ij;kl}$, and higher matrices in the expansion, are *invariant* tensorial operators, i.e.,

$$F_{ij;kl}=\sum_{abcd} U_{ia}U_{ld}F_{ab;cd}U_{bj}^\dagger U_{ck}^\dagger \qquad (2.6)$$

for any transformation $\mathbf{U}$ of SU_3. It may be shown that $\mathbf{F}$ has the general form,

$$\mathbf{F}=\lambda^{(1)}\mathbf{P}^{(1)}+\lambda^{(10)}\mathbf{P}^{(10)}+\lambda^{(\overline{10})}\mathbf{P}^{(\overline{10})}+\lambda^{(27)}\mathbf{P}^{(27)} +\lambda^{(D)}\mathbf{P}^{(D)}+\lambda^{(F)}\mathbf{P}^{(F)}+\eta\mathbf{N}+\eta'\mathbf{N}^\dagger, \qquad (2.7)$$

where the eight parameters λ and η are real functions of $g\Lambda^2$, and the $\mathbf{P}^{(\alpha)}$ are projection operators for the various irreducible families of 8×8 matrices. $\mathbf{P}^{(D)}$ projects onto the completely symmetric 8_D, while $\mathbf{P}^{(F)}$ projects onto the completely antisymmetric 8_F. Because the two eights are equivalent, the operator $\mathbf{N}$ mapping 8_D onto corresponding members of 8_F (and also its adjoint $\mathbf{N}^\dagger$, mapping 8_F onto 8_D) also appears.

All members of the 27 are, thus, degenerate eigenmatrices of $\mathbf{F}$ with eigenvalue $\lambda^{(27)}$, and analogously for the members of the 10, the $\overline{10}$, and the singlet. Only if R symmetry, as well as unitary symmetry, characterizes the dynamics is the situation for the eights so simple. Then, $\eta=\eta'=0$, and the 8_D (8_F) are eigenmatrices of $\mathbf{F}$ belonging to $\lambda^{(D)}(\lambda^{(F)})$. In general, the supermatrix referring to the two eights may be triangularized, and the existence is assured of at least one family of eight matrices, irreducible under SU_3, whose members are degenerate eigenmatrices of $\mathbf{F}$. By group theory alone, we cannot further determine these eigenmatrices, for the result depends upon the detailed dynamics.

III. THE MASS FORMULA

In the basis where M_{ij} is diagonal, (2.5) becomes

$$m_i=\sum_j f_{ij}(g\Lambda^2)m_j+O(m^3\Lambda^{-2}). \qquad (3.1)$$

Define the generators both of hypercharge Y, and of electrical charge $T_3+\frac{1}{2}Y$, so that they are diagonal in this basis. It is only a matter of convention that no breakdown of these conservation laws results from the asymmetry of the masses. This is not so for the total isospin, but we may look for solutions to (3.1) which are at least approximately invariant under the isospin subgroup, thus ignoring the possible existence of other solutions in which isospin is grossly violated. There are just four diagonal 8×8 matrices giving masses compatible with isospin and hypercharge conservation (accommodating the four isotopic submultiplets within the unitary octet): the unit matrix, two matrices with octet transformation properties, and one member of the 27-plet. We denote these diagonal 8×8 matrices by four-dimensional vectors

$$\mathbf{u}=(u_1,u_2,u_3,u_4), \qquad (3.2)$$

whose entries refer, respectively, to N, Λ, Σ, and Ξ, and the isotopic multiplicities are implicit so that the norm is defined by

$$|\mathbf{u}|^2=2u_1^2+u_2^2+3u_3^2+2u_4^2. \qquad (3.3)$$

To the unitary singlet corresponds the normalized vector

$$\mathbf{u}^{(1)}=8^{-1/2}(1,1,1,1), \qquad (3.4)$$

and to the $\mathbf{T}=0$, $Y=0$ member of the 27-plet corresponds

$$\mathbf{u}^{(27)}=(3/40)^{1/2}(1,\,-3,\,-1/3,\,1). \qquad (3.5)$$

The remaining two-dimensional subspace normal to $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(27)}$ has octet transformation properties.

From (3.1), with the neglect of nonlinear terms in masses, it follows that m_i must be an eigenvector of f_{ij} belonging to eigenvalue one. This requirement determines the coupling strength g. Barring an accidental degeneracy between $\lambda^{(1)}$ and either $\lambda^{(8)}$ or $\lambda^{(27)}$, the only physically admissible solution in this approximation is $\mathbf{m}\sim\mathbf{u}^{(1)}$, corresponding to complete degeneracy, for the other eigenvectors of f_{ij} have negative entries. Only when nonlinearities are included do we obtain an equation determining the value of the degenerate mass. We conclude that to order $(m/\Lambda)^2$ the only meaningful solutions to (3.1) are completely degenerage.

A more satisfactory result is obtained from the expansion of (2.2) about the mean baryon mass $\bar{m}$. We find

$$\bar{m}+\delta_i=h(g\Lambda^2,(\bar{m}/\Lambda)^2)\bar{m}+\sum_j \tilde{f}_{ij}(g\Lambda^2,(\bar{m}/\Lambda)^2)\delta_j +O(\bar{m}\delta^2\Lambda^{-2}), \qquad (3.6)$$

where $\bar{m}=m_i-\delta_i=\frac{1}{8}(2m_1+m_2+3m_3+2m_4)$, and $\boldsymbol{\delta}$ is normal to $\mathbf{u}^{(1)}$ with the norm (3.3). The discussion of Sec. II applies equally well to $\tilde{f}_{ij}$, so that we conclude from the linearized approximation to (3.6) that

$$h(g\Lambda^2,(\bar{m}/\Lambda)^2)=1, \tag{3.7}$$

and that $\boldsymbol{\delta}$ must be an eigenvector of $\tilde{f}_{ij}$ normal to $\mathbf{u}^{(1)}$ belonging to eigenvalue one. These two requirements determine the mean mass $\bar{m}$ and the coupling strength g. There are three possibilities:

(i) $\boldsymbol{\delta}=0$ This gives the fully symmetric and supposedly unstable solution to (2.2).

(ii) $\boldsymbol{\delta}\sim\mathbf{u}^{(27)}$ For this type of solution, N and Ξ remain degenerate.

(iii) $\boldsymbol{\delta}\cdot\mathbf{u}^{(27)}=0$ In this case, $\boldsymbol{\delta}$ has octet transformation properties. Comparison with (3.3) and (3.5) gives the Gell-Mann mass rule.

This approximation [to all orders in $(\bar{m}/\Lambda)^2$, and linear in $\boldsymbol{\delta}$] does not determine the magnitude of $\boldsymbol{\delta}$, but from our earlier discussion we know that

$$\delta/\bar{m}=O(\bar{m}/\Lambda)^2. \tag{3.8}$$

Deviations from the mass rule are comparable to the neglected nonlinear terms of (3.4), and are of order $\bar{m}(\delta/\Lambda)^2$. Both the size of the observed mass splittings and of the deviations from the mass formula are compatible with a cutoff energy of several BeV. Of course, we have not demonstrated that spontaneous asymmetries exist, but only that when they do they must satisfy an approximate sum rule.

Electromagnetic mass splittings are of similar size to deviations from the mass formula. Perhaps violation of isospin, and even electromagnetism, is *already implicit* in the model we have considered, possibly along the lines discussed by Bjorken.[6]

We also remark on the derivation of the mass rule for the masses of mesons μ, or for the masses of multiplets of resonances m^*. Such derivative phenomena should satisfy *inhomogeneous* equations of the form

$$m_i^*=\sum_j \mathfrak{F}_{ij}^* m_j^*+\sum_k \mathfrak{F}_{ik}' m_k, \tag{3.9}$$

$$\mu_i^2=\Lambda^2\mathfrak{K}+\sum_j \mathfrak{K}_{ij}\mu_j^2+\sum_j \mathfrak{G}_{ij}m_j^2, \tag{3.10}$$

and an analysis similar to that of (2.2) shows that m_i^* and μ_i^2 satisfy analogous mass formulas.[7]

It must be emphasized that our approach depends hardly at all upon the use of a field-theoretic model. The starting point, Eq. (2.2), could equally well have arisen from reasoning akin to that of Zachariasen and Zemach,[8] wherein the eight nucleon masses (through their symmetric interactions) are required to support themselves self-consistently. In this case, no cutoff appears in the bootstrap equations analogous to Eq. (2.2).

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[6] J. Bjorken (to be published).

[7] Analogous, but not identical, sum rules characterize the m_i^* and the μ_i. Thus, if $m_i-\bar{m}$ has octet transformation properties, so must $m_i^*-\bar{m}^*$. For the 10-plet of $J=3/2^+$ resonances, this gives an equal-spacing rule. See reference 5 and S. L. Glashow and J. J. Sakurai, Nuovo Cimento **26**, 622 (1962).

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