of the parabola depend on the values of a and b, which in turn depend on m_W, ϵ, and x. A qualitative idea of the dependence of R on μ_W can be obtained from Fig. 2. The first curve represents a choice of ϵ and x in region (i) and the second in region (ii).

A similar argument can explain the variation of R with m_W shown in Table III. For fixed μ_W, ϵ, and x, the ratio of the differential decay rates, $R(\epsilon,x)=d^2\Gamma(m_W)/d^2\Gamma(\infty)$, is quadratic in the variable $z'=\omega(K-1+\omega)^{-1}$:

$$R(\epsilon,x)=1-(2-\mu_W)a'z'+(2-\mu_W)^2b'z'^2. \quad (17)$$

$R(\epsilon,x)$ has a minimum at $z'_{\min}=a'[2(2-\mu_W)b']^{-1}$. Depending on the value of μ_W, $z'_{\min}$ can lie inside or outside the allowed range for z', i.e., $0\leq z'\leq 1$. As μ_W becomes increasingly negative, $z'_{\min}$ moves into the interval $[0,1]$ from the right, so that when $\mu_W=-1$ the ratio R has a minimum near $m_W=1.2m_K$.

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Threshold Motion of Regge Poles*

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The motion of Regge poles as $E\to 0$ is examined in detail. It follows from the well-known threshold law of the S matrix that infinitely many poles approach $l=-\frac{1}{2}$, from the first and the third quadrants as $E\to 0+$, and from the second and third as $E\to 0-$. A possible way of including these poles in a representation of S is indicated.

IT is well known that for potentials that fall off sufficiently rapidly at infinity the S matrix for an angular momentum l has the behavior[1]

$$S=1+O(k^{2l+1}) \quad \text{as} \quad k\to 0.$$

If the S matrix is expressed in terms of a single Regge pole,[2] or a finite sum of Regge poles the threshold dependence is clearly not satisfied.[3] We want to point out an intimate connection between this threshold behavior and the infinitely many Regge trajectories that arrive at $l=-\frac{1}{2}$ as $E\to 0$. As $E\to 0+$, there are infinitely many poles which approach $l=-\frac{1}{2}$ from the upper right-half and the lower left-half of the l plane, the approach being essentially independent of the potential. As $E\to 0-$, the poles approach $l=-\frac{1}{2}$ in complex conjugate pairs from the left-half l plane. The present authors had indicated recently that $l=-\frac{1}{2}$ is the low-energy end point of infinitely many Regge trajectories.[4,5] In this article, we give further details about the threshold motion of the poles. We shall also indicate the possible way in which an S matrix can be expressed so as to correctly take into account its threshold properties.[6] We consider only nonrelativistic potential scattering but we believe these results, coming as they do from the threshold dependence, should also hold in the relativistic case. We also find another class of infinitely many Regge poles in the left-half l plane whose energy dependence we have derived. The behavior of these poles is found to be analogous to the right-hand Regge poles associated with bound states and resonances.

An S matrix unitary for real λ $(=l+\frac{1}{2})$ can be written near $E=0$ as[4,7]

$$S(\lambda,k)=\frac{1-k^{2\lambda}e^{i\pi\lambda}C(\lambda)}{1-k^{2\lambda}e^{-i\pi\lambda}C(\lambda)}, \quad (1)$$

where $C(\lambda)$ is a meromorphic function of λ[2,7] and is real for real λ.[8,9]

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[1] Here k is the momentum, $E=k^2$ is the energy.
[2] T. Regge, Nuvo Cimento **14**, 951 (1959).
[3] For instance, the threshold behavior of a single right-hand Regge pole is $S-1=O(k^{2\alpha(0)+1})$, where $\alpha(0)$ is the $k=0$ position of the pole and $-\frac{1}{2}<\alpha(0)<\frac{1}{2}$.
[4] B. R. Desai and R. G. Newton, Phys. Rev. **129**, 1445 (1963).
[5] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962), had indicated that there are conjugate poles approaching $l=-\frac{1}{2}$ as $E\to 0-$ in the relativistic case.
[6] Note that a Regge-pole representation of $(S-1)/k^{2l+1}$ satisfies threshold dependence of S but clearly does not get rid of the infinite number of poles at $l=-\frac{1}{2}$.
[7] R. G. Newton, J. Math. Phys. **3**, 867 (1962).
[8] This follows from the relation given in reference 7,

$$S(\lambda,k)e^{-2\pi i\lambda}+S^{-1}(\lambda,ke^{-i\pi})=1+e^{-2\pi i\lambda}.$$

[9] We have only written the first two dominant terms in k^2 for $\lambda<1$. In general, one has, both in the numerator and the denominator, terms of the form $a_1(\lambda)k^2+a_2(\lambda)k^4+\cdots+b_1(\lambda)k^{2\lambda+2}+\cdots$. For $\lambda>1$, the $k^{2\lambda}$ term should be replaced by the k^2 term.

In general, we have[10,11]

$$C(\lambda)=2^{-2\lambda}\frac{\Gamma(1-\lambda)}{\Gamma(1+\lambda)}\frac{\gamma\int_0^\infty dr\, r^{\frac{1}{2}+\lambda}\phi_0(\lambda,r)V(r)}{2\lambda+\gamma\int_0^\infty dr\, r^{\frac{1}{2}-\lambda}\phi_0(\lambda,r)V(r)},\qquad (2)$$

where ϕ_0 is the zero-energy wave function for angular momentum $l=\lambda-\frac{1}{2}$ which at $r=0$ satisfies the boundary condition

$$\lim_{r\to 0} r^{-\frac{1}{2}-\lambda}\phi_0(\lambda,r)=1$$

and γ is the potential strength.

The poles of the S-matrix are given by

$$1-k^{2\lambda}e^{-i\pi\lambda}C(\lambda)=0 \quad \text{as} \quad E\to 0+,$$
$$1-|k|^{2\lambda}C(\lambda)=0 \quad \text{as} \quad E\to 0-.$$

The properties of $C(\lambda)$ are important in determining the motion of the poles. From (2) we note that $C(0)=1$.[10] Expanding $C^{-1}(\lambda)$ near $\lambda=0$, we obtain, as $E\to 0+$,

$$\exp\lambda(-|\ln E|-i\pi)=1+A\lambda,\qquad (3)$$

where $A=-C'(0)$ and is $\sim\gamma^{-1}$ for small γ. Consequently, the zeros $\lambda=\lambda_r+i\lambda_i$ are determined by the equations

$$\tan(\lambda_i|\ln E|+\lambda_r\pi)=-\lambda_iA/(1+\lambda_rA)$$

and

$$\exp2(-\lambda_r|\ln E|+\lambda_i\pi)=1+2\lambda_rA+(\lambda_r^2+\lambda_i^2)A^2\qquad (4)$$

which, as $E\to 0+$, have solutions of the form

$$\lambda_i\simeq 2n\pi/|\ln E|,\quad \lambda_r\simeq\pi\lambda_i/|\ln E|=2n\pi^2/|\ln E|^2,$$

or

$$\lambda^{(n)}=(2n\pi/|\ln E|)e^{i\phi_n},\quad \tan\phi_n=|\ln E|/\pi,\qquad (5)$$

where $n=\pm1,\ \pm2,\ \ldots,$ with $|n|\ll|\ln E|/2\pi$. As E decreases we have more and more solutions of the form (5). For a fixed small energy E the poles of $S(\lambda,k)$ lie evenly spaced on a ray from the point $\lambda=0$ which makes an angle $\pi/|\ln E|$ with the imaginary axis. The poles lie both in the upper right-half as well as the lower left-half of the λ plane. As $E\to 0+$ the ray turns counterclockwise while the poles move along the ray toward $\lambda=0$; infinitely many poles arrive at $\lambda=0$ when $E=0$. The approach of the poles is independent of the potential.[12] As $E\to\infty$ they will end up either at negative half-integral values λ, or at infinity, or in the region of nonanalyticity of $S(\lambda,k)$.[7] We now take $E\to 0-$. In that case the Regge poles near $\lambda=0$ are the solutions of

$$\tan[\lambda_i|\ln(-E)|]=-\lambda_iA/(1+\lambda_rA)$$

and

$$\exp[-2\lambda_r|\ln(-E)|]=1+2\lambda_rA+(\lambda_r^2+\lambda_i^2)A^2,\qquad (6)$$

that is,

$$\lambda_i\simeq 2n\pi/|\ln(-E)|,\quad \lambda_r\simeq -A^2\lambda_i^2/2|\ln(-E)| = -2A^2n^2\pi^2/|\ln(-E)|^3$$

or

$$\lambda^{(n)}=[2n\pi/|\ln(-E)|]e^{i\phi_n},\quad \tan\phi_n=-|\ln(-E)|^2/n\pi A^2,\qquad (7)$$

where $n=\pm1,\ \pm2,\ \cdots,$ with $|n|\ll|\ln(-E)|/2\pi$.[12]

Therefore, as $E\to 0-$ these poles of $S(\lambda,k)$ in the left-half-plane are complex and, as they must, occur in complex conjugate pairs.[4] Their phases depend on the potential. The existence of an infinity of $E=0$ poles at $l=-\frac{1}{2}$ and the complexity for $E\to 0-$ were both indicated in reference 4.

In any reliable low-energy representation of an S matrix (or a scattering amplitude) in terms of Regge poles the above threshold-poles should be taken into account.[13] It was shown by the authors that it is possible to write the product representation[4]

$$\bar{f}(\lambda,-k)=\bar{f}(0,-k)e^{a(k)\lambda}\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n(k)}\right)e^{\lambda/\lambda_n(k)},$$

where $\bar{f}=f/\Gamma(\frac{1}{2}+\lambda)$ is an entire function,[14] and $\lambda_n(k)$ are the zeros of f. Any convenient point can be chosen for normalization instead of $\lambda=0$ chosen here. We find that for $E\to 0+$ the product involving the zeros near $\lambda=0$ can be written compactly in the form

$$\sin[\pi\lambda/p(k)]/[\pi\lambda/p(k)],$$

[10] For an amplitude satisfying a double dispersion relation, plus unitarity, in the sense of S. Mandelstam, Ann. Phys. **21**, 302 (1963), we obtain

$$C(\lambda)=\int_a^\infty dk\, g(\lambda,ik)k^{-2\lambda}f(\lambda,-ik)\Bigg/\left[i\pi\sin\pi\lambda+\int_a^\infty dk\, g(\lambda,ik)f(\lambda,-ik)\right],$$

where f is the Jost function (i.e., the denominator function), g is essentially the partial wave projection of the absorptive part in the momentum transfer variable, and a the reciprocal of the range of interaction. Note that $C(0)=1$.

[11] In the expression (5.1) of reference 4, $C(0,\gamma)$ should be replaced by a function $D(\lambda,\gamma)$ where $D(0,\gamma)=C(0,\gamma)$. We will then have $C(\lambda)=C(\lambda,\gamma)/[D(\lambda,\gamma)-\lambda]$.

[12] In (5) the first iteration actually gives

$$\lambda_i=(2n\pi/|\ln E|)(1-A/|\ln E|),$$

and depends very weakly on A. For small γ, A is very large and, therefore, smaller energies are required to bring the poles closer to $\lambda=0$. This is understandable since for weak potentials the poles would like to remain in their Born approximation positions which are farther away from $\lambda=0$.

[12] If we add a $B\lambda^2$ term in the expansion of $C^{-1}(\lambda)$, then in (7) A^2 should be replaced by A^2-2B, i.e., we obtain $\lambda_r\simeq-\lambda_i^2(A^2-2B)/2\times|\ln(-E)|$ (we are grateful to Dr. A. Bincer for pointing this out to us). As there cannot be any complex conjugate poles in the right-half-plane (see references 2 and 7); therefore, A^2 must be $\geqslant 2B$.

[13] It is worthwhile to remark, however, that because of the $(\ln E)^{-1}$ dependence, the threshold-poles move very rapidly near $\lambda=0$. For weak Yukawa potentials, it is found that as E is increased, the threshold-poles in the right-half-plane quickly turn around and move into the left-half-plane for extremely small values of E [W. Carnahan (private communication)].

[14] If we assume all derivatives of $rV(r)$ to exist at $r=0$, then according to reference 7, $\bar{f}$ is an entire function.

where $p(k)=\lambda^{(n)}(k)/n$, $\lambda^{(n)}(k)$ being defined in (5). A procedure which approximately takes into account the poles near $\lambda=0$ and a few right-hand Regge poles will be presented later.

We now turn to the $E=0$ poles in the left-half λ plane. Clearly these will come from the zeros of $C(\lambda)$. A necessary condition for their existence is, therefore, $C(\lambda)=0$. In general, this will have an infinitely many solutions real as well as complex. If $\lambda_0(<0)$ is a zero of $C(\lambda)$, we expand

$$C(\lambda)=(\lambda-\lambda_0)C'(\lambda_0)+\cdots.$$

The position λ_0 will, obviously, depend on the potential. As $E\rightarrow 0+$ the corresponding Regge poles are

$$\lambda(k)=\lambda_0+Be^{i\pi\lambda_0}k^{-2\lambda_0}, \tag{8}$$

where

$$B=C'(\lambda_0)^{-1}.$$

Since B can have either sign, λ may approach λ_0 from above or below from left or right. As $E\rightarrow 0-$ we have similarly

$$\lambda(k)=\lambda_0+B|k|^{-2\lambda_0}. \tag{9}$$

This means that a real λ_0 may be approached from either side, but always along the real axis. Furthermore, for real λ_0 the two equations (8) and (9) imply a universal relation (depending only on λ_0) between the direction of approach as $E\rightarrow 0-$ and $E\rightarrow 0+$. If λ_0 is complex, then the pole spirals in towards it, both as $E\rightarrow 0+$ and as $E\rightarrow 0-$.

The Regge poles mentioned above are the analog of the poles in the right-half-plane associated with bound states and resonances. These right-hand poles are obtained directly from the poles $\lambda_0(>0)$ of $C(\lambda)$, and are given for $\lambda_0<1$ by[7,15]

$$\begin{aligned}\lambda(k)&=\lambda_0-be^{-i\pi\lambda_0}k^{2\lambda_0} \quad \text{as} \quad E\rightarrow 0+,\\ &=\lambda_0-b|k|^{2\lambda_0} \quad \text{as} \quad E\rightarrow 0-.\end{aligned} \tag{10}$$

Here it is known that λ_0 is real and that b is positive.[2,7] λ_0, of course, depends on the potential and there is at least one pole if the potential is attractive.[7] For $\lambda_0>1$, the above formula contains a term proportional to k^2,[9] i.e., for all $\lambda_0>1$, the real part of λ is given for real $\bar{b}$ by

$$\begin{aligned}\lambda(k)&=\lambda_0+\bar{b}k^2 \quad \text{as} \quad E\rightarrow 0+,\\ &=\lambda_0-\bar{b}|k|^2 \quad \text{as} \quad E\rightarrow 0-,\end{aligned} \tag{11}$$

while the imaginary part is still given by (10).

Notice that if $\lambda_0<1$, the slope $\lambda'(0)(=d\lambda/dk^2$ at $k^2=0)$ is infinite.[7,16] It was remarked earlier that a single Regge pole does not give the correct threshold behavior. This also means that it does not necessarily give a finite value to $\lambda'(0)$, the "effective range." To overcome these difficulties, therefore, one should include in any Regge-pole representation of the S matrix the infinitely many poles near $\lambda=0$ mentioned earlier.

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[15] This formula has also been derived by A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

[16] In determining high-energy cross sections it is usually asserted that λ' is finite and furthermore that it is of the order of the radius of interaction [see G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962)]. This is clearly incorrect if at $E=0$, λ happens to be <1 (or $\alpha=\lambda-\frac{1}{2}<\frac{1}{2}$). In other words, it may very well happen that some of the conjectured Regge poles arrive at $E=0$ (through positive values of E) at a point below $\alpha=\frac{1}{2}$ and because of the infinite slope there, enter the physical region of the crossed channel with a considerably reduced value.