neutron total cross-section measurements that the level spacings are large.[14] Thus, the energy resolution of this experiment may cover only a few levels. A low density of the higher spin states in the compound nucleus at an excitation energy of ~10.9 MeV could very easily explain the minimum observed there.

[14] E. G. Bilpuch, K. K. Seth, C. D. Bowman, R. H. Tabony, R. C. Smith, and H. W. Newson, Ann. Phys. (N. Y.) **14**, 387 (1961).

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Neutron-Proton Scattering below 20 MeV*

H. Pierre Noyes
Stanford Linear Accelerator Center, Stanford University, Stanford, California
(Received 18 January 1963)

Critical examination and analysis of existing *n-p* scattering data below 20 MeV reveal that they provide quantitative information only about the *S*-wave scattering lengths and effective ranges, which are found to be $a_t=5.396\pm0.011$ F; $a_s=-23.678\pm0.028$ F; $r_t=1.726\pm0.014$ F; $r_s=2.51\pm0.11\pm0.043$ F; where the second error quoted for r_s is a conservative estimate of the uncertainty due to departures from the shape-independent approximation. The correlations in error are $\langle\delta a_t\delta a_s\rangle=-0.7828\delta a_t\delta a_s$; $\langle\delta a_t\delta r_s\rangle=-0.8547\delta a_t\delta r_s$; $\langle\delta a_s\delta r_s\rangle=0.7029\delta a_s\delta r_s$. An estimate of the contribution to the total cross section from scattering in higher angular momentum states, based on model calculations, *p-p* phases, and the $\cos\theta$ term in the differential cross section, allows the deviation from the shape-independent approximation to be computed at 14.1 and 19.665 MeV from total cross-section measurements. It is shown on theoretical grounds that this must come almost entirely from the $^1S_0$ state, and extreme limits to this variation are established. The value found is close to zero at both energies, in accord with theoretical expectations, but the uncertainty is so large that it barely excludes the extreme limits. Some qualitative evidence for or against the existence of the long-range one pion exchange interaction in this state could be obtained by improving the experiments below 5 MeV, but the uncertainty arising from the non-*S* wave scattering precludes any but qualitative results. It is shown that this uncertainty cannot be removed by improved measurement of the differential cross section because 8 independent pieces of experimental information are required. We conclude that the energy variation of the *S* waves below 20 MeV cannot be measured without recourse to experiments which separate the spin states of the particles, such as spin-correlation, triple scattering, polarized-beam polarized-target, etc. If some information is taken from *p-p* scattering and some from theory, a single such measurement in each system might suffice; this minimal program is briefly discussed.

I. INTRODUCTION

ALTHOUGH the neutron-proton interaction has been the subject of intensive experimental and theoretical study since the discovery of the neutron in 1932, and was correctly interpreted by Yukawa as due to the exchange of quanta of finite mass in 1935, until very recently there has been no basic theoretical model capable of accounting for all the qualitative features revealed by the experimental investigations. The discovery of two- and three-pion resonances showed immediately[1,2] that at least an important part of the problem could be understood, and connected with earlier speculations about "vector mesons."[3–5] It had already been conclusively demonstrated[6] that the long-range part of the interaction in high-angular momentum states is quantitatively described by the exchange of single pions. The ω, and to a lesser extent the ρ, account for the strong short-range repulsion in the nucleon-nucleon system, the spin-orbit interaction, and the strong short-range attraction in the nucleon-antinucleon system. If the $^1S_0$ scattering length is fitted, single-pion exchange is too weak to account for the effective range[7,8] even in the absence of a short-range repulsion, so something must give a strong attraction

* Work performed under the auspices of the U. S. Atomic Energy Commission, in part, while the author was at the Lawrence Radiation Laboratory, Livermore.

[1] H. P. Noyes, in *Proceedings of the Rutherford Jubilee Conference, Manchester, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1962), p. 749.

[2] G. Breit, see reference 1, p. 756.

[3] G. Breit, Phys. Rev. **51**, 248 (1936); S. Share and G. Breit, *ibid.* **52**, 546 (1937); G. Breit, *ibid.* **53**, 153 (1938); G. Breit and J. R. Stehn, *ibid.* **53**, 459 (1938); G. Breit, Proc. Natl. Acad. Sci. U. S. **46**, 746 (1960); Phys. Rev. **120**, 287 (1960).

[4] Y. Nambu, Phys. Rev. **106**, 1366 (1957).

[5] J. J. Sakurai, Phys. Rev. **119**, 1784 (1960); Ann. Phys. (N. Y.) **11**, 1 (1960).

[6] For a review of this evidence and references to earlier work cf. M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. **11**, 95 (1961).

[7] J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Progr. Theoret. Phys. (Kyoto) **15**, 86 (1956); **16**, 472 (1956).

[8] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959).

in this state. Whether the ABC phenomenon[9] is due to a strong $I=0$ S-wave pion-pion interaction, or is actually a resonance at a value somewhat above threshold,[10] it would act in the nuclear-force problem like the exchange of an $I=0$ scalar meson and provide this attraction; other resonance phenomena in this mass range (η,ζ) could either strengthen or weaken this attraction, depending on their quantum numbers, but we know from the $^1S_0$ parameters that the over-all effect must be attractive. We conclude that a minimal description of the nucleon-nucleon interaction must contain the exchange of the pion, of an $I=0$ scalar meson with a mass somewhat greater than two pion masses, and of two ($I=0$ and $I=1$) vector mesons with about 5 pion masses. Such models have been shown by several authors[11–13] to give all the qualitative features found in n-p and p-p scattering below 350 MeV in the approximation which interprets the single-particle exchange terms as the Fourier transform of a potential. This agreement with experiment is improved if the interaction is described by a relativistic formalism (which necessitates that the ρ and ω be treated as Regge poles rather than as particles with a discrete mass and angular momentum); then only 9 parameters, five of which can already be roughly estimated from other phenomena, are needed to make this agreement nearly quantitative over the entire energy range.[14]

If, as this author believes, this signal success is due to the fact that the most important physical phenomena responsible for the two-nucleon interaction have finally been isolated and partially understood, and not just a misleading accident, future work on the two-nucleon problem will differ radically from the generally frustrating confusion which has characterized this field in the past.[15] For one thing, there will now be considerably more incentive for including the mesonic degrees of freedom in the study of nuclear matter, and some hope of success. As noted by Teller[16] some time ago, the fact that the spin-flip isospin-flip one-pion exchange is forbidden to first order by the Pauli principle in nuclear matter, implies that the dominant interactions will be due to the scalar and vector meson exchanges we discussed above; consequently, Duerr's[17] interpretation of the Teller-Johnson model[18] has finally been connected with elementary-particle physics in a qualitative way, and pursuit of the quantitative connections might prove revealing. A second area where work will now go forward is the determination of the parameters of the resonances from nucleon-nucleon scattering data, and calculation, or at least estimation, of the nonresonant background.[19] Unfortunately it appears unlikely at present that these parameters can be computed from pion-nucleon or pion-pion scattering to the accuracy required for a quantitative fit to the nucleon-nucleon data; consequently this work will provide a consistency check rather than a quantitative test of the theory. To make quantitative tests of the model it will be necessary to tie down the short-range parts of the interaction (coming from kaon, hyperon, n-pion, $\cdots$, exchanges) by phenomenological parameters determined at low energy, and test the theory by comparing the energy variation of the scattering amplitudes predicted by the longer range parts of the interaction with experiment. Unfortunately this requires more parameters than the S-wave scattering lengths. Breit and Hull[20] have shown that centrifugal shielding of the P waves is incomplete, and consequently that the $^3P_{0,1,2}$ and $^1P_1$ phase shifts cannot be accurately computed from a knowledge of the long-range part of the interaction at *any* energy; we, therefore, will require four P-wave scattering lengths to be determined from experiment. Because of the strong tensor force, the $^3S_1-{}^3D_1$ coupling parameter[21] ϵ^1 and $^3D_1$ phase shift $\delta_{2,1}$ will also be influenced by the short-range part of the interaction in the $^3S_1$ state, and we will need two empirical constants for these states. We conclude that in order to utilize scattering data at high energy for quantitative tests of the theory of the n-p interaction it is first necessary to determine 8 constants from low-energy scattering experiments. The remainder of this paper is devoted to the study of what constants can be determined from existing experiments, and what additional experiments might be required for this purpose.

II. THE SHAPE-INDEPENDENT APPROXIMATION

Since n-p scattering below 20 MeV is dominated by the two S waves, our first concern will be to isolate

[9] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **7**, 35 (1961).

[10] B. Richter, Phys. Rev. Letters **9**, 217 (1962). R. Del Fabbro, M. De Pretis, G. Marini, A. Odian, G. Stoppini, L. Tau, and R. Visentin, report at the 1962 Annual International Conference on High Energy Physics at CERN (unpublished).

[11] R. S. McKean, Jr., Phys. Rev. **125**, 1399 (1962).

[12] D. B. Lichtenberg, J. S. Kovacs, and H. McManus, Bull. Am. Phys. Soc. **7**, 55 (1962).

[13] R. A. Bryan, C. R. Dismukes, and W. Ramsay, UCLA report (unpublished).

[14] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963), and private communication.

[15] To quote M. L. Goldberger: "There are few problems in modern theoretical physics which have attracted more attention than that of trying to determine the fundamental interaction between two nucleons. It is also true that scarcely ever has the world of physics owed so little to so many. . . . In general, in surveying the field, one is oppressed by the unbelievable confusion and conflict that exists. It is hard to believe that many of the authors are talking about the same problem, or in fact that they know what the problem is." *Proceedings of the Midwest Conference on Theoretical Physics* (Purdue University, Lafayette, Indiana, 1960), pp. 50–63.

[16] E. Teller (private communication).

[17] H. P. Duerr, Phys. Rev. **103**, 469 (1956).

[18] M. H. Johnson and E. Teller, Phys. Rev. **98**, 783 (1955).

[19] The nonresonant effects are certainly required for a consistent treatment of the problem and the calculations of D. Amati, E. Leader, and B. Vitale, Nuovo Cimento (to be published), and of W. N. Cottingham and R. Vinh Mau, Phys. Rev. (to be published) indicate that this contribution to the interaction could be comparable in magnitude to that coming from the resonances.

[20] G. Breit and M. H. Hull, Jr., Nucl. Phys. **15**, 216 (1960).

[21] Throughout we use the nuclear-bar parameters $\delta_{l,J}$ and ϵ^J as defined by H. P. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 311 (1957).

these two amplitudes and characterize them as accurately as possible. The first step is to assume that they are given by the shape-independent approximation,

$$T\equiv k\cot\delta_{0,1}\approx -1/a_t+\tfrac{1}{2}r_tk^2,\qquad S\equiv k\cot\delta_0\approx -1/a_s+\tfrac{1}{2}r_sk^2, \tag{1}$$

so called because any interaction containing two adjustable parameters reasonably sensitive to the over-all strength and range (or dependence on momentum transfer) can be fitted to this energy variation at sufficiently low energy. The scattering lengths a_t and a_s can be determined directly by the measurement of the coherent neutron-hydrogen scattering length, $a_{n\mathrm{H}}$, and the total n-p cross section, σ_0, measured at energies just above the point where molecular effects become significant sources of uncertainty, since

$$a_{n\mathrm{H}}=\tfrac{1}{2}(a_s+3a_t)=-\tfrac{1}{2}a,\qquad \sigma_0=\pi(a_s^2+3a_t^2)=\pi\Sigma. \tag{2}$$

In connection with the analysis of their precision n-p total cross-section measurements at 0.4926 and 3.205 MeV, Engelke, Benenson, Melkonian, and Lebowitz[22] have made a critical survey of the existing measurements of σ_0 and conclude that the best value to adopt is that given by Melkonian[23] of $\sigma_0=2036\pm5$ F^2. After discussion with Engelke,[24] it appears that the best value of $a_{n\mathrm{H}}$ is the weighted mean given by Wilson[25] as $a_{n\mathrm{H}}=3.744\pm0.010$ F. Hence, from Eq. (2), we have that

$$a_t=\tfrac{1}{4}(s-a)=5.397\pm0.011\ \mathrm{F},\qquad a_s=-\tfrac{1}{4}(3s+a)=-23.679\pm0.028\ \mathrm{F}, \tag{3}$$

with

$$s=[(4\Sigma-a^2)/3]^{1/2}.$$

The correlation in error is given by

$$\frac{\langle\delta a_t\delta a_s\rangle}{\langle\delta a_t^2\rangle^{1/2}\langle\delta a_s^2\rangle^{1/2}}=\frac{[\delta\Sigma^2-\frac{1}{4}(3s+a)(s-a)\delta a^2]/12s^2}{\{[\delta\Sigma^2+\frac{1}{4}(s-a)^2\delta a^2]/4s^2\}^{1/2}\{[\delta\Sigma^2+\frac{1}{4}(3s+a)^2\delta a^2]/36s^2\}^{1/2}}=-0.7803.$$

In order to determine the effective ranges r_s and r_t, we must make use of experimental information at finite values of k^2, where $\hbar k$ is the momentum of either particle in the c.m. system. For neutrons incident on a stationary proton target, this is given by[26]

$$\hbar^2k^2=2K_{\mathrm{lab}}M_n^2M_p^2/M_n(M_n+M_p)^2,\qquad k^2=0.0120484K_{\mathrm{lab}}\ \mathrm{F}^{-2}, \tag{4}$$

where K_{lab} is the energy of the incident neutron in MeV. Since the deuteron corresponds to a pole[27] in the scattering amplitude $e^{i\delta_{0,1}}\sin\delta_{0,1}=1/(T-ik)$ at $\hbar^2k_0^2=-2M_nM_p\epsilon_d/(M_n+M_p)$, where ϵ_d is the binding energy, we can evaluate the triplet effective range as

$$r_t=2(1-1/a_tk_0)/k_0=1.727\pm0.014\ \mathrm{F}, \tag{5}$$

where we have used the latest measurement of the binding energy of the deuteron by Knowles[28] of $\epsilon_d=2224.52\pm0.20$ keV. The accuracy of this measurement is so high that the uncertainty in r_t arises solely from the uncertainty in a_t, and it is easy to show that the uncertainty in T due to ϵ_d is less than 0.13% of the uncertainty due to a_t at *any* energy; we can, therefore, take k_0 as exactly known in what follows.

In order to obtain a reliable value of r_s from total cross-section data at low energy, it is crucial to select from the mass of existing data[29] those experiments which are most likely to be free from systematic error. This thankless task has been performed for me by Hafner[30]; the experiments, selected on the basis that they are known to be free of systematic error due both to in scattering and to neutrons degraded in energy by other processes, are given in Table I, to which have been added the new measurements of Engelke *et al.*[22]; (reference 22 contains a detailed discussion of the sources of systematic error in this type of experiment.) Hafner also provided a larger selection containing about 20 more measurements, and the analysis presented below has also been carried through for these using various selections; since the results are insensitive to

[22] C. E. Engelke, R. E. Benenson, E. Melkonian, and J. M. Lebowitz, Phys. Rev. **129**, 324 (1963).

[23] E. Melkonian, Phys. Rev. **76**, 1744 (1949); we follow Engelke *et al.* in using the standard deviation rather than twice that figure as quoted by the author.

[24] C. E. Engelke (private communication). I am indebted to Dr. Engelke for informing me of his results prior to publication and for detailed discussion of our somewhat different analyses.

[25] R. Wilson in a forthcoming book on nucleon-nucleon scattering experiments [(to be published) Interscience Publishers Inc., New York]; I am indebted to Professor Wilson for sending portions of the manuscript and for discussion of several points.

[26] We use the values given by W. H. Barkas and A. H. Rosenfeld, Lawrence Radiation Laboratory Report, Berkeley, UCRL-8030 (revised) of $M_pc^2=938.213$ MeV, $M_nc^2=939.507$ MeV, $\hbar c=197.32$ MeV F; note that using the average mass is not quite accurate enough as it would change r_s by about 0.02 F. However, the exact relativistic expression, which is obtained by multiplying Eq. (4) on the right by $(1+K/2M_nc^2)/[1+2M_pK/(M_n+M_p)^2c^2]$, since it is proportional to the n-p mass difference (if K is the difference between the total neutron energy and its rest energy), differs by only about one part in 10^6 from Eq. (4) in this energy range.

[27] M. L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. (N. Y.) **2**, 226 (1957).

[28] J. W. Knowles, Can. J. Phys. **40**, 257 (1962).

[29] R. Howerton, Lawrence Radiation Laboratory Report, Berkeley, UCRL Rept. 5226 (unpublished).

[30] E. M. Hafner (private communication).

TABLE I. Neutron-proton total cross-section measurements below 5 MeV and the estimated contribution to the total cross section coming from angular momentum states other than $^1S_0$ and $^3S_1$.

Point	Reference	Energy (MeV)	Cross section (mb)		$l\neq 0$ contribution (mb)
1	a	0.4926	6202	11	0.03
2	b	1.005	4228	18	0.17
3	c	1.078	4060	30	0.22
4	d	1.315	3675	20	0.29
5	c	1.578	3330	20	0.39
6	e	2.540	2525	9	0.69
7	a	3.205	2206	7	0.89
8	b	4.749	1690	6	1.20

[a] Engelke *et al.*, reference 22.
[b] E. M. Hafner, W. F. Hornyak, C. E. Falk, G. Snow, and T. Coor, Phys. Rev. **89**, 204 (1953).
[c] E. E. Lampi, G. Freier, and J. H. Williams, Phys. Rev. **76**, 188 (1949).
[d] C. L. Storrs and D. H. Frisch, Phys. Rev. **95**, 1252 (1954).
[e] R. E. Fields, R. L. Becker, and R. K. Adair, Phys. Rev. **94**, 389 (1954).

the addition of these measurements to those given in Table I, and the errors are not significantly improved, we give results only for the smaller selection. Wilson[25] concurs with the selections made by Hafner and their evaluation.

The values of a_t, a_s, and r_s are determined by adding the two experiments already discussed and minimizing

$$\chi^2=[(a_s+3a_t)-a_{nH}]^2/\delta a_{nH}^2 +[\pi(a_s^2+3a_t^2)-\sigma_0]^2/\delta\sigma_0^2 +\sum_\sigma\left[\frac{\pi}{S^2+k^2}+\frac{3\pi}{T^2+k^2}-\sigma\right]^2\Big/\delta\sigma^2. \quad (6)$$

The error matrix is obtained by calculating the inverse to $\frac{1}{2}(\partial^2\chi^2/\partial x_i\partial x_j)$, where x_i, x_j run over the three parameters and the second derivative is computed at the minimum. We find that the values of a_s, a_t and their errors are essentially unchanged from those given above (final values are given in the abstract), and that

$$r_s=2.51\pm0.11 \text{ F} \quad (7)$$

with error correlations

$$\langle\delta r_s\delta a_t\rangle=-0.8547\langle\delta r_s^2\rangle^{1/2}\langle\delta a_t^2\rangle^{1/2},$$
$$\langle\delta r_s\delta a_s\rangle=0.7029\langle\delta r_s^2\rangle^{1/2}\langle\delta a_s^2\rangle^{1/2}. \quad (8)$$

Before turning to a discussion of the uncertainties in these values due to departures from the shape-independent approximation, we wish to discuss this value for r_s. To begin with, we note that it is significantly different from the *p-p* effective range $r_{pp}=2.78$ F given by Heller.[31] This deviation from charge independence is greater than one would expect from the $3\frac{1}{2}\%$ $\pi^\pm-\pi^0$ mass difference, but since we have seen above that the "scalar meson" is more important than single-pion exchange in determining the singlet effective range, it will be impossible to discover whether this is a real failure of the charge-independence hypothesis until the structure of this object is sufficiently well understood to allow a calculation of the electromagnetic corrections to its effective mass and coupling constant in the *n-p* and *p-p* systems. It is perhaps worth noting that if this charge-dependent value of $r_s^{np}=2.51$ F is accepted, the discrepancy between the computed and observed value of the *n-p* thermal capture cross section nearly disappears.[32] We note further than the two new measurements at 0.4926 and 3.205 MeV by themselves would give[33] $r_s=2.43\pm0.11$ F while the six earlier measurements gave 2.64 ± 0.12 F. This looks a little large for a statistical fluctuation and suggests that additional precise measurements of *n-p* total cross sections in this energy range would be of value. As we will see in the next section, however, there is no point in pushing the precision of such measurements beyond the point already achieved by Engelke *et al.*, unless the precision of a_{nH} and the very low energy cross section is also improved.

III. DEVIATIONS FROM THE SHAPE-INDEPENDENT APPROXIMATION

Until we can set *a priori* limits to the deviations from the shape-independent approximation as a function of energy we can neither assess the reliability of the parameters determined in the last section nor determine the requisite accuracy for experiments at higher energy which would give significantly new information about the energy dependence of the S waves. These deviations will come from two sources. In the first place, we can anticipate k^4 and higher terms in the exact expressions for S and T, and must estimate the magnitude of their coefficients. Since these terms will cause the total cross section to deviate from the approximation to order k^4, and ϵ^1 and the P phase shifts will contribute terms of the same order, we must also be able to estimate the contribution to the total cross section from higher angular momentum states. We start with this second problem.

As discussed in the Introduction, six phase parameters other than the S phase shifts cannot be predicted from one-pion exchange (OPE) at any energy. Since existing data below 20 MeV consist only of differential and total cross sections, we cannot, at present, take these from experiment. For the $^3S_1-^3D_1$ state we believe that the models of Glendenning and Kramer[34], which consist of an OPE tail and an inner phenomenological part fitted to the deuteron, and which are in rough agreement with *n-p* scattering analyses at high energy, should give a

[31] L. Heller, Phys. Rev. **120**, 627 (1960). Recent work by the author [H.P.N., (unpublished)] indicates a somewhat lower value, but definitely greater than 2.7 F.

[32] N. Austern and E. Rost, Phys. Rev. **117**, 1506 (1960), Eq. (16).

[33] The two values are 2.448 and 2.374 F at 0.4926 and 3.205 MeV, respectively; the first differs from the value of 2.46 F quoted in reference 21 because we have used a different value for a_{nH}; comparison shows that the difference is due solely to this difference in input.

[34] N. Glendenning and G. Kramer, Phys. Rev. **126**, 2159 (1962).

TABLE II. Estimate of $l\neq 0$ contribution to the total n-p cross section between 14 and 20 MeV. $^3S_1-{}^3D_1$ state taken from Glendenning (reference 35), $^3P_{0,1,2}$ from Stapp *et al.* (reference 36) and P_1 from OPE; the error in $^1P_1$ is estimated from differential cross section measurements at 14.1 and 17.9 MeV.

Energy (MeV)	Triplet contribution (mb) ($^3S_1-{}^3D_1$, $^3P_{0,1,2}$)	OPE (mb) $l\geqq 2$	(mb) $^1P_1$	(mb) $\sigma_{l\neq 0}$
14.1	1.27±0.17	0.76	3.85±1.92	5.83±2.10
17.9	2.47 0.10	1.10	4.39 2.19	7.95 2.29
19.665	2.81 0.15	1.25	4.56 2.28	8.63 2.43

reasonable estimate. Glendenning[35] has kindly supplied me with phase shifts for these models at 1, 5, 10, and 14.4 MeV, and I find that the contribution to the total cross section from ϵ^1 and $\delta_{2,1}$ differs very little between the various models. For the ^3P phases, we assume charge independence and take them from the energy-dependent phase-shift analyses of Stapp *et al.*[36] The contribution from these triplet phases is given in Table II. While the phase-shift values themselves are not particularly reliable, we see that the spread between the cross-section contributions is so small that we can perhaps believe the order of magnitude of the total cross-section prediction.

For the $^1P_1$ state, we note that existing models and theories agree that whatever interaction is present in addition to OPE is also predominantly repulsive. Since in effect this additional interaction simply strengthens the centrifugal barrier we can expect much smaller deviations from the OPE value than if either it or the short-range interaction were attractive. It is possible to make a rough check on this theoretical prediction in the following way. The strong tensor force in the triplet state leads to much more isotropic scattering than would be expected from impact parameter arguments.[37] Consequently, to a first approximation the angular variation of the differential cross section is dominated by the $\cos\theta$ term arising from $^1S_0-{}^1P_1$ interference. As the $^3S_1$ phase shift is accurately given by the shape-independent approximation in this energy range (cf. below) we can obtain δ_0 from the total cross section and hence evaluate δ_1 from the $\cos\theta$ term in the differential cross-section measurements at 14.1 [38,39] and 17.9[40] MeV. We have actually carried through this analysis,[41] finding that, in fact, the phase shifts used for Table II do give the expected approximate isotropy in the triplet scattering, and obtained values of δ_1 at these two energies which are consistent with OPE. Unfortunately the error is about 35% in $\sin\delta_1$, which precluded a reliable extrapolation to 19.665 MeV. We, therefore, feel it more reliable to use the OPE value, but assign an experimental uncertainty of 50%, to $\sin^2\delta_1$ which we believe to be conservative. Since we need in addition values only below 5 MeV, we made a rough extrapolation by assuming that the energy *variation* of the phases was the same as for OPE and obtained the values given in Table I. Since this estimate is only 20% of the experimental error for the highest energy in the table, we believe we have successfully eliminated this source of uncertainty from the analysis given in the last section.

Most discussions of the departure of the S waves from the shape-independent approximation make use of a "shape parameter" P defined by adding a term $-Pr^3k^4$ to the expressions given in Eq. (1). This is inadequate for our purposes since the interaction due to OPE gives a branch point in S and T at $k^2=-(m_\pi c/2\hbar)^2$ corresponding to a "laboratory" energy of -10 MeV; consequently the expansion in powers of k^2 about $k^2=0$ diverges beyond 10 MeV and is quantitatively unreliable at much lower energies. As was shown by Noyes and Wong,[8] it is possible to take account of OPE exactly and extend the *a priori* radius of convergence to 40 MeV at the cost of solving a nonsingular integral equation of the Fredholm type. In the approximation which replaces the multiparticle exchange branch cuts by a single pole whose position and residue are adjusted to fit the observed scattering length and effective range, the solution to this integral equation is very accurately represented (to better than 7% up to 40 MeV) by a simple expression derived independently by Cini, Fubini, and Stanghellini (CFS)[42] from fixed angle dispersion relations. This is equivalent to replacing the OPE branch cut by a single pole of known residue at $k^2=-\frac{1}{2}(m_\pi c/\hbar)^2=-\frac{1}{2}r_0^{-2}$ and, hence, to the expression

$$kr_0\cot\delta=a+b(kr_0)^2+c(kr_0)^4/[1+d(kr_0)^2],$$

$$d=\frac{2-f^2M(\frac{3}{2}\sqrt{2}+4a-b)}{1-f^2M(\frac{1}{4}\sqrt{2}+a)}, \qquad (9)$$

$$c=-(1-\tfrac{1}{2}d)(2\sqrt{2}-2b+4a),$$

where M is the ratio of the nucleon to the pion mass and $f^2(\approx 0.08)$ the pion-nucleon coupling constant. Since we are given the triplet scattering length and deuteron binding energy rather than the triplet effective range, it is convenient in the triplet case to introduce the pole at $k^2=-q_0^2/r_0^2$ explicitly, which can be done by taking $b=(1/q_0)\times(1+a/q_0)+cq_0^4/(1-dq_0^2)$. Taking for m_π a third of the neutral pion mass and two-thirds of

[35] N. Glendenning (private communication).

[36] H. P. Stapp, H. P. Noyes, and M. J. Moravcsik (unpublished).

[37] R. S. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950).

[38] H. L. Poss, E. D. Salant, G. A. Snow, and L. C. L. Yuan, Phys. Rev. **87**, 11 (1952).

[39] R. B. Day, R. L. Mills, J. E. Perry, Jr., and F. Scherb, Phys. Rev. **114**, 209 (1959).

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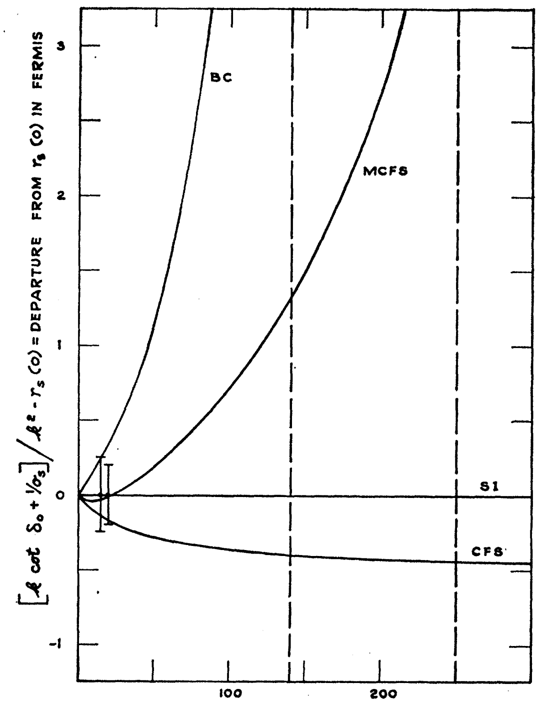

FIG. 1. Predictions for the effective range function defined by $r_s(k^2)=2(k\cot\delta_0+1/a_s)/k^2$. The expected behavior for models which have the long-range OPE interaction and sufficient parameters to fit a_s, $r_s(0)$, and the zero in δ_0 at 250 MeV is given by the curve MCFS. The three other curves are CFS [Cini, Fubini, and Stanghellini (reference 42)], SI (shape-independent approximation), and BC [boundary condition (references 45, 46)].

the charged pion mass, and

$$f^2M=14.4[m_\pi^2/2(M_p+M_n)]=0.52938,$$

we find for the triplet case

$$r_0=1.4292\text{ F},\quad a=-0.26481,\quad b=0.60350,\quad c=-0.04389,\quad d=1.8439. \tag{10}$$

We, therefore, predict for the triplet shape parameter the small value $P_t=-c/8b^3=0.025$, in addition to an important damping of this term by the denominator $1+d(kr_0)^2$ in the 10–20 MeV range.

We have compared the phase shifts computed by Glendenning[35] with the shape-independent approximation, and in all cases find the deviation to lie within the theoretical estimate; this strengthens our belief in the small size of the shape-dependent term. In making this comparison it was important to note that the values of the scattering length differ for the different models and to use the appropriate value in each case. Since these models are in reasonable agreement with experiment at high energy, we believe that the magnitude (although not the sign) of the triplet shape dependence is conservatively estimated by Eqs. (9) and (10).

For the $^1S_0$ state, there is no longer a pole at $k^2=-k_0^2$ but instead a pole on the second Riemann sheet corresponding to a virtual rather than actual bound state. We, therefore, take $b=\frac{1}{2}r_s/r_0$ and, for $r_s=2.544$ determined by a least-squares adjustment (cf. below) find that

$$a=0.060358,\quad c=-0.28312,\quad b=0.89015,\quad d=1.5629, \tag{11}$$

predicting a singlet shape parameter $P_s=0.050$. This prediction is considerably less reliable than for the triplet state because, as already noted, the singlet state is much more strongly influenced by the scalar and vector mesons than by OPE, the repulsion due to the vector mesons causing the singlet phase shift to change sign at 250 MeV, a behavior not predicted by the CFS approximation. If two additional empirical constants are added to the integral equation and fitted to high-energy phase shifts we find that close to $k^2=0$, the estimate of P_s is still approximately correct, but that $k\cot\delta_0$ crosses the shape-independent approximation around 20 MeV. This behavior is sketched in Fig. 1.

Lacking the requisite experimental information, we again turn to model calculations for confirmation of this prediction and find that models which have either a hard core or boundary condition to fit the 250-MeV singularity, the OPE tail, and something else to fit the values of a_s and r_s do indeed behave in this way[43,44] but that the cross-over point is sensitive to the details of the model and may occur anywhere between 10 and 40 MeV. We believe that this shows that the CFS curve gives a conservative estimate of the amount by which the actual curve is likely to fall below the shape-independent approximation at low energy.

In order to estimate the amount by which we can expect the curve to lie above the shape-independent approximation in the extreme case, we make use of the two-parameter model which has *no* potential (OPE) tail, but consists of an energy-independent boundary condition on the wave function at finite radius[45,46]

$$k\cot(\delta+k\bar{r})=A. \tag{12}$$

Making a least-squares adjustment, we find that $\bar{r}=1.1701$ F and $A=0.040245$ F^{-1}. This gives a curve with a singularity at about 135 MeV as indicated in Fig. 1, and again gives what we believe to be a conservative estimate of the deviation. This is confirmed by adding an energy-dependent term to Eq. (12) to move the singularity out to 250 MeV, which results in a curve which lies everywhere between the boundary condition (BC) curve and the shape-independent (SI) approximation and always above the latter. We can, therefore, obtain an estimate of the shape dependence of the value of r_s determined in the last section by

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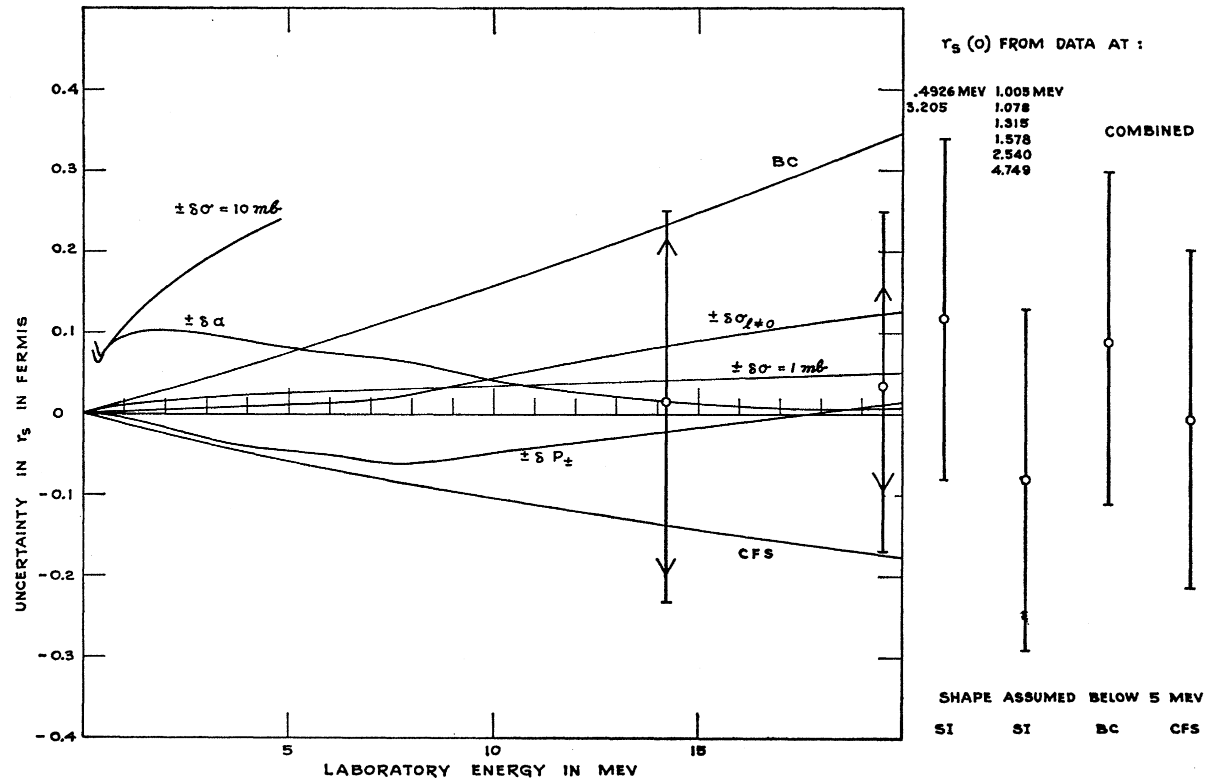

FIG. 2. Contributions to the uncertainty in the value of r_s computed from total cross-section measurements at a single energy: $\pm\delta a$ due to uncertainty in the scattering lengths, $\pm\delta\sigma_{l\neq 0}$ due to the scattering in other angular momentum states, $\pm\delta P_t$ due to the triplet S shape dependence as estimated by the CFS formula, $\pm\delta\sigma$ due to an experimental uncertainty of one or 10 mb in the total cross section. The curves BC and CFS give the maximum theoretically allowed deviation from a constant value (cf. text). The arrowheads on the two experimental points show the uncertainty arising from the experimental error in the total cross section. The point at 19.665 MeV is repeated to the right of the graph to show the effect of the data selected below 5 MeV to determine $r_s(0)$, and of the extreme variation arising from the shape dependence assumed in that analysis.

adjusting either the CFS or the BC models to the same data. Since we have added no more free parameters, the error analysis is unaltered, and we find

$$r_s^{\mathrm{CFS}}=2.544\pm 0.11\ \mathrm{F},\quad r_s^{\mathrm{BC}}=2.458\pm 0.11\ \mathrm{F}. \tag{13}$$

We conclude that in addition to the experimental error of 0.11 F we should add an additional uncertainty due to shape dependence which is at most ± 0.043 F. For data below 5 MeV of the precision of those analyzed here, the use of Eqs. (9) or (12) rather than the usual shape-dependent term $-Pr_s^2k^4$ turns out to be an unnecessary refinement. This would no longer be true if the precision of the experiments were increased or the energy range extended to 10 MeV.

The results of this discussion are summarized in Fig. 2 where we plot separately the various sources of uncertainty in the determination of r_s. These are (a) uncertainty due to the scattering lengths, (b) uncertainty due to the triplet shape dependence, (c) uncertainty due to the contribution from other angular momentum states, (d) uncertainty due to the singlet shape dependence, and (e) for reference, the uncertainty due to a 1-mb error in the measurement of the total cross section. Several conclusions follow immediately from this curve. As already discussed by Engelke *et al.*,[22] their low-energy point is located at the optimum energy for the determination of r_s, and their experimental uncertainty has been reduced to the point where it is equal to the experimental uncertainty arising from other causes (mainly the scattering lengths). Any improvement will, therefore, require in addition remeasurement of the scattering lengths to higher precision. As we go to higher energy we see that the 15–20 MeV region is the most favorable from the point of view of the error due to the scattering lengths; this is due to the fact that $k\cot\delta_{0,1}$ goes through zero at about 17.8 MeV so that the sensitivity to a_t is very small, and that this is already at a high enough energy so that the sensitivity to a_s is nearly negligible. We further see that the 2–2½ mb error in our knowledge of the scattering in other angular momentum states is serious and precludes much more than a qualitative distinction between the two extreme singlet curves. From our discussion given above it is also clear that this uncertainty can only be slightly reduced by more accurate measurements of the $\cos\theta$ term in the differential cross section, and consequently that *measurements which distinguish the polarization states of the neutron and proton and lead to a*

TABLE III. Contributions to the n-p total cross section at 14.1 and 19.665 MeV.

Laboratory energy (MeV)	r_s from	Shape-independent prediction (mb)	Triplet shape uncertainty (F)	Scattering for $l\neq 0$ (mb)
14.1	BC	684.832±2.327	±0.598	5.834±2.096
	SI	683.460		
	CFS	682.696		
19.665	BC	487.423±2.308	±0.260	8.627±2.43
	SI	486.288		
	CFS	485.657		

phase-shift analysis are required to make a quantitative determination of the variation of the S waves away from the shape-independent approximation. This is the most important result obtained in this paper. We discuss briefly what is required in the next section.

Finally, we ask what the cross section at 14.1 MeV of 689±5 mb measured by Poss, Salant, Snow, and Yuan[38] and at 19.665 MeV of 494.2±2.5 mb measured by Day, Mills, Perry, and Scherb[39] can tell us about the deviation of the S waves from the shape-independent approximation. As already discussed, this cannot give us the shape parameter defined at $k^2=0$, so instead we test the result against the three extreme two-parameter models (BC, SI, CFS) by computing the shape function[47] at this energy from the deviation of the total cross section from the shape-independent approximation. Clearly, the formula is

$$P(k^2)\approx\frac{(S^2+k^2)^2}{2\pi Sk^4r_s{}^3}(\sigma_{\rm tot}-\sigma_{\rm SI}-\sigma_{l\neq 0}). \tag{14}$$

For consistency we must compute $\sigma_{\rm SI}$ separately for each model due to the differences in r_s, and we must be careful to include the correlations in error in calculating the uncertainty. We also include the uncertainty due to the triplet shape parameter. The predictions and errors are collected in Table III. The corresponding predictions and observations of the shape function at these energies are given in Table IV. We see that there is no significant deviation from the shape-independent approximation.

While the value of $P(k^2)$ close to zero is in accord with our theoretical expectations, we see that the errors are still too large to give any significant discrimination between the extreme models (cf., Fig. 2). We note also that if the value of r_s were 2.43 F, these results would strongly favor the BC model, while if it were 2.64 F we would say that the BC model was pretty conclusively excluded, emphasizing again the necessity for improving our confidence in r_s by new measurements at low energy. Finally, we reiterate that improved total and differential cross-section measurements below 20 MeV can at best decide between the extreme models and can never give the detailed energy variation of $P_s(k^2)$ needed to test theories of the $^1S_0$ scattering in this energy range.

[47] H. P. Noyes, Bull. Am. Phys. Soc. **7**, 504 (1962) and references 1 and 8. The numbers quoted there have changed somewhat due to changes in input data. The positive value of P given is obtained primarily because the $^1P_1$ phase shift at 14.1 MeV was taken from the differential cross-section analysis rather than OPE; we now believe the latter to be more reliable for the reasons given in the text.

IV. CONCLUSIONS

Evaluation of all the important uncertainties reveals that our knowledge of the n-p S-wave scattering lengths and effective ranges could be improved by at most a factor of three over the values given by this analysis, if the accuracy of total cross-section measurements below 5 MeV, of the very low energy total cross section, and of the coherent neutron-hydrogen scattering lengths were improved by that amount. Such additional measurements would also be desirable because the spread in existing measurements is somewhat larger than is to be expected on purely statistical grounds and makes it questionable whether the mean value of r_s obtained by this analysis can be trusted to the quoted statistical accuracy. If the low-energy analysis is accepted, the total cross-section measurements at 14.1 and 19.665 MeV give a very small singlet shape effect at these energies, in accord with theoretical expectations; this confirmation would become more convincing if the above improvement in the lower energy measurements were achieved. However, our current lack of knowledge of the scattering in other angular momentum states at these energies is comparable to the experimental uncertainty in the total cross sections, and we have shown that this uncertainty cannot be removed by improved measurement of the differential cross section. We conclude that quantitative information about the departure of the S waves from the shape-independent approximation can be achieved only by performing enough new types of experiments to lead to a unique phase-shift analysis.

The n-p cross section at low energy depends on eight phase parameters which cannot be evaluated from one-pion exchange, so in principle eight independent pieces of experimental information are needed at each energy where these phases are to be determined. If we are willing to assume charge independence, the $^3P_{0,1,2}$ phases could be taken from p-p scattering. Since in that system the Coulomb interference terms in the differential cross section give three independent pieces of information,[48] one needs in addition one experiment

TABLE IV. Shape function for $^1S_0$ n p scattering at 14.1 and 19.665 MeV.

Model	14.1 MeV Predicted	14.1 MeV Observed	19.665 MeV Predicted	19.665 MeV Observed
BC	−0.041	−0.012	−0.042	−0.012±0.027
SI	0	−0.002±0.045	0	−0.005
CFS	0.026	0.004	0.024	−0.001

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such as C_{nn} (90°) to obtain $^1S_0$ and these three P phases, as has been discussed by Iwadare.[49] Actually the P phases are still given only up to a fourfold trigonometric ambiguity, but since the analyses are unique at higher energy, and in agreement with the theoretical prediction,[50] an additional experiment such as D or A is needed to resolve this ambiguity only to the extent that one distrusts the extrapolation or the theoretical argument. Charge-dependent effects are still big enough at 20 MeV so that we *cannot* reliably use the $^1S_0$ phase determined from *p-p* scattering to assist the *n-p* analysis for experiments of the precision contemplated here; we, therefore, still need five pieces of information from *n-p* experiments. Two of these can certainly be provided by the total cross section and the cosθ term in the differential cross section if improved measurements of the latter are made. The polarization has recently been measured to high precision at 23.1 MeV,[51] so we can count on this for at least one more piece of information. If one accepts the theoretical argument of Wong,[52] ϵ^1 can be calculated to the requisite accuracy from a knowledge of $^3S_1$ and the OPE interaction, so a minimal program would require only one of the difficult experiments (spin correlation, triple scattering, polarized target, etc.), but detailed examination of the requisite accuracy will not be attempted here. Preliminary calculations[53] indicate that the spin-correlation and to a lesser extent the depolarization experiments are more sensitive to small variations in the phase shifts than the R or A parameters. Since phase shifts computed from the Hamada-Johnston model[54] are in excellent agreement with the polarization measurement,[51] they should provide a reliable starting point for the optimization of the experimental design. Ultimately one hopes that a sufficient variety of experiments will be performed to lead to unique phase parameters from *n-p* experiments alone and, hence, to the eight low-energy empirical constants discussed in the Introduction.

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