(iv) Using $1/(1+4a^2)\leqq|R(s,t)|\leqq 1$ and inequality (B7) one can prove by induction that

$$[1/(1+4a^2)](\pi/2)^{i-1}\leqq|R^{(i)}(s,t)|\leqq(\pi-\epsilon)^{i-1}.$$

Thus, there exist constants, b_1 and b_2, such that, for $\mu\leqq 1/\pi$,

$$0<b_1=[1/(1+4a^2)]\frac{1}{1-\mu(\pi/2)}\leqq|\rho(s,t)|$$
$$\leqq\frac{1}{1-\mu(\pi-\epsilon)}=b_2<\infty \qquad (B.13)$$

If there exist constants b_3 and b_4 such that $b_3\leqq v(s)\leqq b_4$, then one obtains from (B10)

$$b_3(1+2ab_1)\leqq u(s)\leqq b_4(1+2a\mu b_2), \qquad (B14)$$

which proves statement (d) of Sec. III, that is if $r(y)$ is bounded above or below, then so is the solution $g(y)$. The proof of statement (c) of Sec. III [that is, if $r(y)$ is positive definite, then so is the solution $g(y)$] follows immediately from (B14) as b_3 can be chosen positive in this case.

II. Numerical Solution

The basic integral equation (B2) has been solved numerically by applying Simpson's rule to the integral on a grid $\{s_i=-a+(i-1)h;\, h=a/n;\, i=1,\cdots,2n+1\}$. This yields a system of $(2n+1)$ linear algebraic equations for the $(2n+1)$ discrete approximate values $u_i\approx u(s_i)$ which can be solved by a standard method. The quadratures involved in calculating γ and e also have been carried out by Simpson's rule. The functions $K(\gamma)$ and $e(\gamma)$ are obtained in parametric form, that is $(K(\lambda),\gamma(\lambda))$ and $(e(\lambda),\gamma(\lambda))$. To obtain μ, the quantities e and γ are evaluated on a sufficiently fine grid of equidistant λ values. Then,

$$\mu=3e-\gamma\frac{de}{d\gamma}=3e-\gamma\frac{de}{d\lambda}\bigg/\frac{d\gamma}{d\lambda}$$

can be calculated by numerical differentiation.

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Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum

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We continue the analysis of the one-dimensional gas of Bose particles interacting via a repulsive delta function potential by considering the excitation spectrum. Among other things we show that: (i) the elementary excitations are most naturally thought of as a double spectrum, not a single one; (ii) the velocity of sound derived from the macroscopic compressibility is shown to agree with the velocity of sound derived from microscopic considerations, i.e., from the phonon spectrum. We also introduce a distinction between elementary excitations and quasiparticles, on the basis of which we give some heuristic reasons for expecting the double spectrum to be a general feature, even in three dimensions, and not an exception.

I. INTRODUCTION

IN the preceding paper[1] we introduced a soluble model of a Bose gas interacting via a repulsive δ-function potential. We discussed the nature of the eigenfunctions and explicitly calculated the ground-state energy and other properties of the ground state in the limit of a large system.

In this paper we discuss the nature of the excitation spectrum for a large system of N particles. The surprising result, as we stated but did not show in I, is that for all values of the potential strength, the most convenient and natural way to view the spectrum is to regard it as a *double* spectrum of elementary boson excitations. While Bogoliubov's perturbation theory[2,3] gives one of the spectra quite accurately for a weak potential, the second spectrum is entirely unaccounted for (see Figs. 3 and 4). The second spectrum exists only for values of the momentum satisfying $|p|\leqq\pi\rho$.

We may summarize the results of this paper as follows: (i) In Sec. II we discuss the nature of the energy spectrum of the problem and show that there are two elementary spectra. These are always well defined and are explicitly calculated. We show that there is no energy gap and that the two spectra have a common slope at $p=0$ which means that they propagate sound at the same velocity. The velocity of sound at absolute zero derived in this way from an atomic picture[4–6] is shown to be identical with the velocity of sound defined by the usual macroscopic considerations [cf. Eqs. (1.1) and (1.4)].

[1] E. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963) (referred to here as I).
[2] See I, Sec. IV.
[3] See, for example, *The Many Body Problem*, edited by C. DeWitt (John Wiley & Sons, Inc., New York, 1958), pp. 347–355.
[4] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).
[5] R. P. Feynman, Phys. Rev. **91**, 1301 (1953).
[6] R. P. Feynman, Phys. Rev. **94**, 262 (1954).

The velocity of sound v_s, derived from the excitation spectrum is given by

$$v_s = \lim_{p\to 0} \partial\epsilon(p)/\partial p, \tag{1.1}$$

where $\epsilon(p)$ is the energy of an elementary excitation of momentum p. By a well-known macroscopic argument,[7] on the other hand,

$$v_s = [(-L/m\rho)\partial P/\partial L]^{1/2}, \tag{1.2}$$

where P is the pressure,

$$P = -\partial E_0/\partial L, \tag{1.3}$$

L is the length of the "box," $\rho = N/L$ is the density, m is the mass per particle ($=\frac{1}{2}$ in our units), and E_0 is the ground-state energy.

In I we defined the dimensionless parameter $\gamma = \rho^{-1}c$, where $2c$ is the strength of the δ function. In terms of γ, E_0 may be written $E_0 = N\rho^2 e(\gamma)$. In terms of these quantities, v_s may be written

$$\begin{aligned} v_s &= 2\rho(3e - \gamma\dot{e} + \tfrac{1}{2}\gamma^2\ddot{e})^{1/2} \\ &= 2(\mu - \tfrac{1}{2}\gamma\dot{\mu})^{1/2}, \end{aligned} \tag{1.4}$$

where "dot" denotes differentiation with respect to γ and μ is the chemical potential given by

$$\mu = \frac{\partial E_0}{\partial N} = \rho^2(3e - \gamma\dot{e}). \tag{1.5}$$

Bogoliubov's formula for $\epsilon(p)$ [I(4.1)] gives

$$v_s = 2\rho\gamma^{1/2}, \tag{1.6}$$

while using Eq. (1.4) and Bogoliubov's expression for $e(\gamma)$[I(4.2)] we get

$$v_s = 2\rho[\gamma - (1/2\pi)\gamma^{3/2}]^{1/2}. \tag{1.7}$$

Equations (1.6) and (1.7) are plotted as dashed curves in Fig. 5 along with the correct result for v_s obtained numerically. It will be noted that Eq. (1.7) is far more accurate than Eq. (1.6). In fact, the difference between Eq. (1.7) and the exact answer is so minute up to $\gamma = 10$ that we are unable to distinguish the two graphically. This is indeed remarkable in view of the fact that Bogoliubov's expression for $e(\gamma)$ is quite bad beyond $\gamma = 3$ and becomes negative for $\gamma > (3\pi/4)^2 \simeq 5.5$.

(ii) Based on the results of this model, we are led to the introduction of two distinct quantities—called elementary excitations and quasiparticles—which are defined differently, and are, in fact, different, but which are apparently somehow related. This is discussed in Sec. III.

The quantities that we calculate are given by the solutions to certain integral equations. While these cannot be obtained in closed form we are able to prove directly the statements made above. The quantitative results have been obtained by Werner Liniger on an IBM 7090 computer at the IBM Research Center. The equations considered here are similar to those considered in I. The reader is referred to Appendix B of that paper for a discussion of the numerical methods employed.

II. EXCITED STATES AND ELEMENTARY EXCITATIONS

In paper I we discussed the Hamiltonian[8]

$$H = -\sum_1^N \partial^2/\partial x_i^2 + 2c\sum_{\langle i,j\rangle}\delta(x_i - x_j), \tag{2.1}$$

and showed that in the subdomain $0 \le x \le \cdots \le x_N \le L$ all eigenfunctions are of the form

$$\psi(x_1,\cdots,x_N) = \sum_P a(P)P\exp(i\sum_{j=1}^N k_j x_j), \tag{2.2}$$

where the summation is over permutations, P, of $\{k\} = k_1, < k_2, < \cdots, < k_N$, and $a(P)$ is a function of P. We found that for any set $\{k\}$ the $a(P)$'s could be uniquely determined. The condition determining the allowed k's is

$$\delta_j \equiv (k_{j+1} - k_j)L = \sum_{s=1}^N (\theta_{sj} - \theta_{s\,j+1}) + 2\pi n_j \qquad (j = 1, 2, \cdots, N-1) \tag{2.3}$$

$$\theta_{sj} = -2\tan^{-1}[(k_s - k_j)/c], \tag{2.4}$$

and n_j are any integers ≥ 1.

The ground state was obtained from Eq. (2.3) by putting all $n_j = 1$. In this case $-k_1 = k_N = K$ where K is a function of γ. The simplest excitation is obtained by putting some one $n_j = 2$ while the rest remain unity. The effect of this is to increase δ_j by approximately 2π, but at the same time all the other δ's will be shifted slightly. Neglecting the change in spacing, the effect of making $n_j = 2$ can be approximately thought of either as moving k_{j+1} to K (or $-K$) leaving a hole behind, or else as pushing apart the k's, leaving a double space at the jth position.

To orient ourselves, let us consider the $\gamma = \infty$ (hard core) case. As Girardeau[9] has pointed out, the energy spectrum of the Bose gas is then exactly the same as for a noninteracting one-component Fermi gas (the wave functions are quite different of course). The problem is this: How can we describe this Fermi spectrum in conventional boson terms? If we regard the energy levels simply as a collection of numbers, the problem is similar to that of classical spectroscopy. Can we find a small number of elementary energy levels from which all others may be deduced by addition?

The conventional description of the Fermi spectrum is to regard the elementary excitation process as taking a particle from the state q ($|q| < K$) to a state k ($|k| > K$). The energy and momentum of this state are

$$\begin{aligned} \epsilon(k,q) &= k^2 - q^2, \\ p(k,q) &= k - q. \end{aligned} \tag{2.5}$$

[7] See, for example, F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1954), Vol. II, p. 83.

[8] $\hbar = 1$, $2m = 1$.

[9] M. Girardeau, J. Math. Phys. 1, 516 (1960).

The difficulty with this classification scheme is twofold: (a) Every excitation must be described in terms of two parameters instead of merely one as one normally hopes to do for a Bose system. There is no unique $\epsilon(p)$ curve. (b) Too many provisos must be added to the original specification (e.g., there can be at most one excitation of any given type; $|q|$ can be greater than K for high-level compound excitations, etc.).

If we are interested only in low-lying excitations, however, difficulties (a) and (b) above can be remedied by the following scheme. Define two types of one parameter elementary excitations by these rules:

Type I. Take a particle from K to $q>K$ (or alternatively from $-K$ to $q<-K$). This excitation has an energy and momentum given by

$$\epsilon_1=q^2-K^2, \tag{2.6}$$

$$\begin{aligned} p&=q-K, \quad q>K \\ &=q+K, \quad q<-K \end{aligned} \tag{2.7}$$

$$\epsilon_1(p)=p^2+2\pi\rho|p|. \tag{2.8}$$

Type II. Take a particle from $0<q<K$ to $K+2\pi/L$ (or alternatively from $-K<q<0$ to $-K-2\pi/L$). The energy and momentum for this state are given to $O(N^{-1})$ by

$$\epsilon_2=K^2-q^2, \tag{2.9}$$

$$\begin{aligned} p&=K-q, \quad 0<q<K \\ &=-K-q, \quad 0>q>-K \end{aligned} \tag{2.10}$$

$$\epsilon_2(p)=2\pi\rho|p|-p^2. \tag{2.11}$$

For the first type of excitation $-\infty<p<\infty$ while for the second type $-\pi\rho\leq p\leq\pi\rho$ only. These two types of excitations must be supplemented by exactly two other excitations, which we shall call *umklapp* excitations, namely take a particle from $-K$ to $K+2\pi/L$ (or alternatively from K to $-K-2\pi/L$). The energies of the umklapp excitations are $2\pi^2\rho/L$ and, hence, are of order N^{-1}. Their momenta, however, are $2\pi\rho$ and $-2\pi\rho$, respectively.

Any of these excitations may be carried out any number of times and we may simultaneously have as many different types of as many different momenta as we please. For example, one type I excitation of momentum p means taking a particle from K to $K+p$. Two excitations of this same momentum means taking one more particle from $K-2\pi/L$ to $p-2\pi/L$. The momentum of the resulting state is exactly $2p$ while the energy is twice that of one excitation to order N^{-1}. Thus, each of these elementary excitations may be regarded as bosons.

By experimenting with a k-space diagram, the reader may easily convince himself that for any specified combination of multiple excitations of type I (or type II)· there corresponds exactly one true state of the system, and conversely. The momentum of the true state is exactly the sum of the momenta of the elementary excitations, while the energy of the true state differs from the sum of the elementary energies only by $O(N^{-1})$ if the total number of excitations is finite (i.e., not of order N).

The difficulty with this classification scheme is that *either* type I *or* type II excitations form a "complete set" in the sense that they are in a one-one correspondence with the true excitation spectrum. Thus, any type II excitation may be thought of as a multiple type I excitation. The correspondence is this: m excitations of type I with momentum $p=2\pi n/L$ corresponds to exactly the same true state as n type II excitations with momentum $p=2\pi m/L$. In particular, one type II excitation of momentum p corresponds to $(1/2\pi)Lp$ type I excitations of momentum $p=2\pi/L$. If p is a fixed momentum independent of L, however, then the corresponding type II excitation may be thought of essentially as an infinite number of type I excitations of vanishing momentum. Since $\epsilon_1(p)$ and $\epsilon_2(p)$ are not proportional to p, the true energy of the type II excitation, $2\pi\rho p-p^2$, is very badly approximated by the $(1/2\pi)Lp$ type I excitations, viz.,

$$(1/2\pi)Lp[2\pi\rho(2\pi/L)-(2\pi/L)^2]\cong 2\pi\rho p.$$

The situation is, indeed, rather complicated and there does not seem to be any very simple way out of the dilemma. On the one hand, one can say that there is only one elementary excitation spectrum (one can choose either type I or type II) which is boson-like and which "generates" all the true states. On the other hand, physical considerations tell us that type I and type II excitations ought to stand on an equal footing, just as holes and particles play a similar role in the theory of the Fermi gas. It is patently absurd to think of a type II excitation as an infinite number of type I excitations of vanishing momentum.

We shall adopt here the "symmetric" point of view which seems to us the more natural way to view the problem. But we must then realize that if we calculate the partition function on this basis we will be double counting. The true free energy is that of the Fermi gas which one can calculate by standard methods. To leading order,

$$F=E_0-(kT)^2A, \tag{2.12}$$

where A is given by a well-known integral. If one calculates the free energy using the double boson spectrum [Eqs. (2.9) and (2.11)], one finds that

$$F'=E_0-2(kT)^2A. \tag{2.13}$$

The factor 2 is a precise reflection of the double counting. Either type I or type II alone would give Eq. (2.12). But if one does not double count, one gets a thoroughly distorted view of the nature of the excitation spectrum.

The situation for $\gamma\neq\infty$ is qualitatively the same as for the $\gamma=\infty$ case we have just discussed. Only the $\epsilon(p)$ functions differ; having discussed the over-all structure of the elementary excitations, we proceed now to calculate $\epsilon_1(P)$ and $\epsilon_2(P)$ for $\gamma\neq\infty$.

Type I Excitations ("Particle" States)

We put $n_j=1\,(1\leq j\leq N-2)$ and $n_{N-1}\gg 1$. Let $k_N=q$, where $q>K$, and write

$$k_i'=k_i+(1/L)\omega_i, \quad i=1,\cdots,N-1 \qquad (2.14)$$

where k_i' are the new values of the remaining k's. At this point there are two ways to proceed—both leading of the same result. In Eq. (2.14) we could let $k_i=k$'s in the N-particle ground state, or we could let $k_i=k$'s in the $N-1$ particle ground state (ω_i will, of course, be different in the two cases). We shall adopt the latter definition because it leads to a simpler calculation.

Now,

$$\theta(k_j'-k_i')=\theta(k_j-k_i)+\frac{1}{L}(\omega_j-\omega_i)\frac{-2c}{c^2+(k_j-k_i)^2}. \qquad (2.15)$$

Inserting Eq. (2.15) in Eq. (2.3) or I(2.15) we obtain the integral equation

$$\omega(k)+\pi=2c\int_{-K}^{K}\frac{[\omega(r)-\omega(k)]f(r)}{c^2+(r-k)^2}dr-\theta(q-k), \qquad (2.16)$$

where $f(k)$ is the distribution function of the k's in the ground state, [cf. I(3.12)]. We have chosen the term π (instead of 3π, 5π, etc.) in Eq. (2.16) so that as $q\to 0$ the wave function approaches the ground-state wave function for N particles. If we define

$$\omega(k)f(k)\equiv J(k), \qquad (2.17)$$

Eq. (2.16) becomes

$$2\pi J(k)=2c\int_{-K}^{K}\frac{J(r)dr}{c^2+(k-r)^2}-\pi-\theta(q-k). \qquad (2.18)$$

Using Eq. (2.14) the momentum of the state is

$$P=\sum_{j=1}^{N-1}k_j'+q=\sum_{1}^{N-1}k_i+\frac{1}{L}\sum_{1}^{N-1}\omega_i+q$$
$$=q+\int_{-K}^{K}J(k)dk, \qquad (2.19)$$

while the excitation energy is

$$\epsilon_1=\sum_{j=1}^{N-1}(k_i')^2-E_0(N)+q^2$$
$$=-\mu+q^2+2\int_{-K}^{K}kJ(k)dk, \qquad (2.20)$$

where $E_0(N)$ is the ground-state energy for N particles. To find $\epsilon_1(p)$ we have to eliminate the parameter q between Eqs. (2.19) and (2.20).

As we stated in I, Sec. III, Eq. (2.18) has a solution which is unique and negative definite (because $\pi+\theta(q-k)$ is positive definite). Since J (and, hence, ω) are negative we conclude, as was to be expected, that adding a particle with momentum $q>K$ to the $N-1$ particle system decreases all the other k's.

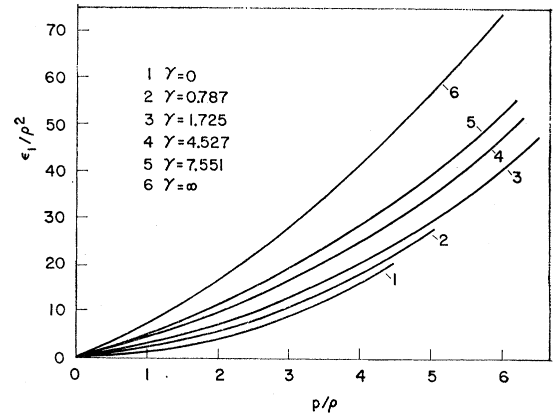

FIG. 1. The type I excitation spectrum, $\epsilon_1(p)$, plotted for small momenta and various values of $\gamma=c/\rho$. When $\gamma=0$, $\epsilon_1=p^2$; when $\gamma=\infty$, $\epsilon_1=p^2+2\pi\rho|p|$.

In Fig. 1 we display the $\epsilon_1(p)$ curve for several values of γ (in dimensionless units). The $\gamma=0$ curve is $\epsilon_1(p)=p^2$ (free particles), while the $\gamma=\infty$ is given by Eq. (2.8). All curves are linear for small p, the region of linearity increasing with γ.

The continuation of these curves for large p is given in the logarithmic plot of Fig. 2. To find an analytic expression for $\epsilon_1(p)$ for large p we expand $\theta(q-k)$ in a Laurent series for large q and retain the leading term. The obvious results are

$$J(k)\simeq-\frac{2\gamma\rho}{q}f(k), \qquad (2.21)$$

$$p\simeq q-2\gamma\rho^2/q, \qquad (2.22)$$

$$\epsilon_1(p)\simeq-\mu+p^2+4\gamma\rho^2. \qquad (2.23)$$

It should be noted that these last equations are true only if γ is finite and $p/\rho\gg\gamma$. If $\gamma=\infty$ on the other

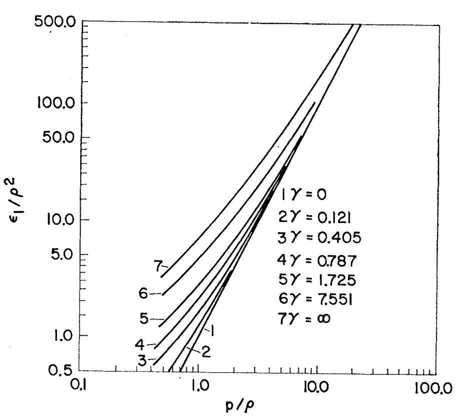

FIG. 2. The type I excitation spectrum for large momentum and various values of $\gamma=c/\rho$. When $\gamma=0$, $\epsilon_1=p^2$; when $\gamma=\infty$, $\epsilon_1=p^2+2\pi\rho|p|$.

hand, $\theta \equiv 0$ and Eqs. (2.6)–(2.8) must be used instead. In this case $J(k) = -\pi f(k) = -\frac{1}{2}$.

Equation (2.21) and (2.23) have a simple physical interpretation. The wave function can be well represented (in an average sense) by

$$\psi_p = \sum_P e^{ipx_N} \psi_0^{N-1}(x_1, \cdots, x_{N-1}), \qquad (2.24)$$

where the summation is on permutations and ψ_0 is the ground state for $N-1$ particles. In second-quantized notation this would be

$$\psi_p = a_p^\dagger |\psi_0^{N-1}\rangle. \qquad (2.25)$$

The energy of this state [Eq. (2.23)] can be interpreted as follows: First, remove a particle from the N-particle ground state. This contributes an energy—μ. Second, put it back in a plane wave state with momentum p—thereby adding p^2 to the energy. But the plane wave also interacts with the remaining $N-1$ particles—this energy is $\rho v(0)$ [$v(p)$ is the Fourier transform (F.T.) of the potential]. In our case this is $2\gamma\rho^2$. Finally, when the wave function is symmetrized, additional terms appear which ordinarily vanish as $p \to \infty$. But one of these terms—the F. T. of the potential times the two-particle correlation function—does not vanish for a δ-function and is in fact just $2\gamma\rho^2$. The total result is Eq. (2.23).

Before turning to excitations of type II ("hole" states) there are several questions to be answered about the behavior of Eqs. (2.18)–(2.20) as $q \to K$.

(a) Do both p and ϵ_1 vanish in this limit? If so, is $\epsilon_1(p)$ linear for small p?

(b) Is Bogoliubov's expression for the velocity of sound correct for small γ?

(c) Do the two expressions, Eqs. (1.1) and (1.4), for the velocity of sound agree for all γ?

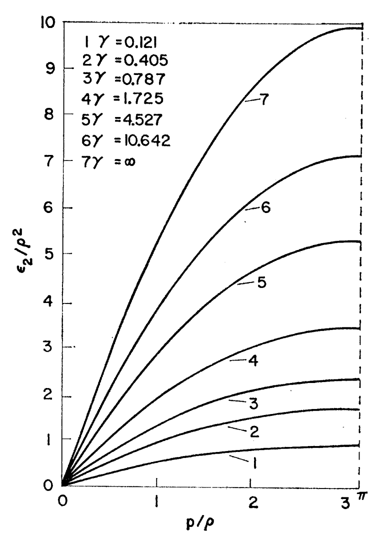

FIG. 3. The type II excitation, ϵ_2, as a function of momentum for various values of $\gamma = c/\rho$. This spectrum exists only up to $|p| = \pi\rho$. When $\gamma = 0$, $\epsilon_2 = 0$; when $\gamma = \infty$, $\epsilon_2 = -p^2 + 2\pi\rho|p|$.

All these questions can be answered in the affirmative, but the proof involves very tedious manipulation of the integral Eqs. I (3.18) and (2.18). These are summarized in the Appendix. Question (b) above is especially significant in view of the great debate as to whether or not there is an energy gap in the excitation spectrum. In our model there is none.

Type II Excitations ("Hole" States)

In this case we set all $n_i = 1$ except for $i = j$ when $n_j = 2$. Let $k_j = q$ (which we may assume to be a continuous parameter) with $0 < q < K$. In analogy with Eq. (2.14) we write

$$\begin{aligned} k_i' &= k_i + (1/L)\omega_i, \qquad (i \leq j) \\ k_i' &= k_{i+1} + (1/L)\omega_i, \qquad (i > j) \end{aligned} \qquad (2.26)$$

where now $k_i = k$'s in the $(N+1)$ particle ground state.

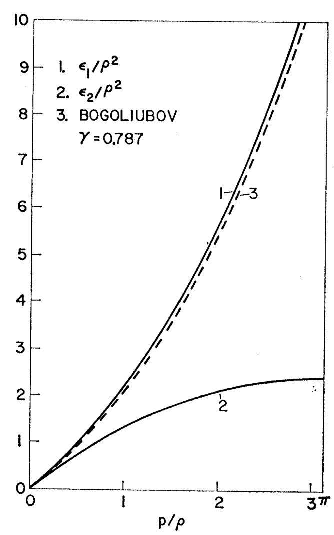

FIG. 4. A comparison plot of the two types of excitations, ϵ_1 and ϵ_2, for $\gamma = 0.787$. The dashed curve is Bogoliubov's spectrum which is quite close to the type I spectrum. The type II spectrum does not exist in Bogoliubov's theory.

As before we derive the equations

$$2\pi J(k) = 2c \int_{-K}^{K} \frac{J(r)dr}{c^2 + (k-r)^2} + \pi + \theta(q-k), \qquad (2.27)$$

$$p = -q + \int_{-K}^{K} J(k) dk, \qquad (2.28)$$

$$\epsilon_2 = \mu - q^2 + 2 \int_{-K}^{K} k J(k) dk. \qquad (2.29)$$

We observe that in this case $J(k)$ is negative definite (to see why this ought to be so, look at the $\gamma = \infty$ limit). When $q = K$, the solution to Eq. (2.27) is the negative of the solution to Eq. (2.18) so that in this limit both p and ϵ_2 vanish. As q decreases both p and ϵ_2 increase until, when $q = 0$, $p = \pi\rho$ [and not $K(\gamma)$ as might have been expected]. The case $q < 0$ need not be considered.

If we think of the Fermi case, what we have done is taken a particle from $q>0$ to K. Taking a particle from $q<0$ to K is, in our picture, to be regarded as the sum of two excitations; taking a particle from $q<0$ to $-K$ followed by an umklapp from $-K$ to K.

In Fig. 3 we show the $\epsilon_2(p)$ curves for various values of γ. The $\gamma=\infty$ case is given by Eq. (2.11). When $\gamma=0$, $\epsilon_2(p)\equiv 0$. Figure 4 compares $\epsilon_1(p)$, $\epsilon_2(p)$ and Bogoliubov's spectrum given in I Eq. (4.1) for $\gamma=0.787$. It will be seen that Bogoliubov's spectrum is a good approximation to $\epsilon_1(p)$ for this value of γ. It will also be noted that ϵ_1 and ϵ_2 have the same slope at $p=0$—a fact which may be easily proved from the integral equations. This common value of the velocity of sound is plotted in Fig. 5 together with Bogoliubov's results, Eqs. (1.6) and (1.7). With regard to the velocity of sound, we have seen that the macroscopic definition, Eq. (1.4), agrees for all γ with the microscopic definition based on the conjecture of Feynman[4–6] that the low-energy excitations are in general longitudinal phonons. But Feynman's additional conjecture that there is only one spectrum is apparently incorrect, at least in the present model. If one excites a sound wave in this system, most probably both types of excitations will be excited, just as in a Fermi gas both holes and particles are excited.

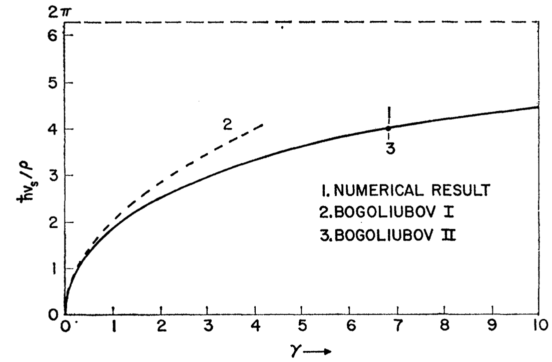

FIG. 5. The velocity of sound, v_s, as a function of $\gamma=c/\rho$. Curve 2 is the result obtained from Bogoliubov's excitation spectrum. Curve 3, which is graphically indistinguishable from the true result up to $\gamma=10$, is the velocity of sound derived from the macroscopic compressibility using Bogoliubov's expression for the ground-state energy.

III. A DISCUSSION OF ELEMENTARY EXCITATIONS AND QUASIPARTICLES

In the literature of the quantum-mechanical many-body problem two phrases are used interchangeably—elementary excitations and quasiparticles. We would like to propose here a distinction between the two.

A gas in a box has certain well-defined energies and wave functions which form a complete set. By definition these do not decay or interact with each other. The energies are real. Furthermore, as we have seen in our model and as is undoubtedly true, in general, the spectrum of low-lying states falls into a pattern. There exists one or more (in our case two) sets of energy vs momentum curves such that: (a) For each point on one of these curves there is an eigenstate; (b) if we add together the energy and momenta corresponding to several points on one or more of the curves, we obtain (to order N^{-1}) a resultant energy and momentum corresponding to an exact wave function of the system. The converse is also true: every state can be thought of as a sum of the elementary states.

These basic energy vs momentum curves we call *elementary excitations*. From this point of view, elementary excitations are a bookkeeping arrangement. There does not exist any simple operator which, acting on the ground state, gives these elementary states, nor can "compound" states be obtained from the elementary ones by simple operators. Nevertheless, when one attempts to diagonalize the many-body Hamiltonian by some method, it is the elementary excitations in the above sense that one is calculating.

Another quantity, called a *quasiparticle* may be defined as a pole in the Fourier transform of a propagator (or Green's function). The one-particle propagator is given in second quantized notation by[10] $G=G_+ + G_-$, where

$$G_+(p,t)=-i\langle\psi_0{}^N|a_p e^{-iHt}a_p{}^\dagger|\psi_0{}^N\rangle e^{iE_0t},\quad t>0,$$
$$G_-(p,t)=\pm i\langle\psi_0{}^N|a_p{}^\dagger e^{iHt}a_p|\psi_0{}^N\rangle e^{-iE_0t},\quad t<0, \tag{3.1}$$

where the $(+)$ sign is for fermions, the $(-)$ sign for bosons. The time Fourier transform of both G_+ and G_- have a branch cut on the real axis and various singularities on the wrong Riemann sheet. The poles nearest the real axis are quasiparticle poles while the other singularities reflect more complex collective modes.

The energy of a quasiparticle (i.e., the position of the pole) is complex, which means that it decays in time. Nevertheless, there is a reputed connection, which has never been made very precise between quasiparticles and elementary excitations in the sense we have used them above. The difference between the two, and the reason one decays and the other does not, lies in the form of the wave functions. The elementary excitations refer to exact eigenfunctions which unfortunately, for an interacting system, do not have the plane wave character of the excitations of a noninteracting gas. The quasiparticle, on the other hand, may be thought of as an attempt to find (inexact) wave functions which do have a plane wave character. The function $a_p{}^\dagger|\psi_0{}^N\rangle$ in Eq. (3.1) is such a plane wave type function.

Elementary excitations are more useful for calculating the free energy or any other quantity where only the energy is involved. Quasiparticles are useful for calculating the response of the system to an external influence, for, by the nature of a physical excitation process, plane waves are generally excited at time $t=0$. The propagator G samples the exact states, so to speak, to find a linear combination which looks most like the plane wave state at $t=0$.

[10] V. M. Galitskii and A. B. Migdal, Soviet Phys.—JETP 7, 96 (1958).

For a noninteracting Bose gas, G_- is identically zero. The pole in G_+ lies on the real axis and gives the spectrum $\epsilon(p)=p^2$. When the interaction is turned on, the extent to which this pole moves off the real axis depends upon the extent to which $a_p^\dagger|\psi_0{}^N\rangle$ is no longer an exact eigenstate of the $N+1$ particle system. It is this pole to which perturbation based calculations direct their attention. In our case we have seen that for large p at least, $a_p^\dagger|\psi_0{}^N\rangle$ approaches an eigenstate [Eq. (2.25)]. Unfortunately, we are not in a position to calculate G, but if one believes that quasiparticle poles and elementary excitations are similar, than our $\epsilon_1(p)$ curve must be connected with this pole in G_+.

From whence arises our second spectrum, $\epsilon_2(p)$? We believe it is connected with G_-. The negative time part of the propagator must also have singularities, but since it vanishes for zero interaction it would be very difficult indeed for perturbation theory to find them. If $\phi_p=a_p|\psi_0{}^{N+1}\rangle$ were an exact eigenstate of the N-particle problem, G_- would have a pole on the real axis. For the noninteracting case $\phi_p\equiv 0$, but if we look at our type II wave functions it will be seen that to a rough approximation they are just ϕ_p. Thus, we are led to the view that our type II spectrum corresponds to a quasiparticle pole of the negative time part of the one body propagator.

We do not propose to elucidate here the connection between elementary excitations and quasiparticles—we merely point out that these two types of objects can be defined. Neither will we attempt to evaluate G. But the above discussion is pertinent to the outstanding question of whether our double excitation spectrum has any relevance in three dimensions.

It is to be noted, first of all, that all the quantities we have calculated are smoothly dependent on γ and behave just as one intuitively expects them to behave in any number of dimensions. Although we have frequently called attention to the analogy of this gas with a Fermi system, and although this Fermi gas analogy clearly breaks down in three dimensions, it should not be supposed that the double spectrum result is a consequence of one dimension forcing the Bose system to look like a Fermi system. While it may be argued that the salient difference between one and three dimensions is that in three dimensions particles can "get around each other," the true state of affairs is expressed by the functions $v(\gamma)$ and $t(\gamma)$ plotted in Fig. 1 of I. As we have already mentioned, the fact that the potential is effectively a kinetic energy barrier for large γ, a result that also holds in three dimensions, means that it is really immaterial to the particles whether they can "get around each other" or merely "through each other."

Thus, there appears to be no truly basic physical (as distinguished from mathematical) distinction between our gas and a similar gas in three dimensions. It is quite possible, therefore, that a double (or perhaps more fold) spectrum exists in three dimensions. Another possibility is that in three dimensions the double spectrum does not exist for small potentials, but makes its appearance when the potential becomes sufficiently strong. In the theory of a lattice of coupled harmonic oscillators,[11] for example, it is known that one imperfect light mass may cause an isolated energy level to appear above the perfect lattice band. In one dimension this always happens; in three dimensions it occurs only if the imperfect mass is sufficiently small. If this second possibility—the onset of a double spectrum for large potentials—were to occur, then clearly perturbation theory would be hard pressed to predict it.

We turn finally to an argument based on the previous considerations of this section. For the very same reasons that it is believed that the positive time part of the propagator has a pole near the real axis, the negative time part too may be presumed to have a similar pole. For what reason is this pole any less "elementary" than its counterpart in the positive time part of the propagator?

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APPENDIX: THE EXCITATION SPECTRUM AT LOW MOMENTUM AND THE VELOCITY OF SOUND

We justify here the claims made in Sec. II about the excitation spectrum of type I (a similar analysis applies to type II). We want to show the absence of a gap and the equivalence of the two definitions of v_s, Eqs. (1.1) and (1.4).

We first transform Eq. (2.18) to dimensionless form by defining

$$k=Kx;\quad J(Kx)=j(x);\quad q=Ks;\quad c=K\lambda;\tag{A1}$$

where $s\geq 1$. Expanding the inhomogeneous term in Eq. (2.18) for $s\sim 1$, we obtain

$$2\pi j(x)=2\lambda\int_{-1}^{1}\frac{j(y)dy}{\lambda^2+(x-y)^2}-\pi+2\tan^{-1}\left(\frac{1-x}{\lambda}\right)+(s-1)\frac{2\lambda}{\lambda^2+(1-x)^2}.\tag{A2}$$

We write

$$j(x,s)=j_0(x)+(s-1)j_1(x),\tag{A3}$$

and define

$$2\phi_1(x)=j_0(x)+j_0(-x),\tag{A4}$$

$$2\phi_2(x)=j_0'(x)+j_0'(-x),\tag{A5}$$

$$2\phi_3(x)=j_1(x)+j_1(-x),\tag{A6}$$

$$2\phi_4(x)=j_1(x)-j_1(-x),\tag{A7}$$

[11] A. Maradudin, P. Mazur, E. Montroll, and G. Weiss, Rev. Mod. Phys. **30**, 175 (1958).

where "prime" means differentiation with respect to x. We easily obtain the equations

$$2\pi\phi_1(x,\lambda)=\int \mathrm{Ker}\phi_1-\pi+\tan^{-1}\left(\frac{1-x}{\lambda}\right)+\tan^{-1}\left(\frac{1+x}{\lambda}\right),\quad \text{(A8)}$$

$$2\pi\phi_2(x,\lambda)=\int \mathrm{Ker}\phi_2-\lambda\left[\frac{1}{\lambda^2+(1-x)^2}+\frac{1}{\lambda^2+(1+x)^2}\right]\times[1+j_0(1)-j_0(-1)],\quad \text{(A9)}$$

$$2\pi\phi_3(x,\lambda)=\int \mathrm{Ker}\phi_3+\lambda\left[\frac{1}{\lambda^2+(1-x)^2}+\frac{1}{\lambda^2+(1+x)^2}\right],\quad \text{(A10)}$$

$$2\pi\phi_4(x,\lambda)=\int \mathrm{Ker}\phi_4+\lambda\left[\frac{1}{\lambda^2+(1-x)^2}-\frac{1}{\lambda^2+(1+x)^2}\right],\quad \text{(A11)}$$

where $\int$ Ker means the integral operator appearing in Eq. (A2).

The function $g(x,\lambda)$ introduced in I satisfies

$$2\pi g(x)=\int \mathrm{Ker}g+1,\quad \text{(A12)}$$

while the function

$$h(x,\lambda)=\left(\frac{\partial}{\partial\lambda}+\frac{x}{\lambda}\frac{\partial}{\partial x}\right)g\quad \text{(A13)}$$

satisfies

$$2\pi h(x)=\int \mathrm{Ker}h-2g(1,\lambda)\left[\frac{1}{\lambda^2+(1-x)^2}+\frac{1}{\lambda^2+(1+x)^2}\right].\quad \text{(A14)}$$

A simple manipulation of Eq. (A12) shows that the function $g'=\partial g/\partial x$ satisfies

$$2\pi g'=\int \mathrm{Ker}g'-2\lambda g(1,\lambda)\left[\frac{1}{\lambda^2+(1-x)^2}-\frac{1}{\lambda^2+(1+x)^2}\right].\quad \text{(A15)}$$

Comparing Eqs. (A10) and (A11) with Eqs. (A14) and (A15) we find that

$$\phi_3(x,\lambda)=-\frac{\lambda}{2g(1,\lambda)}h(x,\lambda),\quad \text{(A16)}$$

and

$$\phi_4(x,\lambda)=-\frac{1}{2g(1,\lambda)}g'(x,\lambda).\quad \text{(A17)}$$

Equation (A8) may be solved by inspection, i.e.,

$$\phi_1(x,\lambda)=-\tfrac{1}{2}.\quad \text{(A18)}$$

Comparison of Eqs. (A9) and (A10), together with the observation that

$$\int_{-1}^{1} dx\, \phi_2(x,\lambda)=j_0(1)-j_0(-1),\quad \text{(A19)}$$

yields the result

$$\phi_2(x,\lambda)=-\phi_3(x,\lambda)\left[1+\int_{-1}^{1} dx\, \phi_3(x,\lambda)dx\right].\quad \text{(A20)}$$

Integrating Eq. (A14) with respect to x and using Eq. (A16) we find that

$$\int_{-1}^{1} dx\frac{\partial g}{\partial\lambda}=-\frac{2}{\lambda}g(1,\lambda)\left[1+\int_{-1}^{1} dx\, \phi_3(x,\lambda)\right]+\frac{1}{\gamma},\quad \text{(A21a)}$$

$$\int_{-1}^{1} dx\, x^2\frac{\partial g}{\partial\lambda}=-\frac{2}{\lambda}g(1,\lambda)\left[1+\int_{-1}^{1} dx\, x^2\phi_3(x,\lambda)\right]+\frac{3}{\lambda}\int_{-1}^{1} dx\, x^2 g(x,\lambda),\quad \text{(A21b)}$$

where we have used the definition of λ/γ given in I (3.20):

$$\frac{\lambda}{\gamma}=\int_{-1}^{1} dx\, g(x,\lambda)=\frac{\rho}{K}.\quad \text{(A22)}$$

Now, since

$$e(\gamma)=\frac{\gamma^3}{\lambda^3}\int_{-1}^{1} g(x,\lambda)x^2dx,\quad \text{(A23)}$$

[cf. I Eq. (3.19)], the definition of μ, Eq. (1.5) together with Eqs. (A21), is equivalent to

$$\mu=K^2\left[1+\int_{-1}^{1} dx\, \phi_3(x)x^2\right]\Big/\left[1+\int_{-1}^{1} dx\, \phi_3(x)\right].\quad \text{(A24)}$$

In terms of the functions ϕ_1 to ϕ_4, Eq. (2.19) for the momentum takes the form (for $s\sim1$)

$$p=K\left[1+\int_{-1}^{1} dx\, \phi_1(x)\right]+K(s-1)\left[1+\int_{-1}^{1} dx\, \phi_3(x)\right].\quad \text{(A25)}$$

The expression for ϵ_1, Eq. (2.20), takes the form (for $s\sim 1$)

$$\epsilon_1=-\mu+K^2+K^2\int_{-1}^{1}dx\,\phi_2(x)(1-x^2)+2K^2(s-1)\left[1+\int_{-1}^{1}dx\,x\phi_4(x)\right] \quad \text{(A26)}$$

Inserting Eqs. (A18), (A20), and (A24) into Eqs. (A25) and (A26) we find that

$$\lim_{s\to 1}p=0;\quad \lim_{s\to 1}\epsilon_1=0; \quad \text{(A27)}$$

thereby proving the assertion that there is no energy gap and that the lower end of the spectrum is in fact $p=0$.

Inserting Eqs. (A17) and (A22) into Eq. (A26), we find that for small s (i.e., small p)

$$\epsilon_1(p)=p\left\{g(1,\lambda)\left[1+\int_{-1}^{1}dx\,\phi_3(x)\right]\right\}^{-1}\rho. \quad \text{(A28)}$$

By Eq. (1.1), the velocity of sound, v_s, is, therefore, the factor $\{\ \}^{-1}\rho$ in Eq. (A28). It is always positive since ϕ_3 is positive. An alternative expression for v_s can be obtained from Eq. (A21a), viz.,

$$v_s=\frac{2}{\lambda}\left[\frac{1}{\gamma}-\int_{-1}^{1}dx\frac{\partial}{\partial\lambda}g(x,\lambda)\right]\rho. \quad \text{(A29)}$$

In I we showed that for λ small, $g(x,\lambda)\sim(1/2\pi\lambda)(1-x^2)^{1/2}$ and $4\lambda^2=\gamma$. Substituting this into Eq. (A29) we obtain

$$v_s\simeq 2\gamma^{1/2}\rho,\quad (\text{small }\gamma), \quad \text{(A30)}$$

thereby confirming Bogoliubov's result, Eq. (1.6).

The last step is to confirm that Eqs. (1.4) and (A29) agree. We must calculate $d\mu/d\gamma=(d\lambda/d\gamma)(d\mu/d\lambda)$. The factor $d\lambda/d\gamma$ can be obtained from Eq. (A22) and (A21a), viz.,

$$\frac{d\lambda}{d\gamma}=\frac{1}{2}\frac{\lambda^2}{\gamma^2}\left\{g(1,\lambda)\left[1+\int_{-1}^{1}dx\,\phi_3(x)\right]\right\}^{-1}. \quad \text{(A31)}$$

To obtain $d\mu/d\lambda$ from Eq. (A24) the essential point is to evaluate $(\partial/\partial\lambda)\phi_3(x,\lambda)$. To this end we define the auxiliary function

$$m(x,\lambda)=\left(\frac{\partial}{\partial\lambda}+\frac{x}{\lambda}\frac{\partial}{\partial x}\right)\phi_3(x,\lambda). \quad \text{(A32)}$$

In analogy with Eqs. (A13) and (A14), this function satisfies

$$2\pi m(x,\lambda)=\int \mathrm{Ker}\,m-[1+2\phi_3(1,\lambda)]\left[\frac{1}{\lambda^2+(1-x)^2}+\frac{1}{\lambda^2+(1+x)^2}\right]+2\left[\frac{1-x}{[\lambda^2+(1-x)^2]^2}+\frac{1+x}{[\lambda^2+(1+x)^2]^2}\right]. \quad \text{(A33)}$$

On the other hand, as we see from Eq. (A15), the function $g''=\partial^2g/\partial x^2$ satisfies

$$2\pi g''=\int \mathrm{Ker}\,g-2\lambda g'(1,\lambda)\left[\frac{1}{\lambda^2+(1-x)^2}+\frac{1}{\lambda^2+(1+x)^2}\right]-4\lambda g(1,\lambda)\left[\frac{1-x}{[\lambda^2+(1-x)^2]^2}+\frac{1+x}{[\lambda^2+(1+x)^2]^2}\right]. \quad \text{(A34)}$$

Comparing Eqs. (A10), (A33), and (A34) we find that

$$m(x,\lambda)=-\frac{1}{2\lambda g(1,\lambda)}g''(x,\lambda)-\frac{1}{\lambda}\left[1+2\phi_3(1,\lambda)+\frac{g'(1,\lambda)}{g(1,\lambda)}\right]\phi_3(x,\lambda). \quad \text{(A35)}$$

In analogy with Eqs. (A21), $d\mu/d\lambda$ may be written in terms of $m(x,\lambda)$ which in turn may be written in terms of ϕ_3, g, g', and g'' by using Eq. (A35). We shall not carry out the tedious manipulation here, but one eventually finds that Eq. (1.4) reduces to Eq. (A29).

An alternative expression for v_s may be obtained by comparing Eqs. (A22), (A28), and (A31). One easily finds that

$$v_s=2\rho\frac{\gamma^3}{\lambda^3}\frac{d\lambda}{d\gamma}=2K-2\gamma\frac{\partial K}{\partial\gamma}. \quad \text{(A36)}$$