Thermodynamic Behavior of the Heisenberg Ferromagnet*†

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A finite-temperature perturbation theory is presented for the Heisenberg model with the object of providing a formalism in which contact can be made with the low-temperature treatment by Dyson, with the random phase approximation of Englert, and, above the Curie point, with high-density treatments of the Ising model. A linked cluster expansion is set up and a simple high-density classification, valid above the Curie point, is applied. The first two terms in the high-density series, tree graphs and ring graphs, yield, respectively, molecular field theory and a form reducing to spin-wave results at low temperatures. A low-temperature classification is then developed which leads to an expansion of the free energy in powers of T in which the terms have the form of those describing bosons with an effective interaction similar to Dyson's $\Gamma_{\rho\sigma}{}^{\lambda}$. The first two terms are the low-temperature approximations of trees and rings, respectively, which justifies the use of the high-density expansion below the Curie point. The next term, including all the effects of spin-wave interactions up to T^4 in the free energy, contains the Born approximation series presented by Dyson. In particular, the cancellation of T^3 terms in the leading Born approximation is demonstrated. A renormalized version of the high-density expansion necessary to treat the region of the Curie point is then considered, and its approximation by an "excluded volume" sum is shown to yield the Curie point of the spherical model, in common with the random phase approximation and with high-density approximations to the Ising model. The extent to which the high-density theory misrepresents the effect of spin-wave interactions is then discussed. In an Appendix an equations-of-motion approach to the random phase approximation and to the interactions between spin waves is presented.

1. INTRODUCTION

RECENTLY, there have been attempts[1,2] to provide an approximate solution, valid throughout the whole range of temperatures, for the thermodynamic behavior of the Heisenberg ferromagnet. The methods of reference 1 were based upon the "random phase approximation" (RPA) in which the commutator of the spin operators is replaced by the average value of the true commutator. Similar results were obtained by Bogolyubov and Tyablikov[2] using a Green's function approach in which the approximation is an assumed factorization of the two-particle Green's function. At low temperatures the results of the spin-wave theory[3] are recovered and at slightly increased temperatures the theory predicts deviations proportional to T^3 in the spin-wave magnetization. As the temperature is raised still higher, the spontaneous magnetization decreases monotonically and in zero field becomes zero for all temperatures above a well-defined critical temperature. The critical temperature of the model turns out to be that of the spherical model of Berlin and Kac,[4] which Brout[5] has shown to be a high-density approximation for the Ising model.

It would, therefore, be desirable to know to what approximation the RPA corresponds. A second question exists: The results of RPA disagree with a low-temperature expansion for the free energy of the Heisenberg model worked out by Dyson.[6] Dyson found that the first deviations of the magnetization from that of the Bloch theory occurred in the T^4 terms in the low-temperature expansion.

The purpose of this paper is to investigate the cause of this disagreement, and to remove the discrepancy by going outside of the scope of RPA as we find to be necessary. We further attempt to investigate the relation between RPA and the high-density theories.[7]

In order to carry out this program, we have developed a formal expansion which allows us to write down, at least in principle, all the contributions to any thermodynamic variable, for instance the energy or magnetization. The terms in the expansion may be classified according to their order with respect to a simple high-density expansion[7] at the Curie point, after the manner of reference 5, and we have also developed a method of

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[5] R. Brout, Phys. Rev. **118**, 1009 (1960); **122**, 469 (1961).
[6] F. J. Dyson, Phys. Rev. **102**, 1217, 1230 (1956).
[7] We shall later find it necessary to differentiate between two types of high-density theory, one a simple high-density theory, the other a renormalized or self-consistent high-density theory. The first type is based on a classification, given in Sec. 3, of the terms in the free energy with respect to their explicit dependence on a high-density parameter. The second type is a modification of the first, necessary for the description of the Curie point. In the version of Horwitz and Callen (see reference 12), the simple high-density classification is used to select a class of irreducible graphs. The renormalized version of the high-density theory then involves the summation of all graphs formed from articulations of these irreducible elements, and is an approximation in which the variational principle is maintained.

classifying graphs with respect to an expansion in powers of the temperature. The classes of graphs which produce the leading and next order in the simple high-density classification turn out to be those giving, respectively, all the terms of leading and next order in the low-temperature expansion. We, therefore, recover molecular field theory, exact in the limit of infinite density, or zero temperature, by summing the leading class of graphs. Summation of the second class of graphs, rings, produces spin-wave theory at low temperatures. In the next order, the two classifications select different sets of graphs; the first spin-wave scattering effects appear in this order in the temperature expansion. Thus, one cannot hope to get a uniform simple extrapolation based on the simple high-density notion that also gives the lowest order effects of spin-wave scattering.

We next consider the Curie point region. Here a renormalized version of the high-density theory is required.[7] The formal expression for the sum of (renormalized) high density terms is difficult to evaluate. We have replaced it by an "excluded volume" sum which approximates it in a way to be discussed. The "excluded volume" summation is shown to lead to the same critical temperature as that obtained by the RPA and possesses other features in common with it. The analysis is for convenience carried out for spin $S=1/2$, but the results may be readily generalized to arbitrary spin.

In Sec. 2 the linked cluster expansion is developed. In Sec. 3, we review the simple high-density classification and derive molecular field theory graphically. In Sec. 4 we present a graphical derivation of spin-wave theory in such a form that the contributing graphs may be easily extrapolated to arbitrary temperature. The low-temperature ordering is not presented until Sec. 5, since it depends on the results of the ring graph summation. Section 6 contains the calculation of the simplest spin-wave–spin-wave scattering graphs, which give Dyson's first Born approximation to the scattering of spin waves. In Sec. 7, the renormalized high-density approximation to the Curie point region is developed and discussed. The conclusion is that for practical purposes the magnetization curve of RPA seems to be adequate and that the spin-wave interaction terms of Dyson are probably never important since the kinematical interactions set in before the dynamical ones achieve any importance. In Sec. 8, the content of the paper is reviewed and the application of the theory to the antiferromagnet is indicated. Finally, the first Appendix contains an equations of motion approach to RPA and to the problem of interactions between spin waves.

2. THE LINKED CLUSTER EXPANSION

In this section[8] we derive a perturbation expansion for the statistical mechanical averages $\langle Q\rangle=\mathrm{Tr}(\rho Q)$ of any dynamical variable Q in the canonical[9] ensemble, where

$$\rho=e^{-\beta H}/(\mathrm{Tr}e^{-\beta H}), \tag{1}$$

H being the Hamiltonian of the system. For the Heisenberg ferromagnet in the presence of an external magnetic field B,

$$H=-h\sum_i S_i^z-\tfrac{1}{2}\sum_{i,j} v_{ij}\mathbf{S}_i\cdot\mathbf{S}_j,$$
$$h=g\mu_B B, \quad v_{ij}>0 \tag{2}$$

$\mathbf{S}_i$ is the spin operator of the ith site and we have chosen the z axis along the direction of the magnetic field. We choose to write the x and y components of $\mathbf{S}$ in terms of the transverse operators

$$S^{\pm}=S^x\pm iS^y, \tag{3}$$

which have the commutation relations

$$[S_i^z,S_j^+]=\delta_{ij}S_j^+, \tag{4a}$$
$$[S_i^z,S_j^-]=-\delta_{ij}S_j^-, \tag{4b}$$
$$[S_i^+,S_j^-]=2\delta_{ij}S_j^z. \tag{4c}$$

The external magnetic field has been introduced for two reasons: First it removes degeneracy, and secondly, it provides a soluble unperturbed Hamiltonian

$$H_0=-h\sum_i S_i^z. \tag{5}$$

Corresponding to this unperturbed Hamiltonian there is a factorable density matrix

$$\rho_0=e^{-\beta H_0}/(\mathrm{Tr}e^{-\beta H_0})=\prod_i \rho_0^{(i)}, \tag{6}$$

where

$$\rho_0^{(i)}=e^{-\beta H_0^{(i)}}/(\mathrm{Tr}e^{-\beta H_0^{(i)}}) \tag{7}$$

and

$$H_0^{(i)}=-hS_i^z. \tag{8}$$

Denoting by H_1 the interaction

$$H_1=H-H_0=-\tfrac{1}{2}\sum_{i,j} v_{ij}\mathbf{S}_i\cdot\mathbf{S}_j, \tag{9}$$

the average of Q becomes

$$\langle Q\rangle=\mathrm{Tr}(\rho Q) \tag{10}$$
$$=\langle e^{\beta H_0}e^{-\beta(H_0+H_1)}Q\rangle_0/\langle e^{\beta H_0}e^{-\beta(H_0+H_1)}\rangle_0, \tag{11}$$

where

$$\langle Q\rangle_0=\mathrm{Tr}(\rho_0 Q) \tag{12}$$

is the unperturbed average of Q. The quantity $\exp(\beta H_0)\times\exp[-\beta(H_0+H_1)]$ can be expanded as a power series in H_1 in the familiar way:

$$\exp(\beta H_0)\exp[-\beta(H_0+H_1)]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta\cdots\int_0^\beta T[H_1(\beta_n)\cdots H_1(\beta_2)H_1(\beta_1)]$$
$$\times d\beta_1 d\beta_2\cdots d\beta_n. \tag{13}$$

[8] The development is similar to that used for the Ising problem by F. Englert, Phys. Rev. **129**, 567 (1963).

[9] This is a canonical ensemble in the sense of there being a fixed number of spins.

Here, the operators $H_1(\beta_i)$ are interaction picture operators defined by

$$Q(\beta')=\exp(\beta' H_0)Q\exp(-\beta' H_0), \tag{14}$$

and the symbol T is the time-ordering operator of Dyson,[10] placing the operators in such a way that $H_1(\beta_i)$ stands to the left of any $H_1(\beta_j)$ with $\beta_j<\beta_i$. In the case when H_0 and H_1 commute, Eq. (13) can be reduced to the exponential series $\sum_n(1/n!)(-\beta H_1)^n$. In general,

$$\langle Q\rangle=\frac{\left\langle \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta\cdots\int_0^\beta T[H_1(\beta_n)\cdots H_1(\beta_1)Q(0)]d\beta_1\cdots d\beta_n\right\rangle_0}{\left\langle \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta\cdots\int_0^\beta T[H_1(\beta_n)\cdots H_1(\beta_1)]d\beta_1\cdots d\beta_n\right\rangle_0}. \tag{15}$$

Any Q will be a product of the operators $\mathbf{S}_i$ in some definite order. It may always be written as $f(i_1\alpha_1,\cdots, i_n\alpha_n)S_{i_1}{}^{\alpha_1}(\beta_1)\cdots S_{i_n}{}^{\alpha_n}(\beta_n)$, where i labels the spin index, α labels the component $(+,-,$ or $z)$ of the spin operator, and f is some c number. Further,

$$\begin{aligned}H_1(\beta_i)&=-\tfrac{1}{2}\textstyle\sum_{i,j}v_{ij}\mathbf{S}_i(\beta_i)\cdot\mathbf{S}_j(\beta_i)\\&=-\tfrac{1}{2}\textstyle\sum_{i,j}v_{ij}[S_i{}^{+}(\beta_i)S_j{}^{-}(\beta_i)+S_i{}^{z}(\beta_i)S_j{}^{z}(\beta_i)],\end{aligned} \tag{16}$$

so that the numerator and denominator of Eq. (15) contain time-ordered averages of the form $\langle T\prod_n(S_{i_n}{}^{\alpha_n}(\beta_n))\rangle_0$. Because ρ_0 is factorable, this separates into products of separate averages in each of which the operators have the same spin index. The spin operators from Q are now averaged together with spins with the same label coming from the interaction. If it were not for these terms, Q could be averaged independently from the interaction terms and the denominator would cancel the interaction term in the numerator of Eq. (15), leaving $\langle Q\rangle_0$ only. The correlation of the spins in Q with those of the same index in the interaction is, of course, essential. Nevertheless, a partial simplification of the above type can be arrived at by using a semi-invariant method.[8,11,12] The classical semi-invariants $M_n(x)$ are defined by

$$\ln\langle\exp(tx)\rangle_0=\sum_{n=1}^{\infty}\frac{t^n}{n!}M_n(x). \tag{17}$$

By replacing tx by $\sum_\nu t^{(\nu)}x^{(\nu)}$ one defines nth order semi-invariants of mixed arguments:

$$\ln\langle\exp(\textstyle\sum_\nu t^{(\nu)}x^{(\nu)})\rangle_0$$

$$=\prod_\nu\left[\sum_{n_\nu=0}^{\infty}\frac{t^{(\nu)n_\nu}}{n_\nu!}\right]M_{\Sigma n_\nu>0}(\prod_\nu x^{(\nu)n_\nu}), \tag{18}$$

so that

$$\prod_\nu(\partial/\partial t^{(\nu)})^{n_\nu}|_{t^{(\nu)}=0}\ln\langle\exp(\textstyle\sum_\nu t^{(\nu)}x^{(\nu)})\rangle_0$$

$$=M_{\Sigma n_\nu>0}(\prod_i x^{(\nu)n_\nu}). \tag{19}$$

In the problem we are considering, the quantities $x^{(\nu)}$ are the spin operators which do not commute. Here it is necessary to introduce time-ordered semi-invariants by a simple generalization of these definitions:

$$\prod_{\alpha i}(\partial/\partial t^{(\alpha i)})^{n_{\alpha i}}|_{t^{\alpha i}=0}\ln\langle T\exp(\textstyle\sum_{\alpha i}S^{(\alpha i)}(\beta_{\alpha i})t^{(\alpha i)})\rangle_0$$

$$=M_{\Sigma n_{\alpha i}>0}(T\prod_{\alpha i}[S^{(\alpha i)}(\beta_{\alpha i})]^{n_{\alpha i}}). \tag{20}$$

Also, we define the "raising" operators $\Delta^{(\alpha i)}(\beta_{\alpha i})$ so that

$$\Delta^{(\alpha i)}(\beta_{\alpha i})M_n(T\prod_{\alpha j}[S^{(\alpha j)}(\beta_{\alpha j})]^{n_{\alpha j}})$$

$$=M_{n+1}(T\prod_{\alpha j\neq\alpha i}[S^{(\alpha j)}(\beta_{\alpha j})]^{n_{\alpha j}}[S^{(\alpha i)}(\beta_{\alpha i})]^{n_{\alpha i}+1}). \tag{21}$$

Then it follows (see Appendix B) that

$$\langle T\prod_{\alpha i}[S^{(\alpha i)}(\beta_{\alpha i})]^{n_{\alpha i}}\rangle_0$$

$$=\prod_{\alpha i}\{M_1(TS^{(\alpha i)}(\beta_{\alpha i}))+\Delta^{(\alpha i)}(\beta_{\alpha i})\}^{n_{\alpha i}}. \tag{22}$$

This formula expresses the product on the left-hand side as a sum of products of semi-invariants where the sum runs over all subdivisions of the product in question. It is an expression analogous to the expression of the W function in terms of products of U functions as

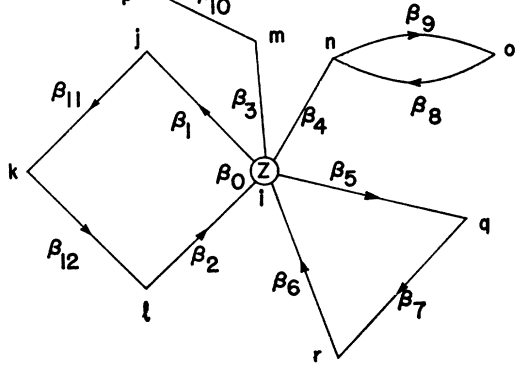

FIG. 1. An example of the way in which the structure at a vertex determines the semi-invariant.

[10] F. J. Dyson, Phys. Rev. **75**, 486 (1949).
[11] R. Brout and F. Englert, Phys. Rev. **120**, 1519 (1960).
[12] G. Horwitz and H. B. Callen, Phys. Rev. **124**, 1757 (1961).

given in Kahn and Uhlenbeck's classical work on the theory of condensation.[13] To illustrate both the application of (22) and the form of the time-ordered semi-invariant, we give the first few members of Eq. (22) remembering that $\langle S_i^{\pm}\rangle_0=\langle S_i^z S_i^{\pm}\rangle_0=0$ and $\langle S^z(\beta_i)\rangle_0$ is independent of β_i:

$$\langle S^z(\beta_i)S^z(\beta_j)\rangle_0=M_2+M_1M_1, \tag{23}$$

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_0$$

$$=M_2(TS_i^+(\beta_i)S_j^-(\beta_j)) \tag{24}$$

$$=e^{-(\beta_i-\beta_j)h}\times\begin{cases}\langle S_i^+S_j^-\rangle_0, & \beta_i>\beta_j\\ \langle S_j^-S_i^+\rangle_0, & \beta_i<\beta_j\end{cases} \tag{25}$$

$$\langle TS^z(\beta_k)S^+(\beta_j)S^-(\beta_i)\rangle_0$$

$$=M_3(TS^z(\beta_k)S^+(\beta_j)S^-(\beta_i))$$

$$+M_1(S^z(\beta_k))M_2(TS^+(\beta_j)S^-(\beta_i)). \tag{26}$$

We now use the semi-invariant development of each of the products that appear in the numerator and denominator of Eq. (15).

We represent by graphs each term which arises from expressing the numerator of Eq. (15) in terms of the semi-invariants. Each interaction is represented by a bond carrying the temperature label and the indices of the spins which interact. In order to distinguish between the longitudinal and the transverse parts of the interaction, the bonds representing the transverse part $-\frac{1}{2}\sum_{ij}v_{ij}S_i^+(\beta_i)S_j^-(\beta_i)$ also carry an arrow directed from the index corresponding to the S^+ to that corresponding to the S^-. The semi-invariants are made up of the averages of the spin operators. The spin operators appear either from H_1 or from Q. If we represent a vertex containing operators S^+, S^-, or S^z from Q by a circle ○ with the appropriate symbols (+, −, or z) written inside, the semi-invariants are then given completely by the structure at the vertices: The order of the semi-invariant associated with a given vertex is the number of bonds jointed at that vertex plus the number of operators from Q associated with the vertex, and the numbers of S^+'s and S^-'s in the semi-invariant are given by the number of bonds carrying arrows directed out from, and into the vertex, respectively, together with the transverse operators from Q. For instance, in the graph shown in Fig. 1, the vertex labeled i corresponds to the time-ordered semi-invariant

$$M_7(TS_i^-(\beta_2)S_i^+(\beta_1)S_i^z(\beta_0)S_i^z(\beta_3)S_i^z(\beta_4)S_i^+(\beta_5)S_i^-(\beta_6)).$$

The building blocks from which the graphs are constructed are the operators from Q and the individual interaction terms $\frac{1}{2}v_{ij}S_i^{\alpha i}S_j^{\alpha j}$ from H_1. The nth order term in the expansion of the numerator of Eq. (15) is a sum of terms each containing a product of n individual interactions. In many of the terms more than one interaction involves a given spin index. Further, a spin index occurring in Q may occur in one or more of the interaction terms. In these cases, the average $\langle\cdots\rangle_0$ is of a product of spins, and using Eq. (22) to write this in terms of the semi-invariants the graphs so constructed are all those that can be made by starting with the isolated bonds, and operators from Q, and joining the indices labeled with the same letter together in all possible ways. Thus, the expansion of the numerator of Eq. (15) into the semi-invariant diagrams produces all possible diagrams that can be drawn from the bonds and vertices ○. The next step is to sum all indices and integrate over all temperatures. Then the same graph appears $n!2^{n_l}/g$ times, where n_l is the number of longitudinal bonds and g is the number of symmetry operations which transform the graph into itself. This statement follows from the fact that each of the n factors $\frac{1}{2}v_{ij}$ appears in any one of the n factors of H_1 giving a factor $n!$. For longitudinal graphs, a bond is undirected and, therefore, can have either of its orientations. Transverse bonds are all oriented. Hence, the factor 2^{n_l}. The factor $2^{n_l}n!$ is clearly redundant if there are g operations which turn a graph into itself, and one must divide by g. g includes rotations, reflections, interchange among the bonds connecting two given vertices, and interchange of identical unlinked parts (the definition of linked is given below).

Clearly, the $n!$ in this factor cancels the $1/n!$ appearing in the numerator of Eq. (15). We now define an unlinked graph as one which separates into two or more parts which are not connected by any bonds (the vertices of Q are to be considered as a linked unit). Because of the cancellation of the $(n!)$'s, the unlinked graph is composed of a product of linked parts each of which contributes independent factors, multiplied by an additional g factor due to the symmetry between identical unlinked parts. Clearly, the total g factor is a product of a g for the part linked to Q multiplied by a g for all the parts unlinked to it. Similar statements apply to a diagram expansion of the denominator. The expansion of numerator and denominator differs only in that the operators of Q provide extra vertices to which we can join bonds. The numerator expansion is, therefore, equivalent to all graphs linked to the operators of Q, multiplied by all graphs unlinked to these graphs. However, this second factor is just the expansion of the denominator. Hence, we arrive at the linked cluster expansion: $\langle Q\rangle$ is given by summing all graphs

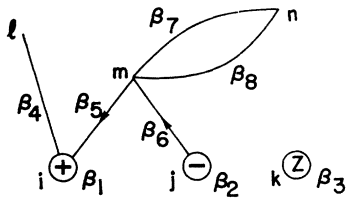

FIG. 2. A typical graph in the linked cluster expansion.

[13] B. Kahn, thesis, Amsterdam, 1938 (N. V. Noord-Hollandsche Uitgeversmaatschappij), Chap. III; B. Kahn and G. E. Uhlenbeck, Physica **6**, 399 (1938); H. D. Ursell, Proc. Cambridge Phil. Soc. **23**, 685 (1927).

linked to one or more of the operators of Q according to the following rules:

(a) Each bond carries labels i, j, β_i, and the contribution $\frac{1}{2}v_{ij}$.

(b) Each vertex specifies a time-ordered semi-invariant whose order is the number of operators of Q involved plus the number of bonds joined to that vertex. The argument of the semi-invariant depends on the type of bonds, and their temperature labeling in the way given above.

(c) Each graph carries a factor $2^{nl}/g$.

(d) The final step consists of free summation over all spin indices, and integration of all temperatures from 0 to β, with due regard to the time ordering.

For example, the graph of Fig. 2 contributes the following expression to $\langle TS_i^+(\beta_1)S_j^-(\beta_2)S_k^z(\beta_3)\rangle$:

$$\tfrac{1}{2}\cdot\tfrac{1}{4}\sum_{lmn}(v_{mn})^2 v_{il}v_{mi}v_{mj}\int_0^\beta d\beta_4\int_0^\beta d\beta_5\int_0^\beta d\beta_6\int_0^\beta d\beta_7\int_0^\beta d\beta_8 M_1(S^z(\beta_4))M_3(TS^+(\beta_1)S^-(\beta_5)S^z(\beta_4))$$
$$\times M_2(TS^z(\beta_7)S^z(\beta_8))M_4(TS^z(\beta_1)S^z(\beta_8)S^+(\beta_6)S^-(\beta_5))M_2(TS^+(\beta_6)S^-(\beta_2))M_1(S^z(\beta_3)).$$

The extra $\frac{1}{2}$ arises from the symmetry of the graph under interchange of the bonds β_7 and β_8. The dummy variables β_4, β_5, β_6, β_7, β_8, l, m, n are superfluous and will in future be omitted from graphs. Finally, it should be noted that because the unperturbed average of a product of spin operators containing unequal numbers of S^+'s and S^-'s vanishes, in a nonvanishing graph the bonds carrying arrows form continuous paths which can only begin or end on a vertex containing transverse operators from Q. It will be convenient in a subsequent section to use the perturbation theory to calculate the free energy F. F is related to the average energy by

$$-\frac{\partial}{\partial\beta}(-\beta F)=\langle H\rangle.$$

It follows that F is, as usual, given by calculating the graphs obtained by closing all energy graphs, with the symmetry factors appropriate to the closed graph.

3. MOLECULAR FIELD THEORY

The most important qualitative features of ferromagnetism are given by the Weiss molecular field theory (or self-consistent field theory). This theory also gives a successful semiquantitative account of the Curie point phenomena apart from critical fluctuations. As has been pointed out repeatedly, this success is due to the fact that the theory is the leading term in a high-density treatment of the ferromagnet. Molecular field theory also gives correctly the first two terms in an expansion in powers of $\exp(-\frac{1}{2}\beta\bar{v})$ for the Ising model, but for the Heisenberg model it must be modified at low temperatures since it fails to give spin waves.

In this section we first present the simple high-density classification of the terms in the linked cluster expansion in the Curie point region. We then isolate terms of leading order in the expansion and show that these terms sum to the molecular field theory. The summation, which proceeds in the manner presented by Horwitz and Callen,[12] is included here for completeness, as well as an introduction to the vertex renormalization procedure which will be used throughout this paper.

Each bond of a graph carries a factor $\frac{1}{2}v_{ij}$ and a label β_i which is eventually integrated from 0 to β. Each vertex is associated with a semi-invariant and a spin index i. Any free spin index (i.e., one labeling a vertex including no spin operators from Q) is summed. For instance, the index k of Fig. 3 will be summed, but only over values labeling spins with which spin S_i^z is coupled through H_1. We shall call the number of such spins z. Then the explicit dependence on $(1/z)^n$ is to be isolated for the simple high-density expansion.

A graph containing x bonds and y free vertices and y' vertices associated with Q carries a factor which is approximately $\beta^z(\bar{v})^x z^y \bar{M}^{y+y'}$ where we have replaced the interactions and the semi-invariants by some average values $\bar{v}$, $\bar{M}$. However, for the molecular field model, the critical temperature is given by

$$\beta_c z\bar{v}S^2=1. \tag{27}$$

This enables us to write $\beta\bar{v}$ in terms of $1/z$ and we obtain for the contribution of the graph:

$$(\beta/\beta_c)^x z^{y-x}\bar{M}^{y+y'}.$$

Thus, for temperatures near the critical temperature or higher, $\beta/\beta_c \lesssim 1$, the order of the graph in the high-density expansion is the number of bonds minus the number of free vertices. The number of bonds is always greater than or equal to the number of free vertices in a linked graph hence the leading term in the high-density expansion comes from the "tree" graphs for which $x=y$. A typical example is shown in Fig. 4.

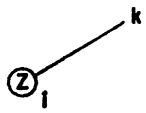

FIG. 3. Coupling of spin i with spin k.

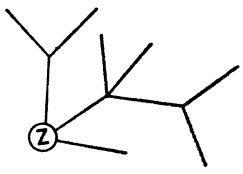

FIG. 4. An example of a tree graph.

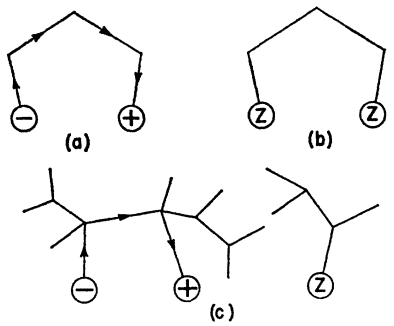

FIG. 5. (a) A simple transverse ring graph. (b) A simple longitudinal ring. (c) A ring with tree renormalization.

Because transverse bonds form closed paths, the graphs of zero order in $1/z$ are made up of only longitudinal bonds. The graphs of order $1/z$ have one more bond than free vertex and simple examples (ring graphs) are given in Figs. 5 (a) and 5 (b). Addition of tree graphs to any vertex will not change the order in the $1/z$ expansion so that the graph of Fig. 5 (c) is also of first order.

We shall now sum the tree graphs to find the magnetization in zero order, which will prove to be that given by Weiss molecular field theory.

The magnetization is $\langle S^z\rangle$ so that we need to sum all tree graphs rooted to $Q=S^z$. These are the graphs of Fig. 4, which contain only the longitudinal part $-\frac{1}{2}\sum_{ij} v_{ij}S_i^zS_j^z$ of the interaction. Since all the operators involved commute, the time-ordered formalism is unnecessary here, and we may omit the β_i labels and perform the time integrations trivially. For the graph with n bonds shown in Fig. 6, the g factor is $n!$, since the same graph is obtained by interchanging bonds in all possible ways. The vectices at the extreme ends are all first-order semi-invariants $\langle S^z\rangle_0$, independent of the spin index. The sum over free indices is immediate, each producing a factor $\frac{1}{2}\sum_j v_{ij}=\frac{1}{2}v(0)$. Here we define the Fourier components of the potential by

$$v(q)=\sum_{ij} v_{ij}e^{iq(r_i-r_j)}. \tag{28}$$

Thus, the contribution of the graph of Fig. 6 is

$$(2^n/n!)[\beta(v(0)/2)\langle S^z\rangle_0]^n M_{n+1}(S^z\cdots S^z).$$

Using the raising operator $\partial/\partial\beta h$ derived in Appendix C, we obtain for the sum of graphs in which all bonds are joined to a common vertex labeled z,

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left[\beta v(0)\langle S^z\rangle_0\frac{\partial}{\partial\beta h}\right]^n M_1 = \exp\left[\beta v(0)\langle S^z\rangle_0\frac{\partial}{\partial\beta h}\right]M_1. \tag{29}$$

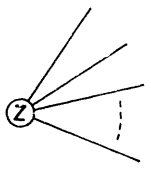

FIG. 6. Articulation of single bonds at a vertex.

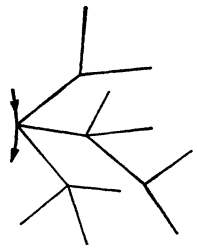

FIG. 7. Renormalization of a second-order semi-invariant by trees.

These are not all the tree graphs; we obtain more by adding to each vertex of Fig. 6 extra bonds just as in Fig. 6 bonds have been added to vertex i. If we continue this process indefinitely, the semi-invariant of each vertex will be changed to the value $\hat{M}_1$, say. This process is called vertex renormalization. But $\hat{M}_1$ will be the value obtained for $\langle S^z\rangle$ from the summation of all tree graphs. Hence,

$$\hat{M}_1=\exp\left[\beta v(0)\hat{M}_1\frac{\partial}{\partial\beta h}\right]M_1. \tag{30}$$

So far, it has not been necessary to specify the magnitude S of the spin. From now on, however, we shall find it convenient to specialize to $S=\frac{1}{2}$. Then $M_1=\frac{1}{2}\times\tanh(\frac{1}{2}\beta h)$, so that

$$\hat{M}_1=\tfrac{1}{2}\tanh[\tfrac{1}{2}\beta(h+v(0)\hat{M}_1)]. \tag{31}$$

The magnetization $\hat{M}_1$ given by (31) is the Weiss molecular field magnetization. For zero h, the magnetization disappears at the temperature T_c given by $1=\beta_c v(0)/4$, which we used in setting up this high-density classification.

4. TERMS OF FIRST ORDER IN THE HIGH DENSITY EXPANSION—SPIN WAVES

In this section we calculate the terms of $O(1/z)$ according to the simple high-density classification presented in the previous chapter. These are the rings with all vertices renormalized by trees. As presented in Sec. 3, except for the case of infinite z, the classification is only valid provided $\beta/\beta_c\lesssim 1$, so that we would not expect to obtain results valid below the Curie point. It will, however, be shown in Sec. 5 that the classes of graphs corresponding to the low-order terms in the simple high-density expansion are the same as those corresponding to the low-order terms in an expansion in powers of the temperature, so that the results are also correct at low temperature.

The result of the ring summation carried out in this section will, at low temperatures, reduce to spin-wave theory.

We sum first the unrenormalized ring 5(a) to obtain the simplest terms in the ring approximation to the transverse propagator $\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle$. The result will be modified later in this section by including the tree renormalization shown in Fig. 5(c).

The unperturbed propagator is

$$\langle TS_i^+(\beta_1)S_j^-(\beta_2)\rangle_0 = \delta_{ij}\frac{e^{-(\beta_1-\beta_2)h}}{e^{\beta h/2}+e^{-\beta h/2}}\begin{cases} e^{\beta h/2}, & \beta_1>\beta_2 \\ e^{-\beta h/2}, & \beta_1<\beta_2. \end{cases} \quad (32)$$

As a function of $(\beta_1-\beta_2)$, this function is periodic with period β and, hence, may be Fourier analyzed:

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_0 = \sum_{k=-\infty}^{\infty}\frac{\tanh\frac{1}{2}\beta h}{\beta(h-i\lambda_k)}e^{-i\lambda_k(\beta_i-\beta_j)}, \quad (33)$$

where $\lambda_k=2\pi k/\beta$. The unperturbed propagator (33) occurs as follows in the sum over rings[14,15]:

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_{\text{unrenormalized rings}} = \sum_{n=0}^{\infty}(\tfrac{1}{2})^n\sum_{i_1\cdots i_{n-1}}v_{ii_1}v_{i_1i_2}\cdots v_{i_{n-1}j}\int_0^\beta\cdots\int_0^\beta d\beta_1\cdots d\beta_n$$
$$\times\langle TS_i^+(\beta_i)S_i^-(\beta_1)\rangle_0\langle TS_{i_1}^+(\beta_1)S_{i_1}^-(\beta_2)\rangle_0\cdots\langle TS_{i_n}^+(\beta_n)S_j^-(\beta_j)\rangle_0. \quad (34)$$

Fourier decomposition of both v_{ij} and the propagator (33) then leads to

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_{\text{unrenormalized rings}} = \sum_q\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}e^{iq(r_i-r_j)}e^{-i\lambda_k(\beta i-\beta_j)}[\beta v(q)/2]^n[\tanh\tfrac{1}{2}\beta h/\beta(h-i\lambda_k)]^{n+1}$$
$$=\sum_q\sum_k e^{iq(r_i-r_j)}e^{-i\lambda_k(\beta i-\beta_j)}\frac{\tanh\frac{1}{2}\beta h}{\beta[h-i\lambda_k-\frac{1}{2}v(q)\tanh\frac{1}{2}\beta h]}. \quad (35)$$

The sum over all integers k may be replaced by a contour integral with respect to z by using the function $\pm\beta[\exp(\pm\beta z)-1]^{-1}$ which has poles with unit residue at the points $z=(2\pi i/\beta)j$. The function which makes the integrand converge for all large z is chosen.[14] The result is

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_{\text{unrenormalized rings}} = \sum_q e^{iq(r_i-r_j)}\tanh\tfrac{1}{2}\beta h\exp[-(\beta_i-\beta_j)(h-\tfrac{1}{2}v(q)\tanh\tfrac{1}{2}\beta h)]\times\begin{cases} g^-(q), & \beta_i>\beta_j, \\ g^+(q), & \beta_i<\beta_j, \end{cases} \quad (36)$$

where

$$g^\pm(q)=\pm\{\exp[\pm\beta(h-\tfrac{1}{2}v(q)\tanh\tfrac{1}{2}\beta h)]-1\}^{-1}. \quad (37)$$

In order to complete the summation of terms of order $1/z$, it is now necessary to include tree renormalization, replacing the unperturbed propagators (second-order semi-invariants M_2) by the sum of all multiple vertices of the type shown in Fig. 7.

The effect of this tree renormalization of each M_2 is given as a special case of the following general theorem:

For a semi-invariant of arbitrary order, tree renormalization causes a displacement of h to $h+v(0)\hat{M}_1$, everywhere that h occurs in the unrenormalized semi-invariant. [This result has already been obtained, Eq. (30), for the semi-invariant of first order.] The proof proceeds as follows: As in the treatment of $M_1(S^z)$, we first obtain the modification of the general semi-invariant $M_{2n+l}(S^+(\beta_1)\cdots S^+(\beta_n)S^-(\beta_{n+1})\cdots S^-(\beta_{2n})S^z(\beta_{2n+1})\cdots S^z(\beta_{2n+l}))$ which results from articulating all single bonds. The semi-invariant associated with the articulation of l' bonds is $M_{2n+l+l'}(TS^+(\beta_1)\cdots S^-(\beta_{2n})S^z(\beta_{2n+1})\cdots S^z(\beta_{2n+l+l'}))$. Using the generating function formalism, we have upon integration over $\beta_{2n+l+1}\cdots\beta_{2n+l+l'}$:

$$\int_0^\beta\cdots\int_0^\beta M_{2n+l+l'}(TS^+(\beta_1)\cdots S^-(\beta_{2n})S^z(\beta_{2n+1})\cdots S^z(\beta_{2n+l+l'}))d\beta_{2n+l+1}\cdots d\beta_{2n+l+l'}$$
$$=\left(\frac{\partial}{\partial\gamma}\right)^{l'}\Big|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2n+l}}\Big|_{\alpha_{2n+l}=0}\ln\left\langle T\exp\left[\gamma\int_0^\beta d\beta'\,S^z(\beta')+\alpha_1S^+(\beta_1)+\cdots+\alpha_{2n+l}S^z(\beta_{2n+l})\right]\right\rangle_0. \quad (38)$$

The sum over the index j labeling the end of each longitudinal bond has the form $\frac{1}{2}\sum_j v_{ij}\langle S_j^z\rangle_0=\frac{1}{2}v(0)M_1$. Further, for l' such bonds the symmetry factor due to interchange of the bonds is $l'!$. Thus, the modified propagator is

$$\sum_{l'=0}^{\infty}(2^{l'}/l'!)[\tfrac{1}{2}v(0)M_1]^{l'}(\partial/\partial\gamma)^{l'}|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2n+l}}\Big|_{\alpha_{2n+l}=0}\ln\langle T\cdots\rangle_0$$
$$=\exp[v(0)M_1\partial/\partial\gamma]|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2n+l}}\Big|_{\alpha_{2n+l}=0}\ln\langle T\cdots\rangle_0. \quad (39)$$

[14] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).
[15] D. J. Thouless, Ann. Phys. (N. Y.) **10**, 553 (1960).

We now show that the differentiation with respect to γ at $\gamma=0$ may be replaced by differentiation with respect to h at its correct value. This is a consequence of the fact that the unperturbed Hamiltonian is $-h\sum_i S_i^z$. Consider the expression

$$\ln\left\langle T\exp\left[\gamma\int_0^\beta d\beta' S^z(\beta')+\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2n+l}S^z(\beta_{2n+l})\right]\right\rangle_0 \equiv \ln\left\langle \exp\left[\gamma\int_{\beta i_1}^\beta S^z(\beta')d\beta'\right]\exp[O_1(\beta_{i_1})]\right.$$
$$\left.\times\exp\left[\gamma\int_{\beta i_2}^{\beta i_1} S^z(\beta')d\beta'\right]\exp[O_2(\beta_{i_2})]\cdots\exp[O_{2n+l}(\beta_{i_{2n+l}})]\exp\left[\gamma\int_0^{\beta i_{2n+l}} S^z(\beta')d\beta'\right]\right\rangle_0, \quad (40)$$

where $\{O_i(\beta_{i_a})\}$ is the sequence of operators $\alpha_i S^+(\beta_i)$, $\alpha_i S^-(\beta_i)$, $\alpha_i S^z(\beta_i)$ labeled in order of decreasing β_i The right-hand side of Eq. (40) is

$$\ln\left\{\frac{\mathrm{Tr}[\exp(\beta hS^z)\exp(\beta\gamma S^z)\exp(-\beta_{i_1}\gamma S^z)\exp(-\beta_{i_1}hS^z)\exp(O_1)\times\exp(\beta_{i_1}hS^z)\cdots\exp(O_{2n+l})\exp(\beta_{i_{2n+l}}hS^z)\exp(\beta_{i_{2n+l}}\gamma S^z)]}{\mathrm{Tr}[\exp(\beta hS^z)]}\right\}$$

because the average is performed with the weighting factor ρ_0, and because

$$\exp[O(\beta)]=\exp[\exp(\beta H_0)O\exp(-\beta H_0)]=\exp(\beta H_0)\exp(O)\exp(-\beta H_0). \quad (41)$$

The logarithm above is, therefore,

$$\exp(\gamma\partial/\partial h)\ln\langle T\exp[\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2n+l}S^z(\beta_{2n+l})]\rangle_0+\ln\left\{\frac{\mathrm{Tr}[\exp(\beta(h+\gamma)S^z)]}{\mathrm{Tr}[\exp(\beta hS^z)]}\right\},$$

where the shifting operator $\exp(\gamma\partial/\partial h)$ has the effect of replacing h everywhere that it occurs by $h+\gamma$. The renormalized semi-invariant (39), therefore, becomes

$$\exp[v(0)M_1\partial/\partial\gamma]|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2n+l}}\Big|_{\alpha_{2n+l}=0}\exp(\gamma\partial/\partial h)\ln\langle T\exp[\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2n+l}S^z(\beta_{2n+l})]\rangle_0$$
$$=\exp[v(0)M_1\partial/\partial\gamma]|_{\gamma=0}\exp(\gamma\partial/\partial h)M_{2n+l}(TS^+(\beta_1)\cdots S^z(\beta_{2n+l}))$$
$$=\exp[v(0)M_1\partial/\partial h]M_{2n+l}(TS^+(\beta_1)\cdots S^z(\beta_{2n+l})). \quad (42)$$

To carry out the complete tree renormalization, it is necessary to add further single bonds to the end of each bond, which converts the bonds into trees. This has the effect of changing M_1 to $\hat{M}_1$ as we saw in the previous section. $\hat{M}_1(S^z(\beta'))$ determined self-consistently by Eq. (31) is not a function of the temperature label β', so that the analysis presented above goes through with M_1 replaced by $\hat{M}_1$ and the complete renormalization of a vertex by longitudinal trees is given by Eq. (42) with M_1 replaced by $\hat{M}_1$, the molecular field value of Eq. (31). This completes the proof of the theorem.

We now insert this renormalization correction into (36) and (37). Each of the factors in (36) which contains h arises from an unrenormalized semi-invariant. Thus, the renormalization with longitudinal trees causes everywhere a translation of h to the value $h+v(0)\hat{M}_1$. Our result is then

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_{\text{rings with tree renormalization}}=\sum_q e^{iq(r_i-r_j)}\tanh[\tfrac{1}{2}\beta(h+v(0)\hat{M}_1)]$$
$$\times\exp\{-(\beta_i-\beta_j)[h+v(0)\hat{M}_1-\tfrac{1}{2}v(q)\tanh[\tfrac{1}{2}\beta(h+v(0)\hat{M}_1)]]\}\times\begin{cases}\tilde{g}^-(q), & \beta_i>\beta_j\\ \tilde{g}^+(q), & \beta_i<\beta_j\end{cases} \quad (43)$$

where

$$\tilde{g}^\pm(q)=\pm\{\exp[\pm\beta\{h+v(0)\hat{M}_1-\tfrac{1}{2}v(q)\tanh[\tfrac{1}{2}\beta(h+v(0)\hat{M}_1)]\}]-1\}^{-1}. \quad (44)$$

For $\hat{M}_1\neq 0$, this expression may be simplified by using Eq. (31) to give

$$\langle TS_i^+(\beta_i)S_j^-(\beta_j)\rangle_{\text{rings}}=\sum_q e^{iq(r_i-r_j)}\langle TS_q^+(\beta_i)S_{-q}^-(\beta_j)\rangle_{\text{rings}}$$
$$=\sum_q e^{iq(r_i-r_j)}\exp[-(\beta_i-\beta_j)\{h+\hat{M}_1[v(0)-v(q)]\}]2\hat{M}_1\times\begin{cases}\tilde{g}^-, & \beta_i>\beta_j\\ \tilde{g}^+, & \beta_i<\beta_j\end{cases} \quad (45)$$

where

$$\tilde{g}^\pm=\pm[\exp\{\pm\beta[h+\hat{M}_1(v(0)-v(q))]\}-1]^{-1}. \quad (46)$$

At low temperatures, $\hat{M}_1 \to \frac{1}{2}$. Defining the frequency

$$\omega(q) = h + \tfrac{1}{2}[v(0) - v(q)], \tag{47}$$

we have

$$\lim_{T\to 0} \langle T S_i^+(\beta_i) S_j^-(\beta_j)\rangle_{\text{rings}}$$

$$= \sum_q e^{iq(r_i - r_j)} \exp[-(\beta_i - \beta_j)\omega(q)] \times \begin{cases} \hat{g}^-, & \beta_i > \beta_j \\ \hat{g}^+, & \beta_i < \beta_j, \end{cases} \tag{48}$$

where

$$\hat{g}^{\pm} = \pm\{\exp[\pm\beta\omega(q)] - 1\}^{-1}, \tag{49}$$

which is, in fact, the Fourier transform of the usual spin wave propagator. Equation (48) then gives for the spin-wave populations

$$n(q) = \langle S_q^- S_{-q}^+\rangle_{\text{rings}} = [\exp\beta\omega(q) - 1]^{-1}. \tag{50}$$

At low temperatures, the magnetization curve can then be determined from the rule

$$\lim_{T\to 0} \langle S^z\rangle_{\text{rings}} = \tfrac{1}{2}\sum_i (1 - 2\langle S_i^- S_i^+\rangle_{\text{rings}})$$

$$= \tfrac{1}{2}\sum_q (1 - 2\langle S_q^- S_{-q}^+\rangle_{\text{rings}}) = \tfrac{1}{2} - O(T/T_c)^{3/2}. \tag{51}$$

This result may also be obtained by direct calculation of $\langle S^z\rangle$ in the same approximation.

Thus, we have recovered spin-wave results[3] by ring summation. The interpretation of these results as a valid low-temperature approximation are given in the next section.

5. LOW-TEMPERATURE ORDERING

In the previous sections we considered the application of a simple high-density ordering whose derivation was valid in the region of, and above, the Curie temperature. In this section we shall complement this high-density ordering by a low-temperature ordering which classifies the graphs according to a power of T or to $\exp[-\frac{1}{2}\beta v(0)]$, which is negligible compared to any power of T at low temperatures. This classification will enable us to select the graphs of lowest orders in the temperature expansion, for comparison with results of Dyson.[6] We shall also see that the low-temperature and the high-density classifications select in low orders the same classes of graphs, so that the results of the high-density summations of Secs. 3 and 4 may be employed *below the Curie point.*

We shall first arrive at a simplification of the longitudinal elements of a graph, by examining the low-temperature values of the longitudinal semi-invariants. The unrenormalized longitudinal semi-invariant of order n can be obtained by operating $n-1$ times with the raising operator $\partial/\partial(\beta h)$ on $M_1 = \frac{1}{2}\tanh\frac{1}{2}\beta h$. We may write M_1 as follows:

$$M_1 = \tfrac{1}{2}[1 - 2\exp(-\beta h) + O(\exp(-2\beta h))], \tag{52}$$

so that $M_{n>1} \propto \exp(-\beta h)$, and for finite field, at low temperatures $M_{n>1}$ becomes "exponentially small" compared to M_1. If the tree renormalization of each M is carried out, $h \to h + v(0)\hat{M}_1$ and $M_{n>1}/M_1$ is proportional to $\exp\{-\beta[\frac{1}{2}v(0) + h]\}$ and the ratio is negligible even in zero field.

Thus, any graph containing a tree renormalized longitudinal semi-invariant M_n with $n>1$ is exponentially small and can be discarded for the purposes of an expansion in powers of T. It also follows that the low-temperature limit of the magnetization, for example, of the Ising model is completely included in those graphs whose vertices are just first-order semi-invariants with tree renormalization, that is, the molecular field trees. Furthermore, this class of graphs contains also all the graphs in which one renormalized semi-invariant is allowed to be of order higher than one, so that we recover from this temperature ordering the well-known result that the molecular-field description of the Ising model is correct at low temperatures apart from corrections of order $\{\exp[-\frac{1}{2}\beta v(0)]\}^2$.

Now we examine the transverse parts of graphs. We found in our examination of the ring summation that it was convenient to Fourier analyze the propagators $M_2(TS^+(\beta_1)S^-(\beta_2))$, because of the convolution nature of the sum with respect to time and space. The Fourier analysis with respect to time has for the low-temperature ordering a further advantage which we shall now develop. It is simply that having placed all the dependence on β_1, β_2 into factors like $e^{i\lambda_k(\beta_1-\beta_2)}$, the integrals over β_1, β_2 which have to be completed to evaluate the ring are of the form $\int_0^\beta d\beta_1\, e^{i\beta_1(\lambda_j+\lambda_k)} = \beta\delta(\lambda_j+\lambda_k)$ since $e^{i\beta\lambda_j} = 1$ for all j. That is, the temperature dependence associated with each of the integrals $\int d\beta_i$ is simply a factor β. The only other place in which a temperature dependence arises is in the Fourier coefficients $M_2(\lambda) = \tanh\{\frac{1}{2}\beta[h + v(0)\hat{M}_1]\}\{\beta[h + v(0)\hat{M}_1 - i\lambda_k]\}^{-1}$, where we have again included tree renormalization. Here it is possible to make a low-temperature approximation by replacing $2\hat{M}_1 = \tanh\{\frac{1}{2}\beta[h + v(0)\hat{M}_1]\}$ by one. Because of the λ, q-conservation around the ring, any ring element is then of the form

$$\sum_{\lambda_j}\sum_q\sum_n \frac{[\frac{1}{2}\beta v(q)]^n}{[\beta(h + \frac{1}{2}v(0) - i\lambda_j)]^{n+1}} = \sum_j\sum_q \{\beta[h + \tfrac{1}{2}(v(0) - v(q)) - 2\pi j]\}^{-1}. \tag{53}$$

Since $\sum_q \propto \int q^2 dq$ and $v(0) - v(q) \propto q^2$, in the limit of zero field this ring element is proportional to $T^{3/2}$.

We now generalize this procedure to the discussion of the temperature dependence of graphs containing more complicated vertices. The general semi-invariant $M_{2n+l}(TS^+(\beta_1)\cdots S^z(\beta'_l))$ is a function of $2n+l$ time labels $\beta_1\cdots\beta_{2n}, \beta'_1\cdots\beta'_l$. The following argument shows

that it is periodic in each of these variables and can, therefore, be Fourier analyzed.[16]

$$M_{2n+l}(TS^+(\beta_1)\cdots S^+(\beta_n)S^-(\beta_{n+1})\cdots \times S^-(\beta_{2n})S^z(\beta'_1)\cdots S^z(\beta'_l)) = \sum_k (-1)^{k-1}(k-1)!\{\nu_k\}, \quad (54)$$

where $\{\nu_k\}$ means the sum of all divisions into k separate averages of the set of operators $S^+(\beta_1)\cdots S^z(\beta'_l)$. Consider any spin operator $S^{(\alpha_i)}(\beta_i)$ of the semi-invariant. Keeping all other times fixed we put β_i in turn equal to 0 and to β: In any part of the semi-invariant, the average in which $S^{(\alpha_i)}(\beta_i)$ appears then changes from having, in the first case $S^{(\alpha_i)}(0)$ to the left of all the other operators, to in the second case having $S^{(\alpha_i)}(\beta)$ to the right of the other operators, that is,

$$\mathrm{Tr}[\exp(-\beta H_0)T(S\cdots S\cdots)S^{(\alpha_i)}(0)],$$
$$\mathrm{Tr}[\exp(-\beta H_0)S^{(\alpha_i)}(\beta)T(S\cdots S\cdots)],$$

respectively. However,

$$\exp(-\beta H_0)S^{(\alpha_i)}(\beta) = [\exp(-\beta H_0)S^{(\alpha_i)}(\beta)\exp(\beta H_0)]\exp(-\beta H_0) = S^{(\alpha_i)}(0)\exp(-\beta H_0), \quad (55)$$

so that the second case, $\beta_i=\beta$, is identical with the first, $\beta_i=0$, using the cyclic invariance of the trace. Thus,

$$M_{2n+l}(\beta_i=\beta)=M_{2n+l}(\beta_i=0), \quad (56)$$

with all other time variables arbitrarily fixed.

Thus, we may write M_{2n+l} as a multiple Fourier series:

$$M_{2n+l}(\beta_1\cdots\beta_n\beta_{n+1}\cdots\beta_{2n}\beta'_1\cdots\beta'_l)=\sum_{\lambda_1\cdots\lambda'_l} M_{2n+l}(\lambda_1\cdots\lambda_{2n},\lambda'_1\cdots\lambda'_l) \times\exp[-i(\beta_1\lambda_1+\cdots+\beta_n\lambda_n-\beta_{n+1}\lambda_{n+1}\cdots-\beta_{2n}\lambda_{2n}+\beta'_1\lambda'_1+\cdots+\beta'_l\lambda'_l)], \quad (57)$$

where

$$M_{2n+l}(\lambda_1\cdots\lambda_{2n},\lambda'_1\cdots\lambda'_l)=\beta^{-(2n+l)}\int_0^\beta d\beta_1\cdots\int_0^\beta d\beta'_l M_{2n+l}(\beta_1\cdots\beta'_l)\exp[i(\beta_1\lambda_1\cdots-\beta_{2n}\lambda_{2n}+\cdots+\beta'_l\lambda'_l)]. \quad (58)$$

The time translational invariance of $M_{2n+l}(\beta_1\cdots\beta_{2n}\beta'_1\cdots\beta'_l)$ implies that $M_{2n+l}(\lambda_1\cdots\lambda'_l)$ vanishes unless

$$\lambda_1+\cdots+\lambda_n-\lambda_{n+1}\cdots-\lambda_{2n}+\lambda'_1\cdots+\lambda'_l=0.$$

Thus, in computing $M_{2n+l}(\lambda_1\cdots\lambda'_l)$ according to Eq. (58) we may discard any parts of the integration which will not lead to a factor $\delta_{\lambda_1+\cdots+\lambda_n-\lambda_{n+1}\cdots-\lambda_{2n}+\cdots+\lambda'_l,0}$ since these parts will cancel among themselves. We may extract the explicit time dependence of $M_{2n+l}(\beta_1\cdots\beta'_l)$:

$$M_{2n+l}(\beta_1\cdots\beta'_l)=\mathfrak{m}_{2n+l}\exp[-h(\beta_1+\cdots+\beta_n-\beta_{n+1}\cdots-\beta_{2n})], \quad (59)$$

where $\mathfrak{m}$ is constant within a given time ordering. Then the time integrations of Eq. (58) are integrations of factors like $\exp[\beta_i(\cdots h\cdots+\cdots i\lambda_i\cdots)]$ within a given time ordering. Only the final integration has limits $(0, \beta)$, and this integration only may give rise to the δ function. It can only do so if the integrand has a term whose only β_i dependence is $\exp[\pm\beta_i(\lambda_1\cdots+\lambda_n-\lambda_{n+1}\cdots+\lambda'_l)]$, that is, containing all the λ's. Such a term will result from doing the preceding integrations and evaluating them at their upper limit only. This provides us with a simple procedure for evaluating Eq. (58). Evidently, the result of the integrations over a given time order can give rise to no additional h dependence beyond that already in $\mathfrak{m}_{2n+l}$. Thus, a further simplification appears: For the purposes of a low-temperature expansion, we may evaluate Eq. (58) by the method outlined above, *neglecting all time orderings which have* $\mathfrak{m}_{2n+l}$ *exponentially small.*

Thus, for example, of the six possible time orderings of

$$M_{2+1}(TS^+(\beta_1)S^-(\beta_2)S^z(\beta'_1))=\langle TS^+(\beta_1)S^-(\beta_2)S^z(\beta'_1)\rangle_0-\langle S^z(\beta'_1)\rangle_0\langle TS^+(\beta_1)S^-(\beta_2)\rangle_0, \quad (60)$$

only $\beta_1>\beta'_1>\beta_2$, for which $\mathfrak{m}_{2+1}=-1$, leads to results for $\mathfrak{m}_{2+1}$ which are not exponentially small. Then

$$M_{2+1}(\lambda_1\lambda_2\lambda'_1)=\beta^{-3}\int_0^\beta d\beta_1\int^{\beta_1}d\beta'_1\int^{\beta'_1}d\beta_2(-1)\exp\{-[\beta_1(h-i\lambda_1)-\beta_2(h-i\lambda_2)-i\beta'_1\lambda'_1]\}$$
$$=-\delta_{\lambda_1-\lambda_2+\lambda'_1,0}[\beta(h-i\lambda_2)\beta(h-i\lambda_2+i\lambda'_1)]^{-1}. \quad (61)$$

We now make the assumption that $\mathfrak{m}_{2n+l}$ is exponentially small [at least of order $\exp(-\beta h)$] except when

$$\mathrm{all}\{\beta_1\cdots\beta_n\}>\mathrm{all}\{\beta'_1\cdots\beta'_l\}>\mathrm{all}\{\beta_{n+1}\cdots\beta_{2n}\}. \quad (62)$$

This is the case for low-order semi-invariants examined in detail (M_2, M_4, M_{2+1}, M_{2+2}) but a general proof has not

[16] A similar result has been obtained for the propagators for systems of fermions or bosons by P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).

yet been found. From the assumption, using the argument above it follows that

$$M_{2n+l}(\lambda_1\cdots\lambda_n\lambda_{n+1}\cdots\lambda_{2n}\lambda'_1\cdots\lambda'_l)$$
$$=\mathfrak{M}_{2n+l}{}^{\max}\beta^{-(2n+l)}\sum_{\{i_k\}}\sum_{\{j_k\}}\int_0^\beta d\beta_{i_1}\cdots\int^{\beta_{i_{n-1}}}d\beta_{i_n}\int^{\beta_{i_n}}d\beta'_{j_1}\cdots\int^{\beta'_{j_{l-1}}}d\beta'_{j_l}\int^{\beta'_{j_l}}d\beta_{i_{n+1}}\cdots\int^{\beta_{i_{2n-1}}}d\beta_{i_{2n}}$$
$$\times\prod_{i=1}^{n}\exp[-\beta_i(h-i\lambda_i)+\beta_{n+i}(h-i\lambda_{n+i})]\prod_{j=1}^{l}\exp[i\beta_j\lambda'_j],\quad(63)$$

where $\{i_k\}$ are all distinct sequences formed from the set of numbers 1 to $2n$, to $\{j_k\}$ are all possible sequences formed from 1 to l, so that the integral is over all time orderings satisfying Eq. (62), and $\mathfrak{M}_{2n+l}{}^{\max}$ is the value of $\mathfrak{M}_{2n+l}$ corresponding to any of these orderings, a constant of order 1. The result of the integrations is

$$M_{2n+l}(\lambda_1\cdots\lambda'_l)=\mathfrak{M}_{2n+l}{}^{\max}\beta^{-(2n+l-1)}\delta_{\lambda_1+\cdots+\lambda_n-\lambda_{n+1}\cdots-\lambda_{2n}+\lambda'_1\cdots+\lambda'_l,0}\sum_{\{i_k\}}\sum_{\{j_k\}}[(h-i\lambda_{i_{2n}})(2h-i\lambda_{i_{2n}}-i\lambda_{i_{2n-1}})\cdots$$
$$\times(nh-i\lambda_{i_{2n}}-\cdots-i\lambda_{i_{n+1}})(D_1+i\lambda'_{j_l})(D_1+i\lambda'_{j_l}+i\lambda'_{j_{l-1}})\cdots(D_1+i\lambda'_{j_l}+\cdots+i\lambda'_{j_1})$$
$$\times(D_2-h+i\lambda_{i_n})(D_2-2h+i\lambda_{i_n}+i\lambda_{i_{n-1}})\cdots(D_2-(n-1)h+i\lambda_{i_n}+\cdots+i\lambda_{i_2})],\quad(64)$$

where

$$D_1=nh-i\lambda_{i_{2n}}-\cdots-i\lambda_{i_{n+1}},\quad(65)$$
$$D_2=D_1+i\lambda'_{j_l}+\cdots+i\lambda'_{j_1}=nh-i\lambda_{i_1}-\cdots-i\lambda_{i_n}.\quad(66)$$

Recombination of the terms in the sum $\sum_{\{i_k\}}$ leads to

$$M_{2n+l}(\lambda_1\cdots\lambda_{2n}\lambda'_1\cdots\lambda'_l)=\mathfrak{M}_{2n+l}{}^{\max}\delta_{\lambda_1+\cdots+\lambda_n-\lambda_{n+1}\cdots-\lambda_{2n}+\lambda'_1\cdots+\lambda'_l,0}[\beta(h-i\lambda_1)\beta(h-i\lambda_2)\cdots\beta(h-i\lambda_n)\beta(h-i\lambda_{n+1})\cdots$$
$$\times\beta(h-i\lambda_{2n})]^{-1}\sum_{\{jk\}}[\beta(D_1+i\lambda'_{j_l})\beta(D_1+i\lambda'_{j_l}+i\lambda'_{j_{l-1}})\cdots\beta(D_1+i\lambda'_{j_l}+i\lambda'_{j_{l-1}}+\cdots+i\lambda'_{j_2})]^{-1}.\quad(67)$$

When $l=0$, the last sum goes over to βD_2, and the semi-invariant is

$$M_{2n}(\lambda_1\cdots\lambda_n\lambda_{n+1}\cdots\lambda_{2n})=\frac{\mathfrak{M}_{2n}{}^{\max}\delta_{\lambda_1\cdots+\lambda_n-\lambda_{n+1}\cdots-\lambda_{2n},0}\beta[(h-i\lambda_1)+\cdots+(h-i\lambda_n)]}{\beta(h-i\lambda_1)\beta(h-i\lambda_2)\cdots\beta(h-i\lambda_n)\cdots\beta(h-i\lambda_{2n})}.\quad(68)$$

Under tree renormalization, $h\to h+v(0)\hat{M}_1$ in the above equations. From now on we assume that tree renormalization of all vertices has already been carried out. Having derived the form of the general semi-invariants, we may deduce the temperature dependence of any graph by the same method as was applied above to the simple transverse ring.

Consider first purely transverse graphs. The most general graph of this type is a set of m transverse semi-invariants of order greater than two joined by n ring elements (with tree renormalization everywhere). Along the ring elements, and at the vertices, there is momentum and frequency conservation, and the value of the graph is then obtained by summing over all Q-independent q and λ variables. The contribution from a tree-renormalized ring element without its final vertices is $[\beta(h+\frac{1}{2}v(0)-i\lambda)]^2\{\beta[h+\frac{1}{2}(v(0)-v(q))-i\lambda]\}^{-1}$ where the momentum and frequency of the ring are q, λ, respectively. The values of the vertices are given by inserting into Eq. (68) the appropriate n's and λ's, and replacing h by $h+\frac{1}{2}v(0)$. In a free-energy graph the denominators from the vertices cancel exactly the factors $[\beta(h+\frac{1}{2}v(0)-i\lambda)]^2$ in the contribution from the ring elements and the final sum over the Q-independent λ's and q's is of a product of n factors like $\{\beta[h+\frac{1}{2}(v(0)-v(q))-i\lambda_i]\}^{-1}$, one for each ring element, multiplied by a product of m sums of single terms $[\beta(h+\frac{1}{2}v(0)-i\lambda_i)]$ from the numerators of the semi-invariants with $n>1$. Each of the Q sums over λ is replaced by a contour integration as in the treatment of the simple ring. The values of the sums are determined by the poles from the product of denominators, as a function of the Q-independent λ's, so that the total sum is a product of Q spin-wave distribution functions $n(q_i)$, multiplied by a product of $(n-Q)$ denominators each of the form $\sum_i\beta\omega(q_i)$.[17] The product of m sums of terms $[\beta(h+\frac{1}{2}v(0)-i\lambda_i)]$ becomes a product of m sums of terms $\frac{1}{2}\beta v(q_i)$ so that the final contribution to $-\beta F$ has the form $\sum_{q_1\cdots q_Q}(\frac{1}{2}\beta\bar{v})^m(\beta\bar{\omega})^{Q-n}(\bar{n})^Q$, which is proportional to $T^{(\frac{1}{2}Q-m)}$.

This rule needs modification in two exceptional cases. The first is when α ($\alpha\geq 2$) of the ring elements have the same momentum (and, therefore, the same frequency), as for instance in the example ($\alpha=2$) given in

[17] Compare J. M. Luttinger, Phys. Rev. **121**, 942 (1961).

[18] Henceforth, tree renormalization, which is everwhere implied, is not shown explicitly in our diagrams. We further simplify the diagrams by omitting the structure (bonds and M_2's) of the ring elements.

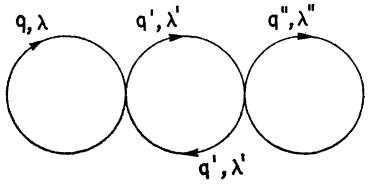

FIG. 8. A transverse graph in which two of the ring elements have the same frequency and momentum.

Fig. 8.[18] Then the product of n denominators $[\beta(h+\omega(q_i)-i\lambda_i)]$ contains $[\beta(h+\omega(q')-i\lambda')]^{\alpha}$. On carrying out the sum over λ', the multiple pole from this factor does not give rise to a factor $[\beta\omega(q')]^{-1}$ but it removes $(\alpha-1)$ factors $[\beta(h+\frac{1}{2}v(0)-i\lambda'+\cdots)] \rightarrow \sum \frac{1}{2}\beta\bar{v}$ from the numerator. Then the contribution to $-\beta F$ is of form

$$\sum_{q_1\cdots q_Q} (\tfrac{1}{2}\beta\bar{v})^{n-\alpha-1}(\beta\bar{\omega})^{Q-n+\alpha-1}(\bar{n})^{Q} \propto T^{[\frac{3}{2}Q-m+(\alpha-1)]}. \quad (69)$$

The second exceptional case is when, because of frequency conservation, the frequency of one ring element appears with opposite sign in the frequency of another (as for instance λ^N does in Fig. 9). Then two poles contribute to the sum over this frequency, one yielding $n(q)$, the other $(1-n)$. The $(1-n)$ term then fails to produce the usual $T^{3/2}$ factor. However, the factor in the denominator is still of the form $\beta\bar{\omega}$, so that the final contribution, instead of being $T^{3/2}$, is T.

Thus, in general, a transverse graph with Q-independent q's, of which Q' appear with both positive and negative sign in the labeling of the ring elements, m semi-invariants of order greater than 2, and p_α groups

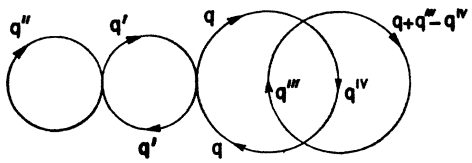

FIG. 9. An example of a graph with $Q=5$, $Q'=1$, $m=4$, $p_2=2$, and, therefore, a contribution to $-\beta F$ proportional to T^5.

of α ring elements with the same q, contributes to $-\beta F$ a term

$$-\beta F(Q,Q',m,p_\alpha) \propto T^{\nu},$$

$$\nu=[\tfrac{3}{2}(Q-Q')+Q'-m+\sum_{\alpha=2}^{\infty} p_\alpha(\alpha-1)]. \quad (70)$$

For example, the graph of Fig. 9 gives a term $\propto T^5$.

In order to carry out a similar analysis for the general case with mixed semi-invariants, we use Eq. (67) to replace the longitudinal bonds in a (tree renormalized) diagram by equivalent vertices V in an otherwise completely transverse diagram. This is represented pictorially for a section with l bonds in Fig. 10, and

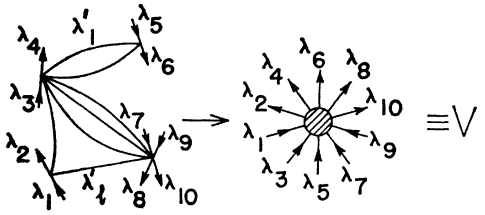

FIG. 10. Replacement of longitudinal bonds by an equivalent transverse vertex V.

corresponds to summing over the l λ'_i's, and inserting l factors of $\frac{1}{2}\beta\bar{v}$.

Consider the general vertex of this type with l longitudinal bonds and d different parts which it joins, with altogether N incoming and N outgoing transverse arrows. From Eq. (67) it is evident that the first $2n$ denominators of each M_{2n+l} contribute to the final vertex a factor which is exactly the $2N$ denominators of $M_{2N}(\lambda_1\cdots\lambda_{2N})$. The d δ functions from the mixed semi-invariants produce λ conservation for the final vertex, and also give $d-1$ restrictions on the l sums over λ'. There then remain the extra [sums of products of $(2l-d)$ denominators] from all the mixed semi-invariants, and the l factors $\frac{1}{2}\beta\bar{v}$ from the bonds. Hence, the final vertex is

$$V\sim\frac{\delta_{\lambda_1\cdots\lambda_{2n},0}(\beta\bar{v}/2)^l}{\beta(h+\frac{1}{2}v(0)-i\lambda_1)\cdots\beta(h+\frac{1}{2}v(0)-i\lambda_{2N})}\sum_{l-d+1 \text{ of the } \lambda''\text{s and } q'\text{s}}[\text{Sums of products of } (2l-d) \text{ denominators}]. \quad (71)$$

Each of the products of $(2l-d)$ denominators consists of d factors, each of the form of the last summand of equation (67) with the appropriate $D_1^{(i)}$, and h replaced by $h+\frac{1}{2}v(0)$. The sum over all independent λ''s will then produce a product of $l-1$ denominators, each of which is just a function of the $D_1^{(i)}$. Furthermore, the denominators must be sums of $\beta D_1^{(i)}$ simply because of λ'-conservation cross the longitudinal bond. Thus, the complete vertex is

$$V\sim\frac{\delta_{\lambda_1\cdots-\lambda_{2n},0}(\beta\bar{v}/2)^l}{\beta(h+\frac{1}{2}v(0)-i\lambda_1)\cdots\beta(h+\frac{1}{2}v(0)-i\lambda_{2N})}[\text{Sums of products of } l-1 \text{ denominators, each of form } \beta\textstyle\sum_{\{i\}} D_1^{(i)}]. \quad (72)$$

When such a vertex occurs in a general graph, it behaves like a transverse semi-invariant of order $2N$, except that it possesses the factor $(\beta\bar{v}/2)^l$ and $l-1$ extra denominators of form $\beta\sum D_1^{(i)}$, and lacks the sum of single terms $[\beta(h+\frac{1}{2}v(0)-i\lambda_i)]$. [Compare Eq. (72) with Eq. (68).] When computing the complete graph by summing over all λ_i, the extra poles that can occur from the denominators $\beta\sum D_1^{(i)}$ can be neglected,

since these lead to distribution functions of the type $\{\exp[\beta\sum\frac{1}{2}v(q)]-1\}^{-1}$. When the values of $\lambda_1\cdots\lambda_{2N}$ determined by the poles of the ring denominators are inserted into $\beta\sum D_1^{(i)}$, a sum of $\beta v(q)$ terms is obtained {as was found in the transverse case for the numerators $[\beta(h+\frac{1}{2}v(0)-i\lambda_i)]$}. The effect of the equivalent vertex is, therefore, the same as that of the $2n$th order semi-invariant except for a factor $(\beta\bar{v}/2)^l/[(\beta\bar{v}/2)(\beta\bar{v}/2)^{l-1}]$, proportional to T^0, independent of l, d, and T.

Thus, the temperature dependence of the contribution to $-\beta F$ from a graph with m vertices of order greater than 2, of which m' are equivalent vertices is again given by Eq. (70).

Since $Q\geqslant m+1$, the lowest order graphs are those with m and p_α small. All graphs with $p_\alpha>0$ are at least of order $(T^{3\frac{1}{2}})$ and any semi-invariant or effective vertex of order greater than 4 requires at least $Q=3$ and is, therefore, also negligible to $O(T^3)$. The only graphs whose contributions to $-\beta F$ are $O(T^3)$ or less are those drawn in Fig. 11. There, the vertex is the sum of M_4 and all equivalent vertices of this order.

This completes the low-temperature ordering. One final important point should be noticed, namely, that

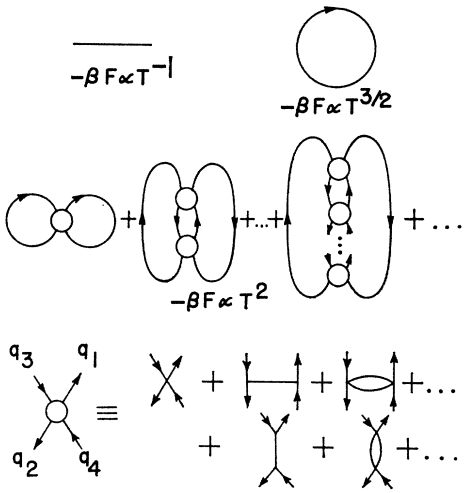

FIG. 11. All graphs contributing to $-\beta F$ a term of order T^3 or less. The vertex is the sum of M_4 and all equivalent vertices of fourth order.

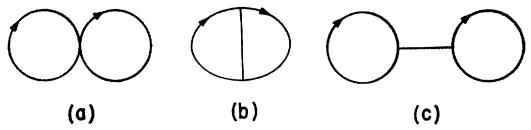

FIG. 12. Graphs yielding the first Born approximation to the scattering of spin waves.

the classification of graphs with respect to powers of the temperature is the same in the first two orders as the simple classification of graphs with respect to powers of $(1/z)$ [the leading term in each expansion, $O(1)$, is the single longitudinal bond]. This explains why the ring summation becomes valid again below the Curie point to $O(T^{3/2})$, as was noted in Sec. 4.

6. SPIN-WAVE INTERACTIONS AT LOW TEMPERATURE

Dyson[6] has very carefully developed a theory which gives the power law deviations from spin-wave theory at low T due to spin-wave interactions. His theory neglects exponential effects and hence becomes inadequate once $\exp[-\frac{1}{2}\beta v(0)]$ becomes of $O(1)$. In the preceding section it was seen that the first corrections to spin-wave theory at low temperatures are given by the free-energy diagrams of Fig. 11 which contain more than one ring element. These are all the contributions to powers less than $T^{4\frac{1}{2}}$ in the low-temperature expansion of F. In this section the formalism we have developed is used to examine the effect of these graphs, for comparison with the results of Dyson.

We shall calculate in detail the simplest of these graphs, Figs. 12(a), (b), and (c), obtained by taking once the simplest part of the vertex of Fig. (11), to show how the T^3 part cancels between these graphs leaving a contribution to F of $O(T^4)$. We then show the equivalence of this set to the first Born approximation to spin-wave scattering calculated by Dyson.

The graph of Fig. 12(a) is composed of transverse semi-invariants of second and fourth order. Examination of the detailed expressions for these, Sec. 2, shows that $\mathfrak{m}_2^{\max}=1$, $\mathfrak{m}_4^{\max}=-2$ so that using Eq. (68), apart from terms of $O(\exp[-\frac{1}{2}\beta v(0)])$,

$$M_2(\lambda_1\lambda_2)=\delta_{\lambda_1\lambda_2}[\beta(h+\tfrac{1}{2}v(0)-i\lambda_1)]^{-1}, \tag{73}$$

$$M_4(\lambda_1\lambda_2\lambda_3\lambda_4)=\frac{-2\delta_{\lambda_1+\lambda_2,\lambda_3+\lambda_4}[\beta(h+\frac{1}{2}v(0)-i\lambda_1)+\beta(h+\frac{1}{2}v(0)-i\lambda_2)]}{\beta(h+\frac{1}{2}v(0)-i\lambda_1)\beta(h+\frac{1}{2}v(0)-i\lambda_2)\beta(h+\frac{1}{2}v(0)-i\lambda_3)\beta(h+\frac{1}{2}v(0)-i\lambda_4)}. \tag{74}$$

Thus, the contribution to the free energy is given by

$$-\beta F^{(1)}=-\tfrac{1}{2}\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\sum_{\lambda,\lambda'}\sum_{q,q'}2[\beta(h+\tfrac{1}{2}v(0)-i\lambda)+\beta(h+\tfrac{1}{2}v(0)-i\lambda')]$$
$$\times\frac{[-\beta v(q)/2]^{n_1}}{[\beta(h+\frac{1}{2}v(0)-i\lambda)]^{n_1+1}}\frac{[-\beta v(q')/2]^{n_2}}{[\beta(h+\frac{1}{2}v(0)-i\lambda')]^{n_2+1}}, \tag{75}$$

apart from terms of $O(\exp[-\frac{1}{2}\beta v(0)])$. Here, n_1 is the number of bonds in its first ring, n_2 is the number in the second ring, and we have used the $q-$, $\lambda-$ conservation to reduce the λ and q sums to two sums. The result of summing over all n_1 and n_2 is

$$-\beta F^{(1)}=-\sum_{\lambda\lambda'}\sum_{qq'}\frac{[\beta(h+\frac{1}{2}v(0)-i\lambda)+\beta(h+\frac{1}{2}v(0)-i\lambda')]}{[\beta(\omega(q)-i\lambda)\beta(\omega(q')-i\lambda')]}\left[\frac{-\beta v(q)/2}{\beta(h+\frac{1}{2}v(0)-i\lambda)}\right]^2\left[\frac{-\beta v(q')/2}{\beta(h+\frac{1}{2}v(0)-i\lambda')}\right]^2. \tag{76}$$

The sums over λ, λ' can be converted to contour integrals with respect to z, z', by inserting into the integrand factors $f^{\pm}(z)=\pm\beta(e^{\pm\beta z}-1)^{-1}$ and $f^{\pm}(z')$, which have poles with unit residue at z, $z'=(2\pi i/\beta)j$. Since there are at least two denominators containing z and at least two containing z', both integrals converge at infinity for either choice of $f^{\pm}$. In either case we find

$$-\beta F^{(1)}=-\sum_{qq'}n(q)n(q')\tfrac{1}{2}\beta[v(q)+v(q')]+O(\exp[-\tfrac{1}{2}\beta v(0)]). \tag{77}$$

The low-temperature values of the graphs of Figs. 12(b) and 12(c) may be calculated in a similar way. As outlined in the previous section, the effect of the longitudinal bond of momentum q'' can be replaced by an equivalent vertex, $M^{l=1}{}_{4\text{ equivalent}}$, which is arrived at by summing $[2\beta v(q'')/2]$ times the two third-order mixed semi-invariants over the frequency λ' carried by the longitudinal bond:

$$M^{1}{}_{4\text{ equivalent}}(\lambda_1\lambda_3\lambda_2\lambda_4)=\beta v(q'')\sum_{\lambda'}M_{2+1}(\lambda_1\lambda_2\lambda')M_{2+1}(\lambda_3\lambda_4-\lambda'). \tag{78}$$

$M_{2+1}(\lambda_1\lambda_2\lambda')$ is given by Eq. (61) or by Eq. (67) with the appropriate value (-1) of $\mathfrak{m}_{2+1}{}^{\max}$ and $h\to h+v(0)\hat{M}_1$. Hence, the vertex is

$$M^{1}{}_{4\text{ equivalent}}(\lambda_1\lambda_3\lambda_2\lambda_4)=\beta v(q'')\delta_{\lambda_1+\lambda_3,\lambda_2+\lambda_4}\times[\beta(h+\tfrac{1}{2}v(0)-i\lambda_1)\beta(h+\tfrac{1}{2}v(0)-i\lambda_2)\beta(h+\tfrac{1}{2}v(0)-i\lambda_3)\beta(h+\tfrac{1}{2}v(0)-i\lambda_4)]^{-1}, \tag{79}$$

where tree renormalization has been included. The contributions of the graphs of Figs. 12(b) and 12(c) are, respectively,

$$-\beta F^{(2)}=\tfrac{1}{2}\sum_{qq'q''}\sum_{\lambda\lambda'}[\beta v(q'')]\delta(q-q'-q'')\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{[-\beta v(q)/2]^{n_1}}{[\beta(h+\frac{1}{2}v(0)-i\lambda)]^{n_1+1}}\frac{[-\beta v(q')/2]^{n_2}}{[\beta(h+\frac{1}{2}v(0)-i\lambda')]^{n_2+1}}, \tag{80}$$

$$-\beta F^{(3)}=\tfrac{1}{2}\sum_{qq'q''}\sum_{\lambda\lambda'}[\beta v(q'')]\delta(q'')\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{[-\beta v(q)/2]^{n_1}}{[\beta(h+\frac{1}{2}v(0)-i\lambda)]^{n_1+1}}\frac{[-\beta v(q')/2]^{n_2}}{[\beta(h+\frac{1}{2}v(0)-i\lambda')]^{n_2+1}}. \tag{81}$$

Again the sums over λ, λ' are carried out by contour integration with the results

$$-\beta F^{(2)}=\textstyle\sum_{qq'}n(q)n(q')\frac{1}{2}\beta v(q-q'), \tag{82}$$

$$-\beta F^{(3)}=\textstyle\sum_{qq'}n(q)n(q')\frac{1}{2}\beta v(0). \tag{83}$$

Summing the results so far obtained,

$$-\beta(F^{(1)}+F^{(2)}+F^{(3)})=\tfrac{1}{2}\beta\textstyle\sum_{qq'}n(q)n(q')[v(0)-v(q)-v(q')+v(q-q')], \tag{84}$$

which is the result of Dyson's first Born approximation to the scattering of spin waves. As was pointed out by Dyson, the free energy in this approximation is of $O(T^4)$ though the constitutent graphs each give a contribution of $O(T^3)$.

We shall not carry out in this paper a complete calculation of the remaining contributions from Fig. 11. But as an indication of the general trend we make the following points.

The first terms in the vertex of Fig. 11, namely, $M_4+M_{4\text{ equivalent}}$, combine to a product of four denominators $[\beta(v(0)-i\lambda_i)]$ multiplied by the following function of the incoming momenta:

$$\hat{V}(q_1q_2q_3q_4)=\delta(q_1+q_2-q_3-q_4)[v(q_1-q_3)-v(q_3)-v(q_4)+v(q_2-q_3)]. \tag{85}$$

[We have replaced the frequencies which occur in the numerator of M_4 by the spin-wave frequencies $\omega(q_i)$ associated with the ring elements to which the vertex is joined.] The interaction of Eq. (85) is the effective interaction $\Gamma_{q_1q_2}{}^{(q_1-q_3)}$ obtained by Dyson, and used by him to obtain the Born approximations.

The remaining factors in the calculation of the graphs of Fig. 11 result from carrying out the sums with respect to λ of the product of spin-wave denominators $(\omega(q_i)-i\lambda_i)$. There is always at least one more denominator than there are independent λ_i. Thus, after completing the sums there remain one or more denominators D which are linear combinations of the $\omega(q_i)$. Such a factor D appears in Dyson's second Born approximation and will appear in all higher ones.

The spin-wave distribution functions $n(q_i)$ which also result from the sums over λ_i cannot be treated so

simply, but the above discussion does show that there is a close correspondence between the successive graphs of Fig. 11 and the successive Born approximations to the spin-wave scattering.

Finally, an alternative method of arriving at the corrections due to spin-wave interactions is given in Appendix A.

7. APPROACH TO THE CURIE POINT IN HIGH-DENSITY APPROXIMATION

In reference 5, simple summation of longitudinal rings was discussed as a possible approximation to the Ising model. This approximation, which has the same critical temperature as the molecular field model, was shown to be inconsistent to $O(1/z)$ and, hence, is not an adequate approximation. Nevertheless, it is a theory that has some of the features of ferromagnetism which go beyond the molecular field theory. We, therefore, first present the corresponding simple theory for the Heisenberg model and then give the modifications required to make it a satisfactory approximation.

The sum of transverse rings with tree renormalization was performed in Sec. 4 with the result [cf. Eq. (43)] that

$$\langle S_{-q}^{-}S_q^{+}\rangle = \frac{\tanh[\frac{1}{2}\beta(h+v(0)\hat{M}_1)]}{\exp[\beta\{h+v(0)\hat{M}_1-\frac{1}{2}v(q)\tanh\frac{1}{2}\beta(h+v(0)\hat{M}_1)\}]-1}. \tag{86}$$

To complete the theory one must add the sum of longitudinal rings with tree renormalization [Appendix D(i)]:

$$\langle S_{-q}^{z}S_q^{z}\rangle=[1-\beta v(q)(\tfrac{1}{4}-\hat{M}_1^{2})]^{-1}. \tag{87}$$

For $\beta v(0)\gg 1$, where $\hat{M}_1\sim\frac{1}{2}$, Eq. (86) leads to spin-wave theory. Further, it contains only exponential deviations from spin-wave theory at low temperatures and in the same limit, Eq. (87) is exponentially small. For $\frac{1}{4}\beta v(0)=O(1)$ and $h\to 0$, when $\hat{M}_1\sim 0$, Eqs. (86) and (87) can be expanded and lead, respectively, to

$$\langle S_{-q}^{-}S_q^{+}\rangle \underset{\hat{M}_1\to 0,\,h\to 0}{\longrightarrow} \tfrac{1}{2}[1-\tfrac{1}{4}\beta v(q)]^{-1}, \tag{88}$$

$$\langle S_{-q}^{z}S_q^{z}\rangle \underset{\hat{M}_1\to 0,\,h\to 0}{\longrightarrow} [1-\tfrac{1}{4}\beta v(q)]^{-1}. \tag{89}$$

Thus, as pointed out in reference 5, for $\hat{M}_1=0$ the quantum and classical theories merge for the class of simple ring graphs. Equations (88) and (89) present a Curie point at $\frac{1}{4}\beta_c v(0)=1$. The difficulty of the above theory is that at the Curie point it leads to infinite fluctuations. Thus, if (86) is used to determine the magnetization from the sum rule, whereas at low T, there is perfectly normal descent of the magnetization curve with temperature increase, when the Curie point region is reached the curve turns up and the magnetization becomes infinite at T_c.

As mentioned in reference 5, this type of inconsistency can be eliminated in the Ising model calculation by passing to the spherical model which keeps the fluctuations finite. A more systematic approach was taken in reference 12 where it was shown that a variational principle based on consistent vertex renormalization eliminates the difficulty. Each of these methods represents a renormalized version of the high-density theory.[7] The difference in these two theories is negligible even for near-neighbor interactions. By consistent vertex renormalization, we mean that if a subset of graphs (rings and trees in this instance) is taken as a basic approximation, then each vertex of a skeleton structure (a ring or a tree) must have articulated to it any number of times the same skeleton structures. In this way it is possible to get a reasonable approach to the Curie point.

In the high-density case, the skeleton graphs are rings and trees, which is fortunate since these are just the graphs which dominate the low-temperature spin-wave region. We now proceed with the formalism required to renormalize vertices with these fundamental elements. Some of this renormalization has already been carried out in Sec. 4. Equations (38) and those that follow give the complete tree renormalization.

Consider now the combined effects of articulating single bonds and longitudinal rings at a vertex. In Appendix D the contribution of an unrenormalized longitudinal ring to $\langle TS^z(\beta')S^z(\beta'')\rangle$ is computed. We denote by $G_2{}^L$ its value excluding the semi-invariants of the final vertices. The self-energy $G_2{}^L$, which is a functional of the semi-invariants $M_2(S^z(\beta_1)S^z(\beta_2))$ is found to be independent of β', β'', so that we may use the same techniques as were used in calculating the renormalization due to single bonds alone. If we articulate ν longitudinal loops and μ longitudinal bonds to the vertex whose unrenormalized semi-invariant is M_{2k+l}, each bond contributes a factor $\frac{1}{2}v(0)M_1$ and raises the order of the semi-invariant by one by introducing an extra factor $\int_0^\beta d\beta'\,S^z(\beta')$ into its argument, and each longitudinal loop contributes a factor $G_2{}^L$ and raises the order of the semi-invariant by 2. The g factor associated with μ bonds and ν loops is $\mu!\,2^\nu\nu!$ so that the value of the vertex now becomes

$$2^{\mu}\frac{(\frac{1}{2}v(0)M_1)^{\mu}}{\mu!}\frac{(G_2{}^L)^{\nu}}{2^{\nu}\nu!}\left(\frac{\partial}{\partial\gamma}\right)^{\mu+2\nu}\Big|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2k+l}}\Big|_{\alpha_{2k+l}=0}$$

$$\times\ln\left\langle T\exp\left[\gamma\int_0^\beta d\beta' S^z(\beta')+\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2k+l}S^z(\beta_{2k+l})\right]\right\rangle_0.$$

The renormalization of the vertex produced by all single bonds and longitudinal loops is, therefore,

$$\exp[v(0)M_1\partial/\partial\gamma+\tfrac{1}{2}G_2{}^L\partial^2/\partial\gamma^2]|_{\gamma=0}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2k+l}}\Big|_{\alpha_{2k+l}=0}$$
$$\times\ln\left\langle T\exp\left[\gamma\int_0^\beta d\beta' S^z(\beta')+\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2k+l}S^z(\beta_{2k+l})\right]\right\rangle_0$$
$$=\exp[v(0)M_1\partial/\partial h+\tfrac{1}{2}G_2{}^L\partial^2/\partial h^2]M_{2k+l}(TS^+(\beta_1)\cdots S^z(\beta_{2k+l})), \quad (90)$$

following the same reasoning that was used in Sec. 4 starting from Eq. (38).

Full renormalization with this class of diagrams, bonds and longitudinal rings will involve the articulation of extra bonds and rings onto the vertices of each bond and ring already articulated. This is equivalent to performing the first renormalization with renormalized rings and bonds. The completely renormalized vertex is, therefore, given by the self-consistent set of equations

$$\tilde{M}_{2k+l}=\exp[v(0)\tilde{M}_1\partial/\partial h+\tfrac{1}{2}\tilde{G}_2{}^L\partial^2/\partial h^2]M_{2k+l}, \quad (91)$$

where

$$\tilde{G}_2{}^L=G_2{}^L\{\tilde{M}_2(S^z,S^z)\}. \quad (92)$$

This is an extension of one of the results of Horwitz and Callen.[12]

The renormalization produced by transverse loops is much more difficult to treat. The main reason for this is that the factor $G_2{}^T(\beta',\beta'')$ obtained by attaching a transverse ring to a vertex is dependent on the temperature labels β', β'' of its extreme bonds (see Appendix D). It also depends on the value of the semi-invariants $M_2(TS^+S^-)$. The articulation of ν transverse loops, therefore, changes the vertex M_{2k+l} to

$$M_{2k+l}{}^T=\frac{1}{\nu!}\prod_{i=1}^{\nu}\left(\int_0^\beta d\beta'_i\int_0^\beta d\beta''_i\, G_2{}^T(\beta'_i,\beta''_i)\right)M_{2k+2\nu+l}\left(TS^+(\beta_1)\cdots S^z(\beta_{2k+l})\prod_{i=1}^{\nu}(S^+(\beta'_i)S^-(\beta''_i))\right). \quad (93)$$

Summing this over all ν and using the generating function for the semi-invariants we, therefore, arrive at a renormalized semi-invariant

$$M_{2k+l}{}^T=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\frac{\partial}{\partial\alpha_1}\Big|_{\alpha_1=0}\cdots\frac{\partial}{\partial\alpha_{2k+l}}\Big|_{\alpha_{2k+l}=0}\prod_{i=1}^{\nu}\left(\int_0^\beta d\beta'_i\int_0^\beta d\beta''_i G_2{}^T(\beta'_i,\beta''_i)\frac{\partial}{\partial\gamma_i}\Big|_{\gamma_i=0}\frac{\partial}{\partial\Gamma_i}\Big|_{\Gamma_i=0}\right.$$
$$\left.\times\ln\langle T\exp[\sum_i\{\gamma_i S^+(\beta'_i)+\Gamma_i S^-(\beta''_i)\}+\alpha_1 S^+(\beta_1)+\cdots+\alpha_{2k+l}S^z(\beta_{2k+l})]\rangle_0\right. \quad (94)$$

Clearly, this is further renormalized by articulating single bonds and longitudinal rings at the same vertex which adds an extra factor $R_L=\exp[v(0)M_1\partial/\partial h+\frac{1}{2}G_2{}^L\partial^2/\partial h^2]$ in the same way as before. We have not been able to reduce the effects of the articulated transverse rings to as simple an operator as in Eq. (91).[19] However, let us suppose that they may be represented in this way by an operator R_T, so that the expression (94) is R_TM_{2k+l}. Obviously, R_T is a functional of $G_2{}^T$. The complete renormalization within the class of rings and chains will then be

$$\mathfrak{M}_{2k+l}=\tilde{R}_T\tilde{R}_LM_{2k+l}, \quad (95)$$

where

$$\tilde{R}_T=R_T(\mathcal{G}_2{}^T), \quad \mathcal{G}_2{}^T=G_2{}^T(\mathfrak{M}_2(TS^+S^-)), \quad (96)$$

and

$$\tilde{R}_L=R_L(\mathfrak{M}_1,\mathcal{G}_2{}^L), \quad \mathcal{G}_2{}^L=G_2{}^L(\mathfrak{M}_2(TS^zS^z)). \quad (97)$$

This provides a formal solution to the problem in zeroth and first order in the renormalized version of the high-density expansion.

The main problem in the investigation of the high-density terms is the evaluation of the operator R_T. The complications appear when one tries to perform the integrations $\int_0^\beta d\beta'_i$, $\int_0^\beta d\beta''_i$ of Eq. (94), for the variables β'_i, β''_i label the position of the operators $S^+(\beta'_i)$, $S^-(\beta''_i)$ in the time-ordered semi-invariant, as well as appearing explicitly in $G_2{}^T$. We are interested in evaluating the renormalization of $M_2(TS^+(\beta_1)S^-(\beta_2))$ and here we remove the complication by means of the following approximation: We replace the $(2\nu+2)$th order semi-invariant of Eq. (93) ($k=1$, $l=0$) by $(-1)^\nu$ times the sum of all products of second-order semi-invariants which do not result in unlinked graphs. Thus,

[19] The renormalization can be performed at low temperatures, using the methods of Sec. 5. Also the renormalization caused by unrenormalized transverse rings can be evaluated for temperatures above the Curie point, where $G_2{}^T$ is independent of β', β''. We there find that R^T is the operator that puts $\lambda_1S^+ + \lambda_2S^-$ in place of hS^z wherever it occurs in the semi-invariant and then operates on the resulting function with $\exp[G_2{}^T(\partial/\partial\lambda_1)(\partial/\partial\lambda_2)]|_{\lambda_1=\lambda_2=0}$. However, further renormalization will introduce a time dependence into $G_2{}^T$ so that the method is not adequate to give the self-consistent renormalization.

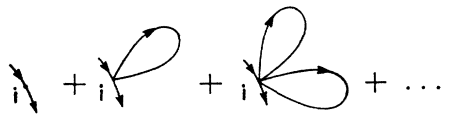

FIG. 13. Articulation of transverse rings at a vertex.

we have replaced the renormalization of the vertex i shown in Fig. 13 by the approximation obtained by taking just the linked graphs from Fig. 14. In the figures, each of the vertices has the same spin index i. The subdiagrams obtained from approximating ν articulated graphs have to be divided by the g factor $\nu!$ associated with the original diagram. If we introduce the dotted line notation of Brout[5] in which a dotted line connecting two vertices i, j denotes the spatial function $-\delta_{ij}$, the approximation sums the diagrams of Fig. 15. The corresponding longitudinal diagrams are the "excluded volume" diagrams summed by Brout.[5] Our approximation is, therefore, the quantum-mechanical analog of Brout's approximation, and the errors made are analogous to the difference between Brout's result and the result of Horwitz and Callen, who summed the classical analog of the series drawn in Fig. 13. The two treatments of the classical model are in close agreement.

The summation of the diagrams of Fig. 15 is similar to the ring summation, and is carried out in Appendix E. There it is shown that if we make this approximation

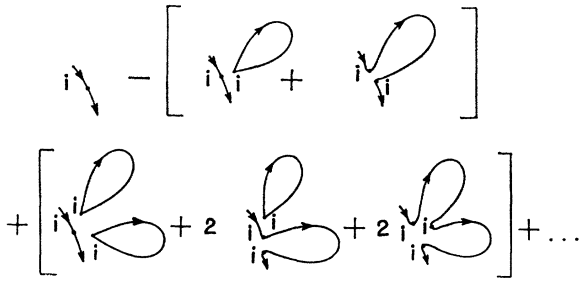

FIG. 14. Approximation of the vertices of Fig. 13 by products of second-order semi-invariants.

to the renormalization with transverse rings self-consistent by renormalizing in a similar fashion the semi-invariants of the added "rings," we obtain the following equation for the renormalized semi-invariant. Writing

$$M_2(\lambda)\mu(\lambda)=\sum_q v(q)\hat{M}_2(q,\lambda), \tag{98}$$

where $\hat{M}_2(q,\lambda)$ is the transform of $\hat{M}_2(TS_i^+(\beta')S_j^-(\beta''))$, we have

$$M_2(\lambda)\mu(\lambda)=\sum_q \frac{v(q)M_2(\lambda)}{1-\frac{1}{2}\beta[v(q)-\mu(\lambda)]M_2(\lambda)}. \tag{99}$$

In the limit of vanishing long-range order, $\mu(\lambda)$ becomes zero for all $\lambda\neq 0$, and the equation satisfied by $\mu\equiv\mu(0)$ is then

$$\mu=\sum_q \frac{v(q)}{1-\frac{1}{2}\beta[v(q)-\mu]}. \tag{100}$$

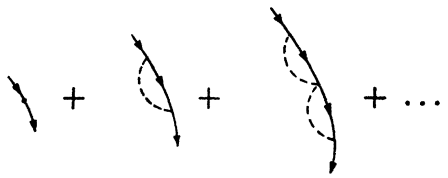

FIG 15. Excluded volume approximation to renormalization with transverse rings.

The energy of the excluded volume ring is

$$E_{\text{E.V.}}=\lim_{\beta'\to 0^+}\sum_{q,\lambda}\frac{e^{i\lambda\beta'}\frac{1}{2}v(q)M_2(\lambda)}{1-\frac{1}{2}\beta[v(q)-\mu(\lambda)]M_2(\lambda)}, \tag{101}$$

which in the limit of zero long-range order becomes

$$E_{\text{E.V.}}=\sum_q \frac{\frac{1}{2}v(q)}{1-\frac{1}{2}\beta[v(q)-\mu]}. \tag{102}$$

After renormalization with all single bonds, which are themselves renormalized with renormalized bonds and excluded volume rings, Eq. (101) gives for $\langle S_q^- S_{-q}^+\rangle$ the value

$$\langle S_q^- S_{-q}^+\rangle=\lim_{\beta'\to 0^+}\sum_k \frac{e^{i\lambda_k\beta'}\tanh[\frac{1}{2}\beta(h+v(0)R)]}{\beta\{h-i\lambda_k+v(0)R-\frac{1}{2}[v(q)-\mu(\lambda_k)]\tanh[\frac{1}{2}\beta(h+v(0)R)]\}}, \tag{103}$$

where R is the magnetization determined self-consistently from (103). Except for the term $\mu(\lambda_k)$ in the denominator this is our previous ring graph summation and hence spin-wave theory at low T. From Eq. (98), it is seen that $\mu(\lambda)$ is $O(\sum_q v(q)n(q))=O(T^{3/2})$. The presence of $\mu(\lambda)$ in the denominator, therefore, represents an approximation to spin-wave–spin-wave interactions due to a diagram of the type shown in Figs. 12(a) or 12(c) but now with arbitrarily many transverse loops articulated on to each vertex of the skeleton graph. At the Curie point this theory extrapolates to the spherical model from Eq. (103). In the "excluded volume" approximation, the longitudinal and transverse loops do not mix. Thus, the energy of the Heisenberg model in the "excluded volume" approximation is the sum of longitudinal and transverse energies, the longitudinal part being given by the spherical model approximation of Brout. At T_c, the sum, therefore, reduces to the spherical model approximation to the Heisenberg model and for low T to spin-wave theory again with incorrect T^3 corrections. The magnetization is given by substituting Eq. (103) into the sum rule (51).

This approximation includes two [Figs. 12(a) and

12(c)] of the three graphs corresponding to Dyson's first Born approximation corrections to spin-wave theory but omits the third graph [Fig. 12(b)]. An exact calculation of this third graph shows that it goes to zero at the Curie point. In an intermediate region, this graph together with those corresponding to higher Born approximations to the scattering of two spin waves may be important. For reasonable z, this intermediate region will not be very extensive. We will present the argument for this at the end of this section.

Since the theory sketched above is extremely similar in its behavior to RPA, derived in Appendix A, the latter theory is probably preferable because of its simplicity. In common with the above theory, RPA gives spin-wave theory correctly to $T^{5/2}$, the spherical model at T_c, and it does not possess the disadvantage of presenting infinite fluctuations at T_c. On the contrary, at T_c it presents the normal infinite slope magnetization curve and has most of the allure of a molecular field theory. For $T>T_c$ this theory gives Ornstein-Zernicke type critical fluctuations which leads to a correlation distance $\propto (T-T_c)^{-1/2}$. All in all, RPA represents the qualitative facts of the Heisenberg ferromagnet very well and probably gives many features with considerable accuracy even for near-neighbor interactions.

We now study the temperature range where spin-wave–spin-wave interactions are most important, since these are misrepresented by theories of the above type. The interactions become most important just below the temperature at which spin-wave theory breaks down and gives way to molecular-field-type theory. One may use as a tool for this study either Eq. (103) or RPA. We shall use RPA. In the form of Eq. (A7), this gives for $h=0$

$$\langle S_q^- S_{-q}^+\rangle = 2\bar{M}_1\{\exp[\beta(v(0)-v(q))\bar{M}_1]-1\}^{-1}. \quad (104)$$

The first thing to notice is that for $z\to\infty$ [i.e., $v(q)\to\delta_{0q}v(0)$] this theory goes over correctly to the Weiss theory. In fact, (A7) becomes

$$\bar{M}_1=\frac{1}{2}\left[1-4\bar{M}_1\sum_q\frac{1}{\exp\beta v(0)\bar{M}_1-1}\right]$$

$$=\frac{1}{2}-\frac{2\bar{M}_1}{\exp\beta v(0)\bar{M}_1-1},$$

or

$$\bar{M}_1=\tfrac{1}{2}\tanh[\tfrac{1}{2}\beta v(0)\bar{M}_1]. \quad (105)$$

Now for finite z, $v(q)$ will cut off at $q\propto z^{-1/3}$. Hence, in the sum of q in (A7) there will be $\propto 1/z$ terms of the "spin-wave" type and $\propto(1-1/z)$ of the molecular-field type. At low T, only the first type gives a contribution; the remaining terms will be exponentially small. Hence, at low T, one will have a pure spin-wave-type behavior. To see this more clearly, let us approximate the potential by

$$v(0)-v(q)\sim v(0)z^{2/3}q^2 \quad \text{for} \quad q<z^{-1/3},$$
$$\sim v(0) \quad \text{for} \quad q>z^{-1/3}.$$

The magnetization is then [using (104) and (A7)]

$$\bar{M}_1=\frac{1}{2}\left[1-\frac{4\bar{M}_1}{2\pi^2 z}\int_0^1\frac{d^3q}{e^{\frac{1}{2}\beta v(0)q^2}-1}-\frac{4\bar{M}_1[1-(1/2\pi^2 z)]}{e^{\frac{1}{2}\beta v(0)}-1}\right]. \quad (106)$$

In the second term of the bracket we have changed variables to $z^{1/3}q$. This term is the spin-wave contribution and the third term is the molecular-field term. These two become equal at a temperature which we denote by T_e. T_e is given approximately by

$$\frac{1}{2\pi^2 z}\left(\frac{2kT_e}{v(0)}\right)^{3/2}=e^{-[v(0)/2kT_e]}\left(1-\frac{1}{2\pi^2 z}\right). \quad (107)$$

For $T<T_e$, the spin-wave contributions dominate, and Dyson's corrections will have a role in this region. For $T>T_e$, the principal effects are a result of the kinematic interaction.

In the limit of infinite z, (107) gives $T_e=0$. For finite z, representative solutions of the transcendental equation are: for $z=10$, $T_e/T_c\sim 0.2$; for $z=100$, $T_e/T_c\sim 0.15$. [We have put $kT_c=\frac{1}{4}v(0)$.] Thus, we find that spin-wave effects dominate the behavior of the model up to about 20% of the Curie point. In this range, Dyson's correction which is of $O((T/T_c)^{5/2})$ of the usual spin-wave theory is quite negligible. Therefore, for the calculation of equilibrium properties it is probably no great error to forsake spin-wave corrections in order to use interpolation formulas either of the RPA type or of the "excluded volume" type, Eq. (103). This will introduce an error of O (5%) at T_c and will extrapolate to a Curie point which is accurate to 5% if Ising-model calculations are to be believed.

It should be emphasized that our arguments are valid for macroscopic properties only. In low-temperature spin-wave resonance or neutron-scattering experiments it should be possible to pick up energy renormalization effects. Such experiments would, indeed, be most valuable.

8. CONCLUSION

In this paper, a finite temperature perturbation theory for the Heisenberg model ferromagnet has been presented. Summation of subsets of graphs has been accomplished by the use of vertex renormalizations. The selection of graphs is determined by high-density and low-temperature classifications. At low temperatures, molecular-field theory, spin-wave theory, and the leading effects of spin-wave interactions are found by summation of graphs. In particular, we have arrived at the Born approximations to the scattering of spin waves and have demonstrated the cancellation of the T^3 terms in the free energy in the first Born approximation.

The simple rings corresponding to spin-wave theory

do not extrapolate in an adequate way to a Curie point since they lead to a divergence in magnetization. Complete vertex renormalization by rings, which would lead to a consistent theory without this divergence,[12] is extremely difficult to carry out in the Heisenberg model, so the "excluded volume" approximation to these articulations is formulated. The ensuing approximation leads to spin-wave theory at low temperatures (with spurious T^3 interaction terms) and extrapolates to the spherical model at T_c. The same behavior is characteristic of RPA which is a simpler theory but without graphical justification in the present approach. For reasonable z, once $kT \gtrsim 0.2kT_c$ the qualitative features of the magnetization are those characteristic of the Weiss field.

Since the first two terms in the simple high-density and low-temperature classifications are the same, one has in low orders the possibility of constructing an extrapolation approximately valid for all T. (The molecular field is the leading approximation of this type.) As remarked above, self-consistency has also to be imposed in order to obtain nondivergent results at the Curie point. The third terms in the low-temperature expansion, which represent the leading effects of spin-wave scattering, are not completely included to any finite order in the high-density expansion, so that an extrapolation to high temperature containing also the spin-wave scattering effects could not be constructed on the basis of a high-density theory.

It should be remarked that a similar analysis can be carried out for the antiferromagnet. Here, of course, the ground state is unknown. However, the graphical approach immediately shows Anderson's spin-wave theory[20] to be valid to $O(1/z)$. If one chooses the zero-order Hamiltonian in this case as a molecular field of value $z\bar{v}$, then the energy denominators in the perturbation expansion are $1/(z\bar{v})$. An nth order ring graph then carries a factor $z^{n-1}\bar{v}^n/(z\bar{v})^{n-1}=\bar{v}$. The "unperturbed energy" is $z\bar{v}$; hence, the ring graphs yield a contribution of $O(1/z)$. The nonring graphs are of still higher order in $1/z$. Summation of rings again yields for the antiferromagnet spin-wave theory. The thermodynamics and approach to the Néel point are now being worked out.

APPENDIX A. EQUATIONS OF MOTION APPROACH TO THE PROBLEM

Here a formalism is presented which allows a simple derivation of random phase approximation (RPA) and of the effects of spin-wave–spin-wave interactions at low temperature.[21] Its limitations will be discussed at the end of this Appendix.

The Hamiltonian, Eq. (2), may be rewritten in Fourier space by defining the operators

$$\mathbf{S}_q=\sum_i e^{iqr_i}\mathbf{S}_i. \tag{A1}$$

Then

$$H=-hS_0^z-\tfrac{1}{2}\sum_q v(q)(S_q^+S_{-q}^-+S_q^zS_{-q}^z),$$

where we have used $\sum_q v(q)=0$. The commutation relations are

$$[S_q^+,S_{-q'}^-]=2S_{q-q'}^z, \tag{A2a}$$

$$[S_q^z,S_{-q'}^\pm]=\pm S_{q-q'}^\pm. \tag{A2b}$$

A simple way to find the spin-wave excitations from the ground state is to operate with the Heisenberg equation of motion for S^- on the ground state, since the spin waves are elementary excitations with $\langle S^z\rangle=\frac{1}{2}N-1$ and, therefore, linear combinations of $S_i^-|0\rangle$. The equations of motion are

$$-i\dot{S}_q^-=[H,S_q^-]=hS_q^--\sum_{q'}[v(q')-v(q-q')]\times S_{q-q'}^zS_{q'}^-. \tag{A3}$$

In arriving at (A3) we have used $v(q)=v(-q)$ and $\sum v(q)=0$. Since $S_{q-q'}^z|0\rangle=\delta_{qq'}\frac{1}{2}|0\rangle$, operating with Eq. (A3) on the ground state produces an equation of the normal mode type $\dot{S}_q^-|0\rangle=i\omega(q)S_q^-|0\rangle$, where $\omega(q)=h+\frac{1}{2}[v(0)-v(q)]$. These are the spin-wave solutions. This simplicity stops when more than one spin wave is present. However, the following generalization allows an approximate extension to higher temperatures. In order to obtain the excitations from the ground state we replaced S_q^z by its "vacuum expectation" value, i.e., $\delta_{0q}/2$. It would then seem reasonable to adopt this idea at all temperatures and replace S_q^z by its thermal average:

$$\langle S_q^z\rangle=\delta_{0q}\bar{M}_1, \tag{A4}$$

and then determine $\bar{M}_1$ self-consistently from (51). This is the random phase approximation. The frequencies are then obtained by replacing $S_{q-q'}^z$ in (A3) by its mean value (A4). This gives rise to

$$\omega\rightarrow\tilde{\omega}(q)=[v(0)-v(q)]\bar{M}_1+h. \tag{A5}$$

The magnetic equation of state is obtained from (51) in the form

$$\bar{M}_1=\tfrac{1}{2}[1-2\textstyle\sum_q\langle S_q^-S_{-q}^+\rangle]. \tag{A6}$$

Taking the expectation value of the commutator in (A2a) shows that in this approximation the boson operators are $S_q^\pm(2\bar{M}_1)^{-1/2}$ since these operators would have the property of giving unit commutator. Thus, we write (A6) in the form [with $\tilde{n}(q)$ the number of excited bosons of type q]

$$\begin{aligned}\bar{M}_1&=\tfrac{1}{2}[1-4\bar{M}_1\textstyle\sum_q\tilde{n}(q)]\\&=\tfrac{1}{2}[1-4\bar{M}_1\textstyle\sum_q[\exp\{\beta[(v(0)-v(q))\times\bar{M}_1+h]\}-1]^{-1}].\end{aligned} \tag{A7}$$

Equation (A7) is the RPA equation of state. At the Curie point, it reduces to the spherical model with $kT_c=\frac{1}{4}v(0)-O(1/z)$. At low T, it gives deviations from spin-wave theory proportional to T^3 in disagreement

[20] P. W. Anderson, Phys. Rev. **86**, 694 (1952).
[21] R. Brout and F. Englert, Bull. Am. Phys. Soc. **6**, 55 (1961).

with Dyson's theory which shows that the correction starts in T^4.

The advantage of this method is that spin-interaction terms are handled in a self-consistent manner through renormalized frequencies. It turns out that this idea is still useful and Dyson's lowest order results can be interpreted in these terms, as first shown by Keffer and Loudon.[22] We pursue the reasoning of Brout and Englert[21] as it gives the form which is most useful for the present paper.

We adopt the point of view of Dyson and invent boson operators $\hat{S}_q^{\pm}$ (the a_q of Dyson) which obey boson commutation relations with $\hat{S}_q^-$ the creation operator:

$$[\hat{S}_q^+,\hat{S}_{-q'}^-]=\delta_{qq'}. \tag{A8}$$

We must then find an operator $\hat{S}_q^z$ which is formed from $\hat{S}_q^+$, $\hat{S}_{q'}^-$ such that the commutation relation (A2b) is given correctly in terms of the new operators. This will insure that the equations of motion (A3) describe the evolution of these new operators. The procedure is clearly equivalent to Dyson's invention of an interaction Hamiltonian in terms of the a_q which has the same matrix elements between the boson states as does the original Hamiltonian. The operator in question is then

$$\hat{S}_q^z=-\textstyle\sum_{q'}\hat{S}_{q-q'}^-\hat{S}_{q'}^+ +\frac{1}{2}\delta_{0q}. \tag{A9}$$

With the term $\frac{1}{2}\delta_{0q}$, the magnetization is correctly given according to the sum rule (51) in terms of the $\hat{S}_q$. Substituting Eq. (A9) into (A3) gives the equations of motion of the $\hat{S}_q$ operators:

$$-i\dot{\hat{S}}_q^-=\omega(q)\hat{S}_q^- -\textstyle\sum_{q'q''}[v(q'')-v(q-q'')]\hat{S}_{q-q'-q''}^-\hat{S}_{q'}^+\hat{S}_{q''}^-. \tag{A10}$$

This equation may be written more symmetrically to resemble Dyson's form

$$-i\dot{\hat{S}}_q^-=\omega(q)\hat{S}_q^- -\tfrac{1}{2}\textstyle\sum_{q'q''}[v(q'')+v(q-q'-q'')-v(q'+q'')-v(q-q'')]\hat{S}_{q-q'-q''}^-\hat{S}_{q'}^+\hat{S}_{q''}^-. \tag{A11}$$

Equation (A11) is the equation of motion that would be derived from the following Hamiltonian involving only operators $\hat{\mathbf{S}}_q$ obeying commutation relations (A8):

$$H=H^0{}_{\text{Dyson}}+H^1{}_{\text{Dyson}}=\textstyle\sum_q\omega(q)\hat{S}_q^-\hat{S}_{-q}^+ -\frac{1}{4}\sum_{qq'q''}[v(q'')+v(q-q'-q'')-v(q'+q'')-v(q-q'')]\times\hat{S}_{q-q'-q''}^-\hat{S}_{q'}^+\hat{S}_{q''}^-\hat{S}_{-q}^+. \tag{A12}$$

This is of the form that Dyson obtained by double commutation.

From the Landau theory of Fermi liquids[23] we know that the low-temperature properties of a quantum liquid are describable in terms of noninteracting quasiparticles whose energy is

$$\epsilon(q)=\omega(q)+t(q), \tag{A13}$$

where $t(q)$ is the forward-scattering amplitude of the particle of momentum q in matter. The lowest order corrections calculated by Dyson may be given a physical interpretation of this kind. Dyson showed that Born approximation is a good approximation to $t(q)$: In the worst case, spin 1/2, first Born approximation accounts for 50% of the corrections. Born approximation in a medium possessing translational invariance is the same as Hartree-Fock theory. This is obtainable either by evaluating $\langle H^1{}_{\text{Dyson}}\rangle$ in one of the eigenstates of $H^0{}_{\text{Dyson}}$ or by contracting two of the operators in the second term of the right-hand side of (A10) into the c numbers $\hat{n}(q)=\langle\hat{S}_{-q}^-\hat{S}_q^+\rangle$, the number of spin waves of type q. We do the latter. The first contraction is obtained by letting $q=q''$. This term recovers RPA in the unsymmetrized form (A10). We now see that the inadequacy of RPA at low T comes from taking expectation values in the equation of motion too soon. The other contraction is obtained from $q''=-q'$ and corresponds to an exchange term. The result of grouping both terms is

$$-i\dot{\hat{S}}_q^-=\omega(q)\hat{S}_q^- +\textstyle\sum_{q'}[v(0)-v(q)-v(q')+v(q+q')]\hat{n}(q')\hat{S}_q^-, \tag{A14}$$

where

$$\hat{n}(q)=\langle\hat{S}_{-q}^-\hat{S}_q^+\rangle=[\exp\beta\omega(q)-1]^{-1},$$

at temperature T. We then find that in Hartree-Fock approximation

$$\epsilon(q)=\omega(q)+\textstyle\sum_{q'}[v(0)-v(q)-v(q')+v(q+q')]\hat{n}(q'). \tag{A15}$$

Assuming an isotropic crystal, we write $v(0)-v(q)=\alpha q^2+\gamma q^4+\cdots$, so that

$$\epsilon(q)=\tfrac{1}{2}\alpha q^2[1-12(\gamma/\alpha)\textstyle\sum_{q'}q'^2\hat{n}(q')]. \tag{A16}$$

Since $\sum\omega(q)\hat{n}(q)\sim\frac{1}{2}\alpha\sum q^2\hat{n}(q)$ is the excitation energy at temperature T, the fractional decrease in $\omega(q)$ is proportional to this energy, which is the result of Keffer and Loudon. For near-neighbor interactions it turns out that the relationship is more precise. Here the fractional decrease in $\omega(q)$ is exactly equal to the energy divided by the ground state energy. In general, this decrease is proportional to $T^{5/2}$. As the number of

[22] F. Keffer and R. Loudon, Suppl. J. Appl. Phys. **32**, 2 (1961).
[23] L. D. Landau, Zh. Eksperim. i. Teor. Fiz. **30**, 1058 (1956) [translation: Soviet Phys.—JETP **3**, 920 (1957)].

spin waves at low T is $\propto T^{3/2}$, the total correction to the magnetization is then T^4. The correction to the total energy is

$$E-E_0=\sum\omega(q)\hat{n}(q)+\tfrac{1}{2}\sum_{qq'}[v(0)-v(q)-v(q')+v(q+q')]\hat{n}(q)\hat{n}(q'). \quad \text{(A17)}$$

It should be stressed that in this treatment we have ignored the "kinematical interaction" of Dyson. The above treatment is not complete enough to justify this, though from the work of Dyson, or the temperature ordering of Sec. 5 it is known to give exponentially small effects at low temperatures.

Finally, because of the interest in the derivation of Dyson's results by Green's function methods, we remark that the above equations of motion [Eq. (A3) or Eq. (A10)] can be immediately cast into Green's function form.[24] However, in order to solve the equations by a simple uncoupling [corresponding to the contraction of (A10) into c numbers $\hat{n}(q)$] valid at low temperatures to T^4 it is necessary to work with the operators $\hat{S}$ so that the inhomogeneous term $\delta(t-t')\times\langle[\hat{S}^+(t),\hat{S}^-(t')]\rangle$ in the Green's function equations gives a factor proportional to 1, and not a factor $\delta(t-t')\langle[S^+(t),S^-(t')]\rangle$ proportional to the average magnetization, which would produce T^3 deviations.

APPENDIX B. PROOF OF EQUATION (22)

From the definition of the semi-invariants it follows that

$$\langle T\prod_{\alpha_i}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}\rangle_0=\prod_{\alpha_i}(\partial/\partial t^{(\alpha_i)})^{n_{\alpha_i}}|_{t^{(\alpha_i)}=0}\,Q(t^{(\alpha_i)}), \quad \text{(B1)}$$

where

$$Q(t^{(\alpha_i)})=\exp\Big[\prod_{\alpha_i}\Big(\sum_{n_{\alpha_i}=0}^{\infty}\frac{(t^{(\alpha_i)})^{n_{\alpha_i}}}{n_{\alpha_i}!}\Big)M_{\Sigma n_{\alpha_i}>0}\Big(T\prod_{\alpha_i}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}\Big)\Big]. \quad \text{(B2)}$$

This is most clearly seen by exponentiating the time-ordered form of Eq. (18). Hence,

$$\frac{\partial Q}{\partial t^{(\alpha_j)}}=\sum_{n_{\alpha_j}=1}^{\infty}\frac{t^{(\alpha_j)n_{\alpha_j}-1}}{(n_{\alpha_j}-1)!}\prod_{\alpha_i\neq\alpha_j}\Big(\sum_{n_{\alpha_i}=0}^{\infty}\frac{t^{(\alpha_i)n_{\alpha_i}}}{n_{\alpha_i}!}\Big)M_{\Sigma n_{\alpha_i}>0}\Big(T\sum_{\alpha_i}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}\Big)Q(t^{(\alpha_i)}) \quad \text{(B3)}$$

$$=Q(t^{(\alpha_i)})\Big[M_1(S^{(\alpha_j)}(\beta_{\alpha_j}))+\prod_{\alpha_i}\Big(\sum_{n_{\alpha_i}=0}^{\infty}\frac{t^{(\alpha_i)n_{\alpha_i}}}{n_{\alpha_i}!}\Big)M_{\Sigma n_{\alpha_i}+1>1}\Big(T\prod_{\alpha_i\neq\alpha_j}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}(S^{(\alpha_j)}(\beta_{\alpha_j}))^{n_{\alpha_j}+1}\Big)\Big]. \quad \text{(B4)}$$

But

$$\Delta^{(\alpha_j)}(\beta_{\alpha_j})Q=\prod_{\alpha_i}\Big(\sum_{n_{\alpha_i}=0}^{\infty}\frac{t^{(\alpha_i)n_{\alpha_i}}}{n_{\alpha_i}!}\Big)M_{\Sigma n_{\alpha_i}+1>1}\Big(T\prod_{\alpha_i\neq\alpha_j}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}(S^{(\alpha_j)}(\beta_{\alpha_j}))^{n_{\alpha_j}+1}\Big)Q(t^{(\alpha_i)}). \quad \text{(B5)}$$

Therefore,

$$\frac{\partial Q}{\partial t^{(\alpha_j)}}=[M_1(S^{(\alpha_j)}(\beta_{\alpha_j}))+\Delta^{(\alpha_j)}(\beta_{\alpha_j})]Q, \quad \text{(B6)}$$

so that

$$Q=\exp\{[M_1(S^{(\alpha_j)}(\beta_{\alpha_j}))+\Delta^{(\alpha_j)}(\beta_{\alpha_j})]t^{(\alpha_j)}\}Q(t^{(\alpha_j)}=0). \quad \text{(B7)}$$

We use the same method to extract the explicit dependence on each $t^{(\alpha_j)}$:

$$Q=\prod_{\alpha_i}\exp\{[M_1(S^{(\alpha_i)}(\beta_{\alpha_i}))+\Delta^{(\alpha_i)}(\beta_{\alpha_i})]t^{(\alpha_i)}\}, \quad \text{(B8)}$$

since $Q(0,0,\cdots 0)=1$. Hence,

$$\langle T\prod_{\alpha_i}(S^{(\alpha_i)}(\beta_{\alpha_i}))^{n_{\alpha_i}}\rangle_0=\prod_{\alpha_i}[M_1(S^{(\alpha_i)}(\beta_{\alpha_i}))+\Delta^{(\alpha_i)}(\beta_{\alpha_i})]^{n_{\alpha_i}}. \quad \text{(B9)}$$

APPENDIX C. RAISING OPERATOR FOR S^2 IN THE CLASSICAL FORMALISM

In this case the semi-invariants are defined by Eq. (17), so that

$$M_n(S^z)=(\partial/\partial t)^n|_{t=0}\ln\langle e^{tS^z}\rangle_0. \quad \text{(C1)}$$

However,

$$\ln\langle e^{tS^z}\rangle_0=\ln\{\mathrm{Tr}[e^{\beta hS^z}e^{tS^z}]\}-\ln\{\mathrm{Tr}[e^{\beta hS^z}]\}, \quad \text{(C2)}$$

so that after the first derivative, $\partial/\partial t$ is equivalent to $\partial/\partial(\beta h)$. Hence,

$$M_n(S^z)=(\partial/\partial\beta h)^{n-1}M_1(S^z). \quad \text{(C3)}$$

[24] R. A. Tahir-Kheli and D. ter Haar, Phys. Rev. **127**, 95 (1962).

APPENDIX D. RING CONTRIBUTIONS

(i) Longitudinal Rings

Here it is not necessary to use the time-ordered formalism.[5] The product of interactions, each now with a factor β, is again of convolution form in Fourier space, and the series is again a geometrical series. The result is

$$\langle S_i^z S_j^z\rangle_{\text{ring}}=\sum_q \frac{e^{iq(r_i-r_j)}}{1-\beta v(q)M_2(S^zS^z)}. \tag{D1}$$

Tree renormalization of each vertex takes M_2 into $(\frac{1}{4}-\hat{M}_1^2)$, with $\hat{M}_1$ given by Eq. (31).

(ii) Self-Energy Parts

These are obtained from the results of Eqs. (36) and (D1) for the propagators by removing the final vertices. The results are

$$G_2{}^T(\beta_1-\beta_2)=\sum_q v(q)\tanh\tfrac{1}{2}\beta h\exp[-(\beta_1-\beta_2)h]\times\begin{cases}\tilde{g}^-(q), & \beta_1>\beta_2,\\ \tilde{g}^+(q), & \beta_1<\beta_2.\end{cases} \tag{D2}$$

$$G_2{}^L=\sum_q \frac{v(q)}{1-\beta v(q)M_2}. \tag{D3}$$

APPENDIX E. EXCLUDED VOLUME DIAGRAMS

The effect of a dotted line in a ring is to put the connected indices equal without affecting the time ordering. Thus, all dotted-line insertions in a ring will have the same time dependence as a ring with no dotted line insertions. Hence, the sum $\mu(\beta_1\beta_2)$ of all dotted-line insertions made between two vertices of a ring depends on β_1, β_2 in the same way as the ring "self-energy" $G_2{}^T$ $(\beta_1-\beta_2)$. It can, therefore, be Fourier analyzed as follows:

$$\mu(\beta_1,\beta_2)=\sum_{\lambda_j} e^{i\lambda_j(\beta_1-\beta_2)}\mu(\lambda_j). \tag{E1}$$

Then the effective interaction between the two vertices is modified to $v_{ij}-\mu(\lambda)\delta_{ij}$ for frequency λ. Calculating $\mu(\lambda)$ self-consistently, this gives for each λ

$$M_2(\lambda)\mu(\lambda)=\sum_q \frac{[v(q)-\mu(\lambda)]M_2(\lambda)}{1-\frac{1}{2}\beta[v(q)-\mu(\lambda)]M_2(\lambda)}+\sum_q \frac{\mu(\lambda)M_2(\lambda)}{1-\frac{1}{2}\beta[v(q)-\mu(\lambda)]M_2(\lambda)}, \tag{E2}$$

and the energy associated with such an excluded volume ring is

$$\lim_{\beta'\to 0^+}\tfrac{1}{2}\sum_{q,\lambda}\frac{e^{i\lambda\beta' v(q)M_2(\lambda)}}{1-\frac{1}{2}\beta[v(q)-\mu(\lambda)]M_2(\lambda)} \tag{E3}$$

$$=\lim_{\beta'\to 0^+}\tfrac{1}{2}\sum_{\lambda} e^{i\lambda\beta'}\mu(\lambda)M_2(\lambda). \tag{E4}$$

Equation (E2) for $\mu(\lambda)$ simplifies at the Curie point, where $M_2(\lambda)\to\delta_{\lambda,0}$. We there obtain $\mu(\lambda)=\mu\delta_{\lambda,0}$, where

$$\mu=\sum_q \frac{v(q)}{1-\frac{1}{2}\beta[v(q)-\mu]}. \tag{E5}$$