# Ising Model with a Long-Range Interaction in the Presence of Residual Short-Range Interactions*

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(Received 9 January 1963)

We investigate in the limit as the range of part of the interspin interaction becomes indefinitely great, but is still small compared to the size of the system, the behavior of Ising models which have, in addition, a residual short-range interaction. We find that the only possible type of transition in this limit is the familiar Bragg-Williams type. We also investigate the passage of the three-dimensional Ising model on the simple cubic lattice to the long-range limit from the short-range limit, dimension by dimension.

## 1. INTRODUCTION

In this paper we extend the results of Siegert[1] on the Ising model in the limit in which all interactions become long ranged to the case in which there are residual short-range interactions. In the fourth and final section of this paper we show that even in the presence of short-range interactions, the existence of *any* interaction of infinitely long range is sufficient to *force* the nature of the transition to be of the Bragg-Williams type, i.e., continuous energy and a discontinuity in the specific heat.

In the second section of this paper we discuss the general case and reduce the limit to a certain short-range problem plus an equal interaction between all spins.

In the third section we illustrate the results of the second section by a number of examples and use the examples to discuss the dimension-by-dimension long-range limit for the Ising model on a simple cubic lattice. We find that taking the long-range limit in one direction alone is sufficient to obtain the qualitative nature of the complete long-range limit. We find also that increasing the range of the force raises the transition temperature, just as with the spherical Ising model.

## 2. THE GENERAL CASE

Siegert has shown[1] that the Weiss-Bragg-Williams approximation is equal to the limiting case of an Ising model in which the range of the interaction becomes infinite in all directions in such a way that, although tending to infinity, it is still small compared to the total size of the system. The maximum interaction energy per spin is held fixed as the range becomes infinite. In this section we extend his result to the case where part of the interaction becomes infinitely long ranged and part remains "short" ranged. We find again that the limits of system size tending to infinity and the range of the force tending to infinity may be interchanged. Also, the shape of the long-range part (in the infinite limit) is not important in leading order but only its total strength.

Let us consider an Ising model with both long- and short-range interactions. Let the energy be given by

$$E/kT=-\tfrac{1}{2}\sum_{i,j}\nu_i(A_{ij}+B_{ij})\nu_j, \tag{2.1}$$

where $A_{ij}$ is the short-range and $B_{ij}$ is the long-range interaction. Following our development in a previous paper[2] we may write the partition function as

$$Z=(2\pi)^{-N/2}\int_{-\infty}^{+\infty}\cdots\int \exp(-\tfrac{1}{2}\sum_{i=1}^{N}x_i^2)\mathfrak{M}(x_i)\prod_{j=1}^{N}dx_i, \tag{2.2}$$

where $N$ is the number of spins and

$$\mathfrak{M}(x_i)=\prod_{j=1}^{N}\{\cosh[\sum_{i=1}^{N}x_i[(A+B)^{1/2}]_{ij}]\}. \tag{2.3}$$

We shall assume that the indices $\mathbf{k}$ and $\mathbf{l}$ are $d$-dimensional position vectors and that $A_{\mathbf{kl}}$ and $B_{\mathbf{kl}}$ are functions of $(\mathbf{k}-\mathbf{l})$ only. Also, we shall assume a ferromagnetic interaction, i.e., $A_{\mathbf{kl}}\geq 0$, $B_{\mathbf{kl}}\geq 0$. We may then diagonalize $A$ and $B$ by introducing the eigenvectors $z_{\mathbf{q}}$ and eigenvalues $a(\mathbf{q})$ and $b(\mathbf{q})$,

$$\begin{aligned} z_{\mathbf{q}}&=N^{-1/2}\textstyle\sum_{\mathbf{l}}\exp(2\pi i\mathbf{q}\cdot\mathbf{l})x_{\mathbf{l}},\\ A_{\mathbf{kl}}&=N^{-1}\textstyle\sum_{\mathbf{q}}a(\mathbf{q})\exp[2\pi i\mathbf{q}\cdot(\mathbf{l}-\mathbf{k})],\\ B_{\mathbf{kl}}&=N^{-1}\textstyle\sum_{\mathbf{q}}b(\mathbf{q})\exp[2\pi i\mathbf{q}\cdot(\mathbf{l}-\mathbf{k})].\end{aligned} \tag{2.4}$$

Equation (2.3) becomes

$$\mathfrak{M}(z_{\mathbf{q}})=\prod_{\mathbf{l}}\{\cosh[N^{-1/2}\sum_{\mathbf{q}}z_{\mathbf{q}}[a(\mathbf{q})+b(\mathbf{q})]^{1/2}\times\exp(-2\pi i\mathbf{q}\cdot\mathbf{l})]\}, \tag{2.5}$$

and Eq. (2.2) goes into

$$Z=(2\pi)^{-N/2}\int_{-\infty}^{+\infty}\cdots\int\exp(-\tfrac{1}{2}\sum_{\mathbf{q}}|z_{\mathbf{q}}|^2)\mathfrak{M}(z_{\mathbf{q}})\prod_{\mathbf{q}}dz_{\mathbf{q}}. \tag{2.6}$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.
[1] A. J. F. Siegert (private communication).
[2] G. A. Baker, Jr., Phys. Rev. **126**, 2072 (1962).

Our next step, following Siegert,[1] is to expand (2.6) about the Weiss field value and by bounding the error, establish the limit as the range of the interaction $B$ becomes infinite. First we may break up (2.6) as

$$Z=(2\pi)^{-N/2}\int_{z_0>0}\int_{-\infty}^{+\infty}\cdots\int \exp(-\tfrac{1}{2}\sum_{\mathbf{q}}|z_{\mathbf{q}}|^2)\mathfrak{M}(z_{\mathbf{q}})\prod_{\mathbf{q}}dz_{\mathbf{q}} + (2\pi)^{-N/2}\int_{z_0<0}\int_{-\infty}^{+\infty}\cdots\int \exp(-\tfrac{1}{2}\sum_{\mathbf{q}}|z_{\mathbf{q}}|^2)\mathfrak{M}(z_{\mathbf{q}})\prod_{\mathbf{q}}dz_{\mathbf{q}}$$

$$=2(2\pi)^{-N/2}\int_{z_0>0}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int \exp(-\tfrac{1}{2}\sum_{\mathbf{q}}|z_{\mathbf{q}}|^2)\mathfrak{M}(z_{\mathbf{q}})\prod_{\mathbf{q}}dz_{\mathbf{q}}, \tag{2.7}$$

as $\mathfrak{M}(z_{\mathbf{q}})$ is manifestly even in all $z_{\mathbf{q}}$ together since $\cosh x$ is an even function of $x$. Now let us introduce a change of variables. Let

$$z_{\mathbf{q}}'=z_{\mathbf{q}}-\{N/[a(0)+b(0)]\}^{1/2}\delta_{\mathbf{q},0}y. \tag{2.8}$$

Now, writing out $\mathfrak{M}$ and $Z$ in the primed variables and dropping the primes we get

$$\mathfrak{M}(z_{\mathbf{q}})=\prod_{\mathbf{l}}\{\cosh[y+N^{-1/2}\sum_{\mathbf{q}}z_{\mathbf{q}}[a(\mathbf{q})+b(\mathbf{q})]^{1/2}\exp(-2\pi i\mathbf{q}\cdot\mathbf{l})]\},$$

$$Z=2(2\pi)^{-N/2}\int_{-y\{N/[a(0)+b(0)]\}^{1/2}}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp(-\tfrac{1}{2}\sum|z_{\mathbf{q}}|^2-\{N/[a(0)+b(0)]\}^{1/2}yz_0 -\tfrac{1}{2}Ny^2/[a(0)+b(0)])\mathfrak{M}(z_{\mathbf{q}})\prod_{\mathbf{q}}dz_{\mathbf{q}}. \tag{2.9}$$

Using the addition formula for $\cosh(\alpha+\beta)$ and defining

$$\Delta(\mathbf{q})=[a(\mathbf{q})+b(\mathbf{q})]^{1/2}-[a(\mathbf{q})]^{1/2}, \tag{2.10}$$

we may rewrite (2.9) as

$$Z=2(2\pi)^{-N/2}\int_{-y\{N/[a(0)+b(0)]\}^{1/2}}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int\exp\{-\{N/[a(0)+b(0)]\}^{1/2}yz_0-\tfrac{1}{2}Ny^2/[a(0)+b(0)]-\tfrac{1}{2}\sum_{\mathbf{q}}|z_{\mathbf{q}}|^2$$
$$+\sum_{\mathbf{l}}\ln\cosh(y+N^{-1/2}\sum_{\mathbf{q}}z_{\mathbf{q}}[a(\mathbf{q})+\delta_{\mathbf{q},0}b(0)]^{1/2}e^{-2\pi i\mathbf{q}\cdot\mathbf{l}})+\sum_{\mathbf{l}}\ln\cosh(N^{-1/2}\sum_{\mathbf{q}\neq 0}z_{\mathbf{q}}\Delta(\mathbf{q})e^{-2\pi i\mathbf{q}\cdot\mathbf{l}})$$
$$+\sum_{\mathbf{l}}\ln[1+\tanh(y+N^{-1/2}\sum_{\mathbf{q}}z_{\mathbf{q}}[a(\mathbf{q})+\delta_{\mathbf{q},0}b(0)]^{1/2}e^{-2\pi i\mathbf{q}\cdot\mathbf{l}})\tanh(N^{-1/2}\sum_{\mathbf{q}\neq 0}z_{\mathbf{q}}\Delta(\mathbf{q})e^{-2\pi i\mathbf{q}\cdot\mathbf{l}})]\}\prod_{\mathbf{q}}dz_{\mathbf{q}}. \tag{2.11}$$

We shall now expand the last two sums in the exponent of (2.11) to second order in the variables $z_q$. It is simple to show that the terms retained form an upper bound to the actual contribution of these terms to the integrand when it is remembered that the argument of every hyperbolic function is real. This expansion is

$$\tfrac{1}{2}\operatorname{sech}^2y\sum_{\mathbf{q}\neq 0}|z_{\mathbf{q}}|^2b(q). \tag{2.12}$$

Differentiating the integrand of (2.11) with respect to $z_{\mathbf{q}}$, we find that if $y$ satisfies

$$y=[a(0)+b(0)]\tanh y, \tag{2.13}$$

then the point $z_{\mathbf{q}}=0$ for all $\mathbf{q}$ is an extremum. If $[a(0)+b(0)]\neq 1.0$, then we may easily show that it is a maximum. By the mean value theorem[3] we may write (2.11) as

$$Z=Z(A^*)\exp\{\tfrac{1}{2}\operatorname{sech}^2y\sum_{\mathbf{q}\neq 0}|\zeta_{\mathbf{q}}|^2b(q)\}, \tag{2.14}$$

where the $\zeta_{\mathbf{q}}$ are appropriate mean values of the $z_{\mathbf{q}}$, and we have identified $Z(A^*)$ as the partition function for an interaction $A^*$ with eigenvalues

$$a^*(\mathbf{q})=a(\mathbf{q})+\delta_{\mathbf{q},0}b(0). \tag{2.15}$$

Since, say, 99% of the contribution to the integral (2.11) defining $Z(A^*)$ comes from values of $|z_{\mathbf{q}}|<M'$, independent of the range of the interaction $B$ and size of the system $N$ (fixed $y\neq 0$), the integral may be truncated with arbitrarily small error $\epsilon$, hence, the $|\zeta_{\mathbf{q}}|$ bounded by, say, $M(\epsilon)$. The contribution of the exponential factor in (2.14) will therefore be bounded, in the limit as $N\to\infty$ by

$$\exp\left\{\tfrac{1}{2}N\operatorname{sech}^2yM^2(\epsilon)\int|b(\mathbf{q})|d\mathbf{q}\right\}. \tag{2.16}$$

According to the arguments given in our previous paper,[2] we may bound

$$|b(\mathbf{q})|\leq M^*\prod_{j=1}^{d}(1+R^{1/d}q_j)^{-1}, \tag{2.17}$$

[3] See, for instance, P. Franklin, *A Treatise on Advanced Calculus* (John Wiley & Sons, Inc., New York, 1949).

where $q_j$ is the $j$th component of $\mathbf{q}$ and $R$ is a range parameter proportional to the number of spins in the range of the interaction, $B$. Hence the contribution of the exponential factor is bounded by

$$\exp\{\tfrac{1}{2}N\,\mathrm{sech}^2 yM^2(\epsilon)M^*[(\ln R)/R^{1/d}]^d\},\quad (2.18)$$

which goes to zero as $R\rightarrow\infty$. Thus,

$$\lim_{R\to\infty}\lim_{N\to\infty}[(\ln Z)/N]=\lim_{N\to\infty}\{[\ln Z(A^*)]/N\}$$
$$=\lim_{N\to\infty}\lim_{R\to\infty}[(\ln Z)/N],\quad (2.19)$$

which is the main conclusion of this section. The second equality follows[3] from the uniform approach as $N\rightarrow\infty$, and existence of the appropriate limits. In case $B$ is not of full dimension $d$, the $d$ in (2.17) and (2.18) is replaced by a smaller number. While the rate is affected, the fact of convergence is not changed.

### 3. SOME EXAMPLES OF THE LONG-RANGE LIMIT

In this section we will work out some examples of the results of Sec. 2. These examples will illustrate the results that the free energy per spin is independent of the shape of the interaction $B$. For the first example, we shall consider an Ising model on a simple quadratic lattice (two dimensional) with nearest-neighbor interactions in one direction and infinitely long range in the other. We will consider the equistrength case, i.e., the maximum interaction energy per spin is the same in both directions, although the nonequistrength case is no harder. The partition function for this model is

$$Z=\sum_{\text{all }\nu_{ij}=\pm1}\exp\left[K\sum_{i=1}^{M}\left\{\frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\nu_{ij}\nu_{ik}+\sum_{j=1}^{N}\nu_{ij}\nu_{i+1,j}\right\}\right],\quad (3.1)$$

where $K=J/kT$. We may obtain a rigorous solution for the energy, etc., for this model by the use of the method of "Gaussian random variables" which was introduced by Kac,[4] and extended by us.[5] Now we know the integration formula,

$$e^{KN\omega_i^2}=(2\pi KN)^{-1/2}\int_{-\infty}^{+\infty}\exp(-KN\bar{\nu}_i^2-2KN\bar{\nu}_i\omega_i)d\bar{\nu}_i.\quad (3.2)$$

Thus we may rewrite 3.1 as

$$Z=\sum_{\text{all }\nu_{ij}=\pm1}(2\pi KN)^{-M/2}\int_{-\infty}^{+\infty}\cdots\int\prod_{i=1}^{M}(d\bar{\nu}_i)\exp\left[-KN\sum_{i=1}^{M}\left(\bar{\nu}_i^2+2\bar{\nu}_i\sum_{j=1}^{N}\nu_{ij}/N\right)-K\sum_{i,j}\nu_{ij}\nu_{i+1,j}\right].\quad (3.3)$$

It is to be noted that the exponent has been partially linearized. We may rearrange (3.3) as

$$Z=\int_{-\infty}^{+\infty}\cdots\int\prod_{i=1}^{M}\left[\frac{d\bar{\nu}_i}{(2\pi KN)^{1/2}}\exp(-KN\sum_{i=1}^{M}\bar{\nu}_i^2)\right]\prod_{j=1}^{N}\Big[\sum_{\text{all }\nu_{ij}=\pm1}\exp(K\sum_{i=1}^{M}2\bar{\nu}_i\nu_{ij}+\nu_{ij}\nu_{i+1,j})\Big].\quad (3.4)$$

The second product depends on $j$ only as a dummy variable, hence if we rename $\nu_{ij}$, $\mu_i$, then our expression for the partition function becomes

$$Z=\int_{-\infty}^{+\infty}\cdots\int\prod_{i=1}^{M}\frac{d\nu_i}{(2\pi KN)^{1/2}}\{\exp(-K\sum_{i=1}^{M}\bar{\nu}_i^2)\sum_{\mu_i=\pm1}\exp[K\sum_{i=1}^{M}(2\bar{\nu}_i\mu_i+\mu_i\mu_{i+1})]\}^N.\quad (3.5)$$

We assume cyclical boundary conditions $\mu_{N+1}=\mu_1$. We recognize the quantity which is raised to the $N$th power as simply the one-dimensional Ising model with a "magnetic field" $\bar{\nu}_i$ at each site. The solution of this problem is well known[6] to be

$$\mathrm{Tr}\{\prod^{M}H_i\},\quad (3.6)$$

where the $H_i$ are $2\times2$ matrices

$$H_i=\begin{pmatrix}\exp(K+2K\bar{\nu}_i) & \exp(-K)\\ \exp(-K) & \exp(K-2K\bar{\nu}_i)\end{pmatrix}.\quad (3.7)$$

The effect of the $N$th power is to permit the evaluation of the partition function per spin by the method of steepest descents[7] in the limit as the number of spins becomes infinite in the long-range interaction direction. Hence, we have effectively reduced the integration in (2.5) to finding the maxima of the quantity which is raised to the $N$th power in (3.5).

We shall now obtain an upper bound. First, Eq. (3.6) is less than

$$2\prod_{i=1}^{M}\lambda_{\max}(i),\quad (3.8)$$

[4] M. Kac, Phys. Fluids **2**, 8 (1959).
[5] G. A. Baker, Jr., Phys. Rev. **122**, 1477 (1961).
[6] See, for example, D. ter Haar, *Elements of Statistical Mechanics* (Rinehart and Company, New York, 1958), Sec. 12.6.
[7] See, for instance, H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, New York, 1950), Chap. 17.

where $\lambda_{\max}(i)$ is the largest eigenvalue of $H_i$. This bound follows by induction from the fact that all elements and eigenvalues of the $H_i$ are positive and if we think of moving a vector from right to left through the matrix products we increase its magnitude at every step by at most a factor of $\lambda_{\max}(i)$, irrespective of its direction. The two comes from the $H_i$ being $2\times 2$ matrices. Using (3.8) we may now factor our upper bound for (3.5) so that the integrations over the $\bar{\nu}_i$ may each be done independently. We have for

$$\lambda_{\max}(i)=e^K\cosh 2K\bar{\nu}_i+[e^{2K}\sinh^2 2k\bar{\nu}_i+e^{-2K}]^{1/2}. \quad (3.9)$$

Thus, there is either one maximum for $\bar{\nu}_i=0$, or, two maxima for $\nu_i=\pm\bar{\nu}(K)$ of $\exp(-K\bar{\nu}_i^2)\lambda_{\max}(i)$. Since the location of the maxima is independent of $i$, let us compare the value of the upper bound obtained above with that of a lower bound obtained by evaluating (3.5) by the method of steepest descents with all $\bar{\nu}_i=\bar{\nu}(K)$. The contribution from (3.6) is[6]

$$\lambda_{\max}{}^M+\lambda_{\min}{}^M, \quad (3.10)$$

where $\lambda_{\min}$ is the other eigenvalue of $H_i$ with $\bar{\nu}_i=\bar{\nu}(K)$. Equation (3.10) is, for large $M$, about $\frac{1}{2}$ of our bound (3.8). Hence, the lower bound here obtained is smaller than the upper by a factor of $2^{-N}$ because of the difference between (3.8) and (3.10). It may also be smaller by a $2^{-M}$ because of the possibility of there being two maxima for $\exp(-K\bar{\nu}_i^2)\lambda_{\max}(i)$. However, as

$$\lim_{N,M\to\infty} 2^{-(N+M\,/(NM)}=1, \quad (3.11)$$

we conclude that the partition function per spin is given by

$$\lambda(K)=\lim_{N,M\to\infty} Z^{1/(MN)} = \max_{-\infty<\bar{\nu}<+\infty}\{\exp(-K\bar{\nu}^2)(e^K\cosh 2K\bar{\nu} +[e^{2K}\sinh^2 2K\bar{\nu}+e^{-2K}]^{1/2})\}. \quad (3.12)$$

Physically speaking, $\bar{\nu}_i$ is proportional to the value of the molecular field and we expect equivalence ($\bar{\nu}_i=\bar{\nu}$) between all rows.

The critical properties of this model follow at once from (3.12). The equation for the critical point, i.e., where the maximum is no longer at $\bar{\nu}=0$, is

$$1=2K_c e^{2K_c}, \quad (3.13)$$

which has the solution $K_c\cong 0.28357164$. When $K$ is greater than $K_c$ we determine $\bar{\nu}$ by differentiating (3.12) with respect to $\bar{\nu}$ and equating the result to zero. We find that $\bar{\nu}$ satisfies

$$\bar{\nu}=e^K\sinh(2K\bar{\nu})/(e^{2K}\sinh^2 2K\bar{\nu}+e^{-2K})^{1/2}, \quad (3.14)$$

which has only the solution $\bar{\nu}=0$ for $K$ less than $K_c$ and $\pm\bar{\nu}(K)$ for $K$ greater than $K_c$. We may calculate the energy. It is

$$E=-J\tanh K, \qquad K<K_c, \quad (3.15a)$$

$$E=J\left[\frac{2e^{-4K}\bar{\nu}^2}{\sinh 2K\bar{\nu}(\sinh 2K\bar{\nu}+\bar{\nu}\cosh 2K\bar{\nu})}-1-\bar{\nu}^2\right], \qquad K>K_c. \quad (3.15b)$$

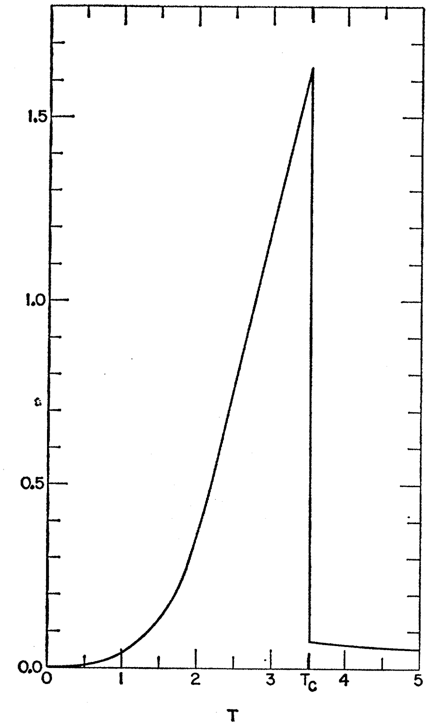


FIG. 1. Sketch of the specific heat for the equistrength, two-dimensional Ising model with long-range interactions in one direction and short-range interactions in the other.

A simple calculation shows $E$ to be continuous and equal to $-0.27620749J$ at the critical point. The specific heat is discontinuous at the critical point, and is

$$C_v(K_c^-)=8kK_c^3(1+2K_c)^{-2}=0.07427813k,$$
$$\Delta C_v=C_v(K_c^+)-C_v(K_c^-) =3k(1+2K_c)^2/[2K_c(1-3e^{4K_c})] =1.5601429k. \quad (3.16)$$

Thus, we see that this model displays a typical Bragg-Williams singularity, with a discontinuity in the specific heat and no discontinuity in the energy. We have illustrated the specific heat in Fig. 1. Above the critical point, the properties are the same as those of a collection of uncoupled, one-dimensional Ising models. This behavior corresponds to the fact that the energy of a system with purely Bragg-Williams-type interaction is zero above the critical point. At the critical point the discontinuity in the specific heat arises from the transition to an ordered state caused by the long-range interactions. It should be noted that the critical point comes at much higher temperature ($K_c=0.28$) than the Bragg-Williams critical point ($K_c=0.5$) in the absence of the short-range interaction, but at a lower temperature than given by the Bragg-Williams approximation to the whole system ($K_c=0.25$). Even so, the numerical value of $\Delta C_v$ is quite close to the value $3k/2$, obtained in the absence of the short-range force.

The spontaneous magnetization, which is proportional to $\bar{\nu}$, is easily shown from (3.14) to be proportional to $(K-K_c)^{1/2}$ at the critical point.

The partition function corresponding to $A^*$ for the first example [$A^*$ is defined by (2.15)] is

$$Z(A^*)=\sum_{\text{all }\nu_{ij}=\pm1}\exp\{[K/(NM)](\sum_{i,j}^{M,N}\nu_{ij})^2+K\sum_{i,j}^{M,N}\nu_{ij}\nu_{i+1,j}\}. \tag{3.17}$$

By use of a formula like (3.2) we may rewrite (3.17) as

$$Z(A^*)=\sum_{\text{all }\nu_{ij}=\pm1}(2\pi KNM)^{-1/2}\int_{-\infty}^{+\infty}d\bar{\nu}\exp\{-KNM\bar{\nu}^2+2KMN\bar{\nu}\sum_{i,j}^{M,N}\nu_{ij}/(MN)-K\sum_{i,j}^{M,N}\nu_{ij}\nu_{i+1,j}\}, \tag{3.18}$$

which may be rearranged [as in (3.5)] as

$$Z(A^*)=(2\pi KNM)^{-1/2}\int_{-\infty}^{+\infty}d\bar{\nu}\,\{\exp(-KM\bar{\nu}^2)\sum_{\mu_i=\pm1}\exp[K\sum_{i=1}^{M}(2\bar{\nu}\mu_i+\mu_i\mu_{i+1})]\}^N. \tag{3.19}$$

Equation (3.19) is, however, of the same sort as (3.5), except now we have the same "magnetic field" $\bar{\nu}$ at every lattice site. Thus, the solution proceeds as before, except without the additional complication of $\bar{\nu}_i$ instead of $\bar{\nu}$. Hence, (3.12) is again the solution. This result is in agreement with (2.19).

For our last example we wish to consider an Ising model on a simple cubic lattice in which there are nearest-neighbor interactions in two dimensions and an infinitely long-range force in the other direction. Again we consider the "equistrength" case where the maximum interaction energy per spin is the same in all directions. As the derivation is almost the same here as in (3.1) to (3.12) above, we shall discuss only the differences. The major difference arises in (3.6) where the $H_i$ are now the $2^{M'}\times2^{M'}$ matrices as discussed by Onsager.[8] Consequently, various 2 are replaced by $2^{M'}$. Also where $\bar{\nu}_i$ was previously a number, it must now be treated as a $M'$-dimensional vector. However, one eventually obtains a "squeezing" equation similar to (3.11) and thus, if $\Lambda(K,H)$ is the partition function per spin for the simple quadratic lattice with $K=J/kT$ and $H$ the applied magnetic field, then

$$\lambda(K)=\max_{-\infty<\bar{\nu}<+\infty}\{\exp(-K\bar{\nu}^2)\Lambda(K,2J\bar{\nu}/m)\}, \tag{3.20}$$

where $m$ is magnetic moment per spin. The equation for the critical point corresponding to (3.13) is easily derived from the requirement that the second partial with respect to $\bar{\nu}$ vanish for $\bar{\nu}=0$ at that point. It is

$$2K_c\chi(K_c)=1, \tag{3.21}$$

where $\chi$ is the reduced magnetic susceptibility $\chi_0kT/m^2$. Using the Padé approximant method[9] to evaluate $\chi(K)$ we may solve for $K_c$. We obtain $K_c\approx0.1889619$. The energy is again continuous and the specific heat discontinuous. The discontinuity is given by

$$\Delta C_v=-12K_c^2\left[\frac{1}{2}\frac{\partial^3}{\partial\bar{\nu}^2\partial K}(\ln\Lambda)-1\right]\Bigg/\left[\frac{\partial^4}{\partial\bar{\nu}^4}(\ln\Lambda)\right]\Bigg|_{K_c,\bar{\nu}=0}. \tag{3.22}$$

The third partial derivative is expressible in terms of $\chi(K)$. The fourth partial may be computed directly by the Padé approximant method from a series expansion in terms of diagrams in which all but 4 of the vertices are the meet of an even number of lines.[10] We have not, however, done so. It should again be noted that the critical point comes out at a much higher temperature than Bragg-Williams critical point ($K_c=0.5$) in the absence of the short-range interaction, or the short-range critical point[8] ($K_c\approx0.4406868$) in the absence of the long-range interaction. It is also at a higher temperature than the short-range limit of all the interactions, i.e., the Ising model on the simple cubic lattice[9] ($K_c\approx0.22172$), but a lower temperature than the Bragg-Williams approximation to it ($K_c\approx0.166667$).

These results, together with corresponding results for the spherical Ising model,[2] enable us to give a plausible discussion of the behavior of the Ising model (we discuss specifically the simple cubic lattice here but other lattices are similar) as the range of the interaction varies from nearest-neighbor to infinitely long. For nearest-neighbor interactions,[11] the specific heat is singular on both sides of $T_c$. The nature of the singularity is probably logarithmic on both sides but of smaller amplitude above $T_c$. Following the results for

[8] L. Onsager, Phys. Rev. **65**, 117 (1944).
[9] G. A. Baker, Jr., Phys. Rev. **124**, 768 (1961).
[10] This result corresponds to that of T. Oguchi [J. Phys. Soc. Japan **6**, 31 (1951)] for the reduced magnetic susceptibility.
[11] G. A. Baker, Jr., Phys. Rev. **129**, 99 (1963); C. Domb, Phil. Mag. Suppl. **9**, 149 (1960).

the spherical model[2] and the one-dimensional Ising model[5] we think that the quantitative nature of the transition does not change as the (strength preserving) range increases, although the transition temperature increases. In the limit in which the range in any one direction becomes infinite, we obtain a typical Bragg-Williams transition. It probably looks much like Fig. 1. The infinite part of the specific heat curve is squeezed to the critical point and disappears in the limit as the range becomes infinite. The nature of the short-range interaction transition is thus completely obscured in this limit. If the long-range limit is taken in another dimension we obtain a nonequistrength version of Fig. 1. If the final long-range limit is taken, the variation is rather minor and the standard Bragg-Williams approximation is obtained. As pointed out in Sec. 2, the shape remains practically unchanged and the transition temperature increases.

## 4. LONG- AND SHORT-RANGE MODEL IN TERMS OF THE SHORT-RANGE PARTITION FUNCTION

We point out in this section that many of the results obtained for the special examples in the previous section are more generally valid. From (2.19) it follows that we may evaluate any long-range limit of the type discussed by considering $Z(A^*)$. Thus, letting $b=b(0)$, we have

$$Z(A^*)=\sum_{\text{all states}} \exp\left[-\tfrac{1}{2}\sum_{i,j} \nu_i A_{ij}\nu_j + b\left(\sum_i \nu_i\right)^2/N\right], \quad (4.1)$$

which may be identically rewritten as

$$Z(A^*)=\sum_{\text{all states}} (2\pi N/b)^{-1/2} \times\int_{-\infty}^{+\infty} d\bar{\nu}\, \exp\left(-N\bar{\nu}^2/b-2\bar{\nu}\sum_i \nu_i\right) \times\exp\left[-\tfrac{1}{2}\sum_{i,j} \nu_i A_{ij}\nu_j\right]. \quad (4.2)$$

If we introduce $\lambda(K)=[Z(A^*)]^{1/N}$, the partition function per spin, and $\Lambda(K,H)=[Z(A)]^{1/N}$, then in the limit as $N\rightarrow\infty$

$$\lambda(K)=\left\{(2\pi N/b)^{-1/2} \times\int_{-\infty}^{+\infty} d\bar{\nu}\, [\exp(-\bar{\nu}^2/b)\Lambda(K,2kT\bar{\nu}/m)]^N\right\}^{1/N}. \quad (4.3)$$

In the limit as $N\rightarrow\infty$ we may evaluate the integral by the method of steepest descents.[7] Hence,

$$\lambda(K)=\max_{-\infty<\bar{\nu}<\infty} \{\exp(-\bar{\nu}^2/b)\Lambda(K,2kT\bar{\nu}/m)\}, \quad (4.4)$$

which is the same result as (3.20). The equation for the critical point is

$$2b\chi(K_c)=1 \quad (4.5)$$

in analogy to (3.21), where $\chi$ is again the reduced magnetic susceptibility. A formula similar to (3.22) holds for the discontinuity in the specific heat. We wish to point out that in all models in which $\chi$ is singular, the smallest amount of long-range force or, *for that matter, of effective long-range force introduced through an approximate solution procedure forces a Bragg-Williams type solution.* If the amount of the long-range force is very small, then at moderate distances from the critical point the specific heat will look very much as it did without any long-range interaction; however, near the critical point the presence of any long-range force is of overriding importance.

## ACKNOWLEDGMENTS

We wish to thank Professor M. Kac for calling the first example of Sec. 3 to our attention. Its investigation was the genesis of this study.