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# High-Frequency Conductivity of Quantum Plasma in a Magnetic Field

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The problem of the electromagnetic absorption coefficient in a quantum plasma in the presence of a uniform magnetic field is investigated by a kinetic description. The finite duration of encounters is taken into account in a self-consistent fashion which includes collective effects properly. This treatment is the quantum extension of an earlier classical study. The application of this theory to heavily doped semiconductors is suggested.

## I. INTRODUCTION

THE absorption of long-wavelength electromagnetic waves in classical plasmas, which properly takes into account collective effects, has been treated in an elementary model by Dawson and Oberman.[1,2] The results of this model are in accord with those of Oberman, Ron, and Dawson,[3] who have given a complete classical treatment, using the Bogoliubov, Born, Green, Kirkwood, and Yvon (BBKGY) hierarchy in the plasma limit. Both the extension of the elementary model of Dawson and Oberman[1] to a quantum plasma and generalization of reference 3 via Green's function techniques has been given by Ron and Tzoar.[4,5] The effect of the presence of a constant magnetic field on the complete classical treatment of reference 3 has been studied by Oberman and Shure.[6]

The purpose of the present work is to adapt the elementary model to include the effect of a uniform magnetic field in the quantum situation. The frequency (high) and wavelength (long) restrictions, as well as the meaning of the plasma expansion parameter, are discussed in the previous papers.

It is to be pointed out that the present treatment does not carry the usual time-scale restrictions inherent in the *transition probability* approach[7] to transport phenomena (just as the classical analog of the present treatment is not restricted by the Bogoliubov assumption[8] of the existence of two time scales).

We obtain a simple statement of the conductivity which reduces in the classical limit to that of reference 6. The results, however, are not valid in the vicinity of the gyrofrequency for the same reasons as those in the classical case. It is likely that our result could find application in the study of the impurity contributions to the absorption of electromagnetic waves (*optical* properties) in heavily doped semiconductors.[9]

## II. PLASMA MODEL AND CONDUCTIVITY

The most complete analog to the classical case as given by Dawson and Oberman[1] is found by using the Wigner distribution function—mixed representation for the density matrix—in coordinate-momentum phase space.[10,11] We describe the electron dynamics by the self-consistent set of equations for the distribution function, regarding the ions as a set of randomly distributed fixed scatterers. Thus, the electrons are treated quantum mechanically as an electron gas. In addition to the self-consistent field there is present a prevailing spatially uniform electric field **E** oscillating in time at the frequency $\omega$, and a static magnetic field **B**. We restrict ourselves to frequencies much greater than the collision frequency $2\pi/\tau$, where $\tau$ is the mean free time

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for particle collisions. This implies that we may *systematically* neglect electron-electron correlations (including their exchange effects). The validity of this neglect has been borne out by the more general treatments of references 3 and 5. In addition, we limit ourselves to magnetic fields of such size that

$$\hbar\omega_c \ll E_F,$$

where $\omega_c$ is the electron gyrofrequency and $E_F$ is the Fermi energy. This is, indeed, not a severe limitation on the size of the field. Any interaction due to the spin of the particles is systematically neglected. If we take the previous restrictions into account, we obtain the equation for the Wigner distribution $F(\mathbf{R},\mathbf{p},t)$,[11] which reads in the rest frame of the ions

$$\frac{\partial F}{\partial t}+\frac{\mathbf{p}}{m}\cdot\frac{\partial F}{\partial \mathbf{R}}-e\left[\mathbf{E}e^{-i\omega t}+\frac{1}{mc}\mathbf{p}\times\mathbf{B}\right]\frac{\partial F}{\partial \mathbf{p}} =\frac{e}{i\hbar}\int d\mathbf{r}\, e^{-i(\mathbf{p}/\hbar)\cdot\mathbf{r}}[\Phi(\mathbf{R}+\tfrac{1}{2}\mathbf{r})-\Phi(\mathbf{R}-\tfrac{1}{2}\mathbf{r})] \times\int\frac{d\mathbf{p}'}{(2\pi\hbar)^3}e^{i(\mathbf{p}/\hbar)\cdot\mathbf{r}}F(\mathbf{R},\mathbf{p}',t). \quad (1)$$

Here $\mathbf{R}$, $\mathbf{p}$ represent the position and momentum of the electrons at the time $t$, $-e$ and $m$ are the charge and mass of the electrons,[12] $h$ is the Planck constant, $c$ is the speed of light, and

$$\Phi(\mathbf{R},t)=-e\int d\mathbf{R}'d\mathbf{p}\;|\mathbf{R}'-\mathbf{R}|^{-1}F(\mathbf{R}',\mathbf{p},t)+U_{\text{ion}}(\mathbf{R}) \quad (2)$$

is the self-consistent field of the electrons and the field due to the presence of ions with charge $Ze$ at the positions $\mathbf{r}_i$

$$U_{\text{ion}}(\mathbf{R})=Ze\textstyle\sum_i|\mathbf{R}-\mathbf{r}_i|^{-1}. \quad (3)$$

To facilitate the solution of the coupled Eqs. (1) and (2), we perform the following transformation:

$$\boldsymbol{\varrho}=\mathbf{R}+\boldsymbol{\zeta}e^{-i\omega t},\quad \mathbf{q}=\mathbf{p}-i\omega m\boldsymbol{\zeta}e^{-i\omega t},\quad t=t, \quad (4)$$

with

$$\boldsymbol{\zeta}=-\boldsymbol{\varepsilon}+\left(1-\frac{\omega_c^2}{\omega^2}\right)^{-1}\left\{i\frac{\omega_c}{\omega}(\boldsymbol{\varepsilon}\times\mathbf{b})-\frac{\omega_c^2}{\omega^2}[\mathbf{b}\times(\boldsymbol{\varepsilon}\times\mathbf{b})]\right\},$$
$$\boldsymbol{\varepsilon}=e\mathbf{E}/m\omega^2,$$
$$\omega_c=eB/mc,\quad \mathbf{b}=\mathbf{B}/B. \quad (5)$$

[12] For the suggested application $e$ and $m$ should be taken as their effective values, see reference 9.

Equations (1) and (2) become

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{q}}{m}\cdot\frac{\partial}{\partial\boldsymbol{\varrho}}-\omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial\mathbf{q}}\right]\tilde{F}(\boldsymbol{\varrho},\mathbf{q},t) =\frac{e}{i\hbar}\int d\mathbf{r}\,e^{-i(\mathbf{q}/\hbar)\cdot\mathbf{r}}[\tilde{\Phi}(\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r},t)-\tilde{\Phi}(\boldsymbol{\varrho}-\tfrac{1}{2}\mathbf{r},t)] \times\frac{1}{(2\pi\hbar)^3}\int d\mathbf{q}'\,e^{i(\mathbf{q}'/\hbar)\cdot\mathbf{r}}\tilde{F}(\boldsymbol{\varrho},\mathbf{q}',t), \quad (6)$$

$$\tilde{\Phi}(\boldsymbol{\varrho},t)=\Phi(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t}) =U_{\text{ion}}(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t})-e\int d\boldsymbol{\varrho}'d\mathbf{q}\;|\boldsymbol{\varrho}-\boldsymbol{\varrho}'|^{-1}\tilde{F}(\boldsymbol{\varrho}',\mathbf{q},t), \quad (7)$$

with

$$\tilde{F}(\boldsymbol{\varrho},\mathbf{q},t)=F(\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{-i\omega t},\,\mathbf{q}+i\omega m\boldsymbol{\zeta}e^{-i\omega t},\,t). \quad (8)$$

We shall now assume that the right-hand side of Eq. (6) causes only a small perturbation on the equilibrium solution of that equation,

$$f_0(p)\equiv\tilde{F}_0(p)=\frac{2}{(2\pi\hbar)^3n_0}\{\exp[\beta(p^2/2m-\mu)]+1\}^{-1}, \quad (9)$$

where $n_0$ is the average electron density, $\mu$ is the chemical potential of the noninteracting electrons, $\beta$ is the inverse temperature in energy units, and $f_0(\mathbf{p})$ is normalized to one upon integration over $\mathbf{p}$ (the factor 2 comes from the summation over the spin components). In other words, we assume that the discrete nature of the ions caused only a small effect, and that the electron motion in the region of frequencies under consideration is largely inertia dominated (the conductivity is mainly reactive). The equations for $f$ and $\psi$, the small departures from equilibrium are

$$\left[\frac{\partial}{\partial t}+\frac{\mathbf{q}}{m}\cdot\frac{\partial}{\partial\boldsymbol{\varrho}}-\omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial\mathbf{q}}\right]f(\boldsymbol{\varrho},\mathbf{q},t) =\frac{1}{i\hbar}\int\frac{d\mathbf{q}'}{(2\pi\hbar)^3}f_0(\mathbf{q}')\int dr\,e^{-i[(\mathbf{q}-\mathbf{q}')/\hbar]\cdot\mathbf{r}} \times[\psi(\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r},t)-\psi(\boldsymbol{\varrho}-\tfrac{1}{2}\mathbf{r},t)], \quad (10)$$

and

$$\psi(\boldsymbol{\varrho},t)=Ze\sum_i|\boldsymbol{\varrho}-\boldsymbol{\zeta}e^{i\omega t}-\mathbf{r}_i|^{-1} -en_0\int d\boldsymbol{\varrho}'d\mathbf{q}\;|\boldsymbol{\varrho}-\boldsymbol{\varrho}'|^{-1}f(\boldsymbol{\varrho}',\mathbf{q},t)-en_0. \quad (11)$$

If we denote the Fourier transform of a function $f(\boldsymbol{\varrho})$ by

$$f(\mathbf{k})=\frac{1}{(2\pi)^3}\int d\boldsymbol{\varrho}\,e^{i\mathbf{k}\cdot\boldsymbol{\rho}}f(\boldsymbol{\varrho}), \quad (12)$$

and Fourier-analyze Eqs. (10) and (11), we have

$$\left[\frac{\partial}{\partial t}-\frac{i\mathbf{k}\cdot\mathbf{q}}{m}-\omega_c(\mathbf{q}\times\mathbf{b})\cdot\frac{\partial}{\partial\mathbf{q}}\right]f(\mathbf{k},\mathbf{q},t)$$
$$=\frac{e}{i\hbar}\psi(\mathbf{k},t)[f_0(\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k})-f_0(\mathbf{q}-\tfrac{1}{2}\hbar\mathbf{k})]. \quad (13)$$

and

$$\psi(\mathbf{k},t)=-\frac{4\pi e n_0}{k^2}\int d\mathbf{q}\, f(\mathbf{k},\mathbf{q},t)+\frac{4\pi eZ}{(2\pi)^3k^2}\sum_i \exp[i\mathbf{k}\cdot(\mathbf{r}_i+\boldsymbol{\zeta}e^{-i\omega t})]. \quad (14)$$

We proceed by choosing a cylindrical coordinate system, with polar axis along $\mathbf{B}$, in terms of which the rectangular components of $\mathbf{k}$, $\mathbf{q}$, and $\boldsymbol{\varepsilon}$ are

$$\mathbf{k}=(k_\perp\cos\alpha,k_\perp\sin\alpha,k_{\shortparallel}),$$
$$\mathbf{q}=(q_\perp\cos\phi,q_\perp\sin\phi,q_{\shortparallel}),$$
$$\boldsymbol{\varepsilon}=(\epsilon_\perp\cos\theta,\epsilon_\perp\sin\theta,\epsilon_{\shortparallel}),$$

where the angles are measured from some direction, the $x$ axis, in a plane perpendicular to $\mathbf{B}$. We now introduce the transformation[6]

$$F(\mathbf{k},\mathbf{q})=e^{-ik_\perp a\sin(\phi-\alpha)}\times\sum_n e^{+in(\phi-\alpha)}J_n(k_\perp a)F^{(n)}(\mathbf{k},\mathbf{q}), \quad (15)$$

with the inversion

$$F^{(n)}(\mathbf{k},\mathbf{q})=\frac{1}{2\pi J_n(k_\perp a)}\times\int_0^{2\pi}d\phi\, e^{ik_\perp a\sin(\phi-\alpha)}e^{-in(\phi-\alpha)}F(\mathbf{k},\mathbf{q}), \quad (16)$$

where $a=-q_\perp c/eB$, and the $J_n$ are Bessel functions of the first kind. If we apply this transformation to Eqs. (13) and (14) we obtain

$$\left(\frac{\partial}{\partial t}-\frac{ik_{\shortparallel}q_{\shortparallel}}{m}+in\omega_c\right)f^{(n)}(\mathbf{k},\mathbf{q},t)$$
$$=\frac{e}{i\hbar}[f_0^{(n)}(\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k})-f_0^{(n)}(\mathbf{q}-\tfrac{1}{2}\hbar\mathbf{k})]\psi(\mathbf{k},t), \quad (17)$$

and

$$\psi(k,t)=-\frac{4\pi e n_0}{k^2}\int d\mathbf{q}\sum_n J_n^2(k_\perp a)f^{(n)}(\mathbf{k},\mathbf{q},t)+\frac{4\pi eZ}{(2\pi)^3k^2}\sum_i \exp[i\mathbf{k}\cdot(r_i+\boldsymbol{\zeta}e^{-i\omega t})]. \quad (18)$$

Since the conductivity is defined by the limit $\boldsymbol{\varepsilon}\to 0$, we shall expand the second term on the right-hand side of Eq. (18)

$$\sum_i \exp[i\mathbf{k}\cdot(\mathbf{r}_i+\boldsymbol{\zeta}e^{-i\omega t})]=\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}(1+i\mathbf{k}\cdot\boldsymbol{\zeta}e^{-i\omega t}). \quad (19)$$

(To verify that this expansion leads to correct results for all $\mathbf{k}$ of interest, see reference 1, Appendix A.) With this linearization we can decompose the solutions of Eqs. (17) and (18) into two parts corresponding to the two-source terms on the right-hand-side of Eq. (19),

$$f^{(n)}(\mathbf{k},\mathbf{q},t)=f_s^{(n)}(\mathbf{k},\mathbf{q})+f^{(n)}(\mathbf{k},\mathbf{q},\omega)e^{-i\omega t},$$
$$\psi(\mathbf{k},t)=\psi_s(\mathbf{k})+\psi(\mathbf{k},\omega)e^{-i\omega t}. \quad (20)$$

The solution for the static part is

$$\psi_s(\mathbf{k})=\frac{4\pi e^2Z}{k^2}\frac{1}{(2\pi)^3}\frac{1}{D(\mathbf{k},0)}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}, \quad (21)$$

and for the dynamic part

$$\psi(\mathbf{k},\omega)=\frac{4\pi e^2Z}{k^2}\frac{1}{(2\pi)^3}\frac{i\mathbf{k}\cdot\boldsymbol{\zeta}}{D(\mathbf{k},\omega)}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}, \quad (22)$$

where $D(\mathbf{k},\omega)$, the dielectric function in a magnetic field, is

$$D(\mathbf{k},\omega)=1-\frac{4\pi e^2n_0}{\hbar k^2}\int d\mathbf{q}\sum_n\frac{J_n^2(k_\perp a)}{k_{\shortparallel}q_{\shortparallel}/m-\omega_c n+\omega+i\nu}\times[f_0^{(n)}(\mathbf{q}+\tfrac{1}{2}\hbar\mathbf{k})-f_0^{(n)}(\mathbf{q}-\tfrac{1}{2}\hbar\mathbf{k})]. \quad (23)$$

Following Oberman and Dawson[1] we obtain the average field on the ions due to the electrons

$$\langle\mathbf{E}(\omega)\rangle_{\rm av}=+\frac{4\pi eZ}{(2\pi)^3}\int d\mathbf{k}\frac{\mathbf{k}}{k^2}\mathbf{k}\cdot\boldsymbol{\zeta}\left[\frac{1}{D(\mathbf{k},0)}-\frac{1}{D(k,\omega)}\right]\times\left\langle\frac{1}{N_i}\sum_{ij}e^{-i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\right\rangle, \quad (24)$$

where $\langle\ \rangle$ stands for the ensemble average over ion positions. Under the assumption that the impurity ions are randomly distributed (see reference 2 for an elementary treatment in case the correlation between impurity ions is significant), the ensemble average over ion positions is just unity.

From the equation of motion for the electrons, now in the ion rest frame, we find

$$-i\omega\mathbf{j}+\omega_c\mathbf{j}\times\mathbf{b}=\frac{e^2n_0}{m}[\mathbf{E}+\langle\mathbf{E}(\omega)\rangle_{\rm av}], \quad (25)$$

where $\mathbf{j}$ is the average current density, and where we have employed the fact that the force on the ions due to the electrons is the negative of the force on the electrons due to the ions, and that this force is invariant under the frame transformation. If we utilize right- and left-polarized components of $\mathbf{E}\perp\mathbf{B}$ we find, with $j_\pm=j_x\pm ij_y$, etc.,

$$j_\pm(\omega)=\sigma_\pm(\omega)E_\pm,$$
$$j_{\shortparallel}(\omega)=\sigma_{\shortparallel}(\omega)E_{\shortparallel}, \quad (26)$$

where

$$\sigma_{\parallel}(\omega)=\sigma_0(\omega)\left(1-\frac{2}{3\pi}\frac{e^2Z}{m\omega^2}I_1(\omega)\right), \tag{27}$$

and

$$\sigma_{\pm}(\omega)=\sigma_0(\omega)\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}\times\left[1-\frac{2}{3\pi}\frac{e^2Z}{m\omega^2}\left(1\pm\frac{\omega_c}{\omega}\right)^{-1}I_2(\omega)\right]. \tag{28}$$

In Eqs. (27) and (28) we denoted by

$$\sigma_0=i\omega_p{}^2/4\pi\omega \tag{29}$$

the dominant reactive conductivity of the free electrons, with

$$\omega_p=(4\pi e^2n_0/m)^{1/2} \tag{30}$$

the plasma frequency. The functions $I_1(\omega)$ and $I_2(\omega)$ are defined by

$$I_1(\omega)=\frac{3}{4\pi}\int d\mathbf{k}\,\frac{k_{\parallel}{}^2}{k^2}\left(\frac{1}{D(\mathbf{k},0)}-\frac{1}{D(\mathbf{k},\omega)}\right), \tag{31}$$

and

$$I_2(\omega)=\frac{3}{8\pi}\int d\mathbf{k}\,\frac{k_{\perp}{}^2}{k^2}\left(\frac{1}{D(\mathbf{k},0)}-\frac{1}{D(\mathbf{k},\omega)}\right). \tag{32}$$

Equations (27) and (28) constitute our general result for the conductivity tensor of the system in the presence of a uniform magnetic field. The absorption coefficients of the electromagnetic waves in the system is simply related to this tensor (see Dawson and Oberman[1] for discussion of this point). It is easy to show that Eqs. (27) and (28) reduce to the results of reference 6 in the classical limit and to the results of references 4 and 5 in the case $\mathbf{B}=0$.

## III. CONCLUSIONS

By means of an elementary model we have computed the high-frequency[13] conductivity (and hence the absorption coefficient of electromagnetic waves) of a quantum plasma embedded in a uniform magnetic field. Our treatment which stems from a proper time-dependent kinetic description does not have the time-scale restriction of the usual transition-probability approach, and does give a proper description of the time-dependent collective response (e.g., dynamic shielding of the ions, etc.). The present theory is *not* valid in the immediate vicinity of the gyrofrequency, but other more usual kinetic-type approaches are then applicable. The separate treatment of this frequency region, as well as numerical plots, are the subject of a future communication. For a critical discussion of the physical ingredients of this model, the reader is urged to read Sec. VI of Dawson and Oberman.[1]

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[13] *Note added in proof*. The results for the conductivity perpendicular to the field are actually valid for low frequencies, $\omega\rightarrow 0$, since high-frequency means with respect to the motion in the rest frame of the electrons.