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Quantized Gravitational Field*

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A gravitational action operator is constructed that is invariant under general coordinate transformations and local Lorentz (gauge) transformations. To interpret the formalism the arbitrariness in description must be restricted by introducing gauge conditions and coordinate conditions. The time gauge is defined by locking the time axes of the local coordinate systems to the general coordinate time axis. The resulting form of the action operator, including the contribution of a spinless matter field, enables canonical pairs of variables to be identified. There are four field variables that lack canonical partners, in virtue of differential constraint equations, which can be interpreted as space-time coordinate displacements. In a physically distinguished class of coordinate system the gravitational field variables are not explicit functions of the coordinate displacement parameters. There remains the freedom of Lorentz transformation. The generators of spatial translations and rotations have the correct commutation properties. The question of Lorentz invariance is left undecided since the energy density operator is only given implicitly.

INTRODUCTION

ELECTRODYNAMICS is characterized by the property of gauge invariance—the freedom to alter the phase of any charge-bearing field arbitrarily at each space-time point while subjecting the electromagnetic potentials to a corresponding inhomogeneous transformation. It is not surprising that Weyl, the originator of the electromagnetic gauge invariance principle, also recognized[1] that the gravitational field can be characterized by a kind of gauge transformation. This is the possibility of altering freely at each point the orientation of a local Lorentz coordinate frame while suitably transforming certain gravitational potentials. Such a transformation is quite distinct from the more familiar global coordinate transformation. In a subsequent development of this conception, Yang and Mills[2] introduced an arbitrarily oriented three-dimensional isotopic space at each space-time point thereby relating a hypothetical vector field to isotopic spin. (The occasional remark that the gravitational field can be viewed as a Yang-Mills field is thus rather anachronistic.)

Due to the great interest in non-Abelian vector gauge fields as a possible foundation for comprehending the strong nuclear interactions, there have been some developments in the formulation of a relativistic quantum field theory of interacting vector fields. It is our intention here to begin the task of applying this experience to the more difficult problem of "quantizing the gravitational field". Since this work is based upon the quantum action principle, there will be areas of contact with the similarly based but differently developed semiclassical considerations of Arnowitt, Deser, and Misner.[3]

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[1] H. Weyl, Z. Physik **56**, 330 (1929).

[2] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[3] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **117**, 1595 (1960).

ACTION PRINCIPLE

The field variables that we shall use[4] to describe the gravitational field are 4×4 $e_a{}^\mu(x)$ and the 4×6 $\omega_{\mu ab}(x) = -\omega_{\mu ba}(x)$. These are vector fields with regard to general coordinate transformations,

$$\bar{e}_a{}^\mu(\bar{x}) = (\partial\bar{x}^\mu/\partial x^\nu)e_a{}^\nu(x),$$
$$\bar{\omega}_{\mu ab}(\bar{x}) = (\partial x^\nu/\partial\bar{x}^\mu)\omega_{\nu ab}(x).$$

The response to a local Lorentz transformation is

$$\bar{e}_a{}^\mu(x) = l_a{}^b(x)e_b{}^\mu(x),$$
$$\bar{\omega}_{\mu ab}(x) = l_a{}^{a'}(x)l_b{}^{b'}(x)\omega_{\mu a'b'}(x) + l_b{}^{b'}(x)\partial_\mu l_{ab'}(x),$$

where

$$l_a{}^{a'}(x)g_{a'b'}l_b{}^{b'}(x) = g_{ab},$$

and g_{ab} is the constant metric tensor of a Minkowski space.

The inhomogeneous term in the gauge transformation of $\omega_{\mu ab}$ must be removed to form a covariant that can be used in the construction of an invariant action operator. This is accomplished with the aid of the local spin transformation

$$L(x)^{-1}[\partial_\mu - \tfrac{1}{2}i\bar{\omega}_{\mu ab}(x)S^{ab}]L(x) = \partial_\mu - \tfrac{1}{2}i\omega_{\mu ab}(x)S^{ab},$$

where

$$L(x)^{-1}S^{ab}L(x) = l^a{}_{a'}(x)l^b{}_{b'}(x)S^{a'b'}.$$

We consider the coordinate-spin commutator (there is no reference here to operator properties of $\omega_{\mu ab}$)

$$[\partial_\mu - \tfrac{1}{2}i\omega_{\mu ab}S^{ab},\ \partial_\nu - \tfrac{1}{2}i\omega_{\nu cd}S^{cd}] = -\tfrac{1}{2}iR_{\mu\nu ab}(x)S^{ab},$$

where

$$R_{\mu\nu ab}(x) = \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} - \omega_{\mu ac}\omega_\nu{}^c{}_b + \omega_{\nu ac}\omega_\mu{}^c{}_b = -R_{\nu\mu ab} = -R_{\mu\nu ba},$$

and observe that

$$\bar{R}_{\mu\nu ab}(x) = l_a{}^{a'}(x)l_b{}^{b'}(x)R_{\mu\nu a'b'}(x).$$

[4] The viewpoint and notation follow a previous paper of the author [J. Schwinger, Phys. Rev. **130**, 800 (1963)].

Thus, $R_{\mu\nu ab}$ is an antisymmetrical tensor with regard to local Lorentz transformations. It is also an antisymmetrical tensor for general coordinate transformations, in virtue of the curl derivative structure. The Jacobi identity obeyed by a double commutator implies a differential identity for the functions $R_{\mu\nu ab}(x)$. This is expressed most compactly with the aid of the dual tensor density

$$*R^{\mu\nu}{}_{ab}(x)=\tfrac{1}{2}\epsilon^{\mu\nu\lambda\kappa}R_{\lambda\kappa ab}(x),$$
$$*R^{01}{}_{ab}=R_{23ab},\ \cdots$$

as

$$\partial_\mu\, *R^{\mu\nu}{}_{ab}-\omega_{\mu a}{}^{c}\, *R^{\mu\nu}{}_{cb}-\omega_{\mu b}{}^{c}\, *R^{\mu\nu}{}_{ac}=0.$$

The term tensor density refers to the general coordinate transformation property

$$*\bar{R}^{\mu\nu}{}_{ab}(\bar{x})=(\det\partial x/\partial\bar{x})(\partial\bar{x}^\mu/\partial x^\lambda)(\partial\bar{x}^\nu/\partial x^\kappa)\, *R^{\lambda\kappa}{}_{ab}(x).$$

There is also a double dual tensor density

$$**R^{\mu\nu ab}=\tfrac{1}{4}\epsilon^{\mu\nu\lambda\kappa}R_{\lambda\kappa cd}\epsilon^{abcd},$$

with the local Lorentz transformation behavior

$$**\bar{R}^{\mu\nu ab}(x)=[\det l(x)]l^a{}_{a'}(x)l^b{}_{b'}(x)\, **R^{\mu\nu a'b'}(x).$$

This object obeys the differential identity

$$\partial_\mu\, **R^{\mu\nu ab}-\omega_\mu{}^a{}_c\, **R^{\mu\nu cb}-\omega_\mu{}^b{}_c\, **R^{\mu\nu ac}=0.$$

To construct an invariant action operator

$$W=\int(dx)\mathfrak{L},$$

we must devise a local function of the gravitational field variables that is a scalar density for general coordinate transformations, and a scalar with respect to proper local Lorentz transformations. The two simplest possibilities are

$$[\det e_\mu{}^a(x)]e^{\mu a}(x)e^{\nu b}(x)R_{\mu\nu ab}(x),$$

and

$$\tfrac{1}{4}**R^{\mu\nu ab}(x)R_{\mu\nu ab}(x),$$

which share the property of reversing sign under an improper local Lorentz transformation. The second choice is constructed entirely from $\omega_{\mu ab}$. It is not an effective contribution to an action operator, however, for the differential identity obeyed by $**R^{\mu\nu ab}$ implies that

$$\delta[\tfrac{1}{4}**R^{\mu\nu ab}R_{\mu\nu ab}]=\partial_\mu[**R^{\mu\nu ab}\delta\omega_{\nu ab}],$$

which is devoid of consequences for field equations.

Let us adopt provisionally the gravitational action operator

$$W=\int(dx)(\det e_\mu{}^a)(-1/2\kappa)R,$$

where

$$R=e^{\mu a}e^{\nu b}R_{\mu\nu ab}=R_{\mu\nu}{}^{\mu\nu},$$

while κ is a constant with the dimensions of length squared, and proceed to use it in a heuristic manner, without regard to precise operator properties. Then, apart from divergence terms,

$$\delta W=(-1/2\kappa)\int(dx)[\det e\,\delta e_a{}^\mu(2R_\mu{}^a-e_\mu{}^aR)+\delta\omega_{\mu ab}K^{\mu ab}],$$

in which

$$K^{\mu ab}=\partial_\nu[\det e(e^{\mu a}e^{\nu b}-e^{\mu b}e^{\nu a})]$$
$$-\omega_\nu{}^a{}_c[\det e(e^{\mu c}e^{\nu b}-e^{\mu b}e^{\nu c})]$$
$$-\omega_\nu{}^b{}_c[\det e(e^{\mu a}e^{\nu c}-e^{\mu c}e^{\nu a})],$$

and

$$R_\mu{}^a=R_{\mu\nu}{}^{ab}e_b{}^\nu=R_{\mu\nu}{}^{a\nu}.$$

We have also written

$$\det e=\det e_\mu{}^a.$$

The functions defined by the variations must obey differential identities as a consequence of the invariance of W under local Lorentz transformations and coordinate transformations. Thus, the infinitesimal local Lorentz transformation

$$\delta e_a{}^\mu=\delta\omega_a{}^be_b{}^\mu,$$
$$\delta\omega_{\nu ab}=\delta\omega_a{}^c\omega_{\nu cb}+\delta\omega_b{}^c\omega_{\nu ac}+\partial_\nu\delta\omega_{ab},$$
$$\delta\omega_{ab}(x)=-\delta\omega_{ba}(x),$$

implies the identity

$$\partial_\nu K^{\nu ab}-\omega_\nu{}^a{}_cK^{\nu cb}-\omega_\nu{}^b{}_cK^{\nu ac}=-\det e\, e^{\mu a}e^{\nu b}(R_{\mu\nu}-R_{\nu\mu}),$$

where

$$R_{\mu\nu}=R_{\mu\lambda\nu}{}^\lambda=e_\nu{}^ae^{\lambda b}R_{\mu\lambda ab}.$$

The infinitesimal coordinate transformation

$$\delta e_a{}^\mu=-\delta x^\nu\partial_\nu e_a{}^\mu+e_a{}^\nu\partial_\nu\delta x^\mu,$$
$$\delta\omega_{\nu ab}=-\delta x^\lambda\partial_\lambda\omega_{\nu ab}-\omega_{\lambda ab}\partial_\nu\delta x^\lambda,$$

gives the identity

$$\partial_\nu[\det e(2R_\mu{}^\nu-\delta_\mu{}^\nu R)]+\det e(2R_\nu{}^a-e_\nu{}^aR)\partial_\mu e_a{}^\nu$$
$$=\partial_\nu(K^{\nu ab}\omega_{\mu ab})-K^{\nu ab}\partial_\mu\omega_{\nu ab}.$$

These identities become more familiar if we set $K^{\nu ab}$ equal to zero, for then

$$R_{\mu\nu}=R_{\nu\mu},$$

while

$$\delta W=\int(dx)(-g)^{1/2}\tfrac{1}{2}\delta g^{\mu\nu}(-1/\kappa)(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R),$$

in which

$$g^{\mu\nu}=e_a{}^\mu g^{ab}e_b{}^\nu,$$

and

$$\partial_\nu[(-g)^{1/2}(R_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu R)]$$
$$+(-g)^{1/2}(R_{\lambda\nu}-\tfrac{1}{2}g_{\lambda\nu}R)\tfrac{1}{2}\partial_\mu g^{\lambda\nu}=0.$$

Thus, $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is Einstein's tensor.

The field equations

$$K^{\mu ab}=0$$

can be presented in the form

$$\Omega^{\mu}{}_{ab}-e^{\mu c}(\omega_{abc}-\omega_{bac})+e_b{}^{\mu}\lambda_a-e_a{}^{\mu}\lambda_b=0,$$

where

$$\Omega^{\mu}{}_{ab}=e_a{}^{\nu}\partial_\nu e_b{}^{\mu}-e_b{}^{\nu}\partial_\nu e_a{}^{\mu},$$

and

$$\lambda_a=(\det e)^{-1}\partial_\nu[(\det e)e_a{}^{\nu}]+\omega^b{}_{ba}.$$

An equivalent version is given by

$$\Omega_{cab}-\omega_{abc}+\omega_{bac}+g_{bc}\lambda_a-g_{ac}\lambda_b=0,$$

and a special consequence of the latter is

$$-\Omega^b{}_{ba}-\omega^b{}_{ba}+3\lambda_a=0.$$

But

$$\Omega^b{}_{ba}=\partial_\nu e_a{}^{\nu}-e_a{}^{\nu}e_\mu{}^b\partial_\nu e_b{}^{\mu}$$
$$=(\det e)^{-1}\partial_\nu[(\det e)e_a{}^{\nu}],$$

and, therefore,

$$\lambda_a=0.$$

That property is still implied by the resulting equation

$$\omega_{abc}-\omega_{bac}=\Omega_{cab},$$

which has the solution

$$\omega_{abc}=\tfrac{1}{2}[\Omega_{bca}+\Omega_{cab}-\Omega_{abc}].$$

This represents a dynamical deduction, based upon $K^{\mu ab}=0$, of the symmetry restriction

$$\Gamma_{abc}=\Gamma_{bac}$$

for the quantities

$$\Gamma_{abc}=\omega_{abc}-(e_a{}^{\nu}\partial_\nu e_b{}^{\mu})e_{\mu c}.$$

Invariance with respect to arbitrary local Lorentz transformations and coordinate transformations implies that the field equations exhibit a corresponding incompleteness in the description of the time evolution of the system. In order to obtain a clear physical interpretation of the formalism one must limit this arbitrariness by restricting the choice of local Lorentz frame and general coordinate system. We shall designate such restrictions as gauge conditions and coordinate conditions, respectively.

TIME GAUGE

The first objective will be to give the time coordinate a physical meaning by locking the time axes of the local coordinate systems to the time axis of the general coordinate system. The time coordinate x^0 can be distinguished by the requirement that $e_a{}^0(x)$ shall be a timelike vector in the local coordinate frames,

$$-e_a{}^0(x)g^{ab}e_b{}^0(x)>0.$$

Then it is possible to choose each local coordinate system so that the spatial components of $e_a{}^0$ vanish. This is the time gauge,

$$e_{(k)}{}^0(x)=0.$$

An equivalent characterization in terms of the inverse system $e_\mu{}^a$ is

$$e_k{}^{(0)}(x)=0.$$

Note also that

$$e_0{}^{(0)}=(e_{(0)}{}^0)^{-1},$$

and

$$e_{(m)}{}^k e_l{}^{(m)}=\delta_l{}^k=e_m{}^{(k)}e_{(l)}{}^m,$$

while

$$e_0{}^{(k)}=-e_0{}^{(0)}e_l{}^{(k)}e_{(0)}{}^l.$$

Furthermore,

$$\det e_\mu{}^a=e_0{}^{(0)}\det e_l{}^{(k)}=[e_{(0)}{}^0\det e_{(l)}{}^k]^{-1}.$$

The gravitational action operator appears in the time gauge as

$$W=(1/\kappa)\int(dx)\det{}_{(3)}e[e^{k(l)}R_{0k(0)(l)}$$
$$-\tfrac{1}{2}e_0{}^{(0)}e^{k(m)}e^{l(n)}R_{kl(m)(n)}+e_0{}^{(0)}e^{k(m)}e_{(0)}{}^lR_{kl(m)(0)}],$$

where

$$R_{0k(0)(l)}=\partial_0\omega_{k(0)(l)}-\partial_k\omega_{0(0)(l)}$$
$$+\omega_{k(l)}{}^{(m)}\omega_{0(0)(m)}+\omega_{k(0)(m)}\omega_0{}^{(m)}{}_{(l)},$$

$$R_{kl(m)(0)}=-\partial_k\omega_{l(0)(m)}+\partial_l\omega_{k(0)(m)}$$
$$+\omega_{k(m)}{}^{(n)}\omega_{l(0)(n)}-\omega_{l(m)}{}^{(n)}\omega_{k(0)(n)},$$

and

$$R_{kl(m)(n)}={}_{(3)}R_{kl(m)(n)}-\omega_{k(0)(m)}\omega_{l(0)(n)}+\omega_{l(0)(m)}\omega_{k(0)(n)}.$$

In the last equation, the notation ${}_{(3)}R_{kl(m)(n)}$ implies the formation of this tensor from the three-dimensional quantities $\omega_{k(m)(n)}$. We have also written

$$\det{}_{(3)}e=\det e_l{}^{(k)}.$$

It will be observed that $\omega_{k(0)(l)}$ and $\det_{(3)}e\, e^{k(l)}$ obey equations of motion. There are no equations of motion for $\omega_{0(0)(l)}$, $\omega_{0(l)(m)}$, $\omega_{k(l)(m)}$, $e_{(0)}{}^l$, or $e_{(0)}{}^0$. When only the gravitational field is considered, the variations of the first three sets of variables give equations of constraint which are, respectively,

$$\partial_k(\det{}_{(3)}e\,e_{(l)}{}^k)-\omega_{k(l)}{}^{(m)}(\det{}_{(3)}e\,e_{(m)}{}^k)=0;$$

$$\omega_{(l)(0)(m)}=\omega_{(m)(0)(l)},$$

where

$$\omega_{(l)(0)(m)}=e_{(l)}{}^k\omega_{k(0)(m)};$$

$${}_{(3)}K^{k(m)(n)}=\det{}_{(3)}e[e^{k(m)}(\omega_{(0)(0)}{}^{(n)}-e^{l(n)}\partial_l\ln e_0{}^{(0)})$$
$$-e^{k(n)}(\omega_{(0)(0)}{}^{(m)}-e^{l(m)}\partial_l\ln e_0{}^{(0)})],$$

in which

$$\omega_{(0)(0)}{}^{(m)}=e_{(0)}{}^l\omega_{l(0)}{}^{(m)}+e_{(0)}{}^0\omega_{0(0)}{}^{(m)},$$

and ${}_{(3)}K^{k(m)(n)}$ is formed from three-dimensional quantities in the manner of $K^{\mu ab}$.

If the first constraint equation,

$$\partial_k(\det{}_{(3)}e\,e_{(l)}{}^k)+\det{}_{(3)}e\,\omega^{(m)}{}_{(m)(l)}=0,$$

is combined with the formula for ${}_{(3)}K^k{}_{(m)(n)}$, the latter simplifies to

$$ {}_{(3)}K^k{}_{(m)(n)} = -\det{}_{(3)}e[{}_{(3)}\Omega^k{}_{(m)(n)} - e^{k(p)}(\omega_{(m)(n)(p)} - \omega_{(n)(m)(p)})], $$

and a second application of this constraint shows that

$$ e_k{}^{(m)}\ {}_{(3)}K^k{}_{(m)(n)} = -\det{}_{(3)}e[{}_{(3)}\Omega^{(m)}{}_{(m)(n)} + \omega^{(m)}{}_{(m)(n)}] = 0. $$

Accordingly,

$$ \omega_{(0)(0)(m)} = e_{(m)}{}^l \partial_l \ln e_0{}^{(0)}, $$

and

$$ {}_{(3)}K^k{}_{(m)(n)} = 0, $$

or

$$ \omega_{(k)(l)(m)} = \tfrac{1}{2}[{}_{(3)}\Omega_{(l)(m)(k)} + {}_{(3)}\Omega_{(m)(k)(l)} - {}_{(3)}\Omega_{(k)(l)(m)}], $$

all of this being a three-dimensional counterpart of the four-dimensional discussion.

In virtue of the symmetry possed by $\omega_{(k)(0)(l)}$, there are six pairs of variables in the time-derivative term of the action integrand. A particularly convenient choice is obtained by introducing

$$ \omega_k{}^{(0)}{}_l = -\omega_{k(0)(m)} e_l{}^{(m)} = \omega_l{}^{(0)}{}_k, $$

and the three-dimensional tensor

$$ {}_{(3)}g_{kl} = e_k{}^{(m)} e_{l(m)} = g_{kl}, $$

together with its inverse

$$ {}_{(3)}g^{kl} = e^{k(m)} e_{(m)}{}^l \neq g^{kl}. $$

Thus,

$$ \det{}_{(3)}e\ e^{k(l)} \partial_0 \omega_{k(0)(l)} = -g^{1/2} e^{k(l)} \partial_0 (\omega_k{}^{(0)}{}_m e_{(l)}{}^m), $$

where

$$ g = \det{}_{(3)}g_{kl}, $$

and this becomes

$$ -g^{1/2}\ {}_{(3)}g^{kl}\partial_0\omega_k{}^{(0)}{}_l - \tfrac{1}{2}g^{1/2}\omega_k{}^{(0)}{}_l\partial_0\ {}_{(3)}g^{kl} = \tfrac{1}{2}g^{-1/2}\omega_k{}^{(0)}{}_l\partial_0(g\ {}_{(3)}g^{kl}) - \partial_0[g^{1/2}\ {}_{(3)}g^{kl}\omega_k{}^{(0)}{}_l]. $$

The time-derivative term may be omitted since the action operator of a given dynamical system can be altered by the addition of boundary terms. The required pairs of variables are

$$ q^{kl} = g\ {}_{(3)}g^{kl}, $$

and

$$ \Pi_{kl} = (1/2\kappa) g^{-1/2} \omega_k{}^{(0)}{}_l. $$

Note that the other terms in $R_{0k(0)(l)}$ are effectively equal to zero by virtue of the constraint conditions, provided that $e_{(0)}{}^k$ and $\partial_k e_0{}^{(0)}$ vanish sufficiently rapidly at remote spatial points.

The resulting form of the action operator is

$$ W = \int (dx)[\Pi_{kl}\partial_0 q^{kl} - e_0{}^{(0)} e_{(0)}{}^k \tau_k - e_0{}^{(0)} g^{-1/2}\tau^0], $$

in which

$$ \tau_k = -\Pi_{lm}\partial_k q^{lm} + \partial_k(2\Pi_{lm}q^{lm}) - \partial_l(2\Pi_{km}q^{lm}), $$

and

$$ \tau^0 = (1/2\kappa) g\ {}_{(3)}R - 2\kappa\Pi_{kl}(q^{kl}q^{mn} - q^{kn}q^{lm})\Pi_{mn}. $$

The explicit structure of

$$ g\ {}_{(3)}R = q^{kl}\ {}_{(3)}R_{kl} $$

is given by

$$ q^{kl}\ {}_{(3)}R_{kl} = \partial_k\partial_l q^{kl} + Q, $$

where

$$ Q = -\tfrac{1}{4}q^{mn}\partial_m q^{kl}\partial_n q_{kl} - \tfrac{1}{2}\partial_m q^{kl} q_{ln}\partial_k q^{mn} - \tfrac{1}{2}q^{kl}\partial_k \ln(q^{1/2})\partial_l \ln(q^{1/2}), $$

and

$$ q = \det q^{kl} = g^2, $$

while

$$ q_{kl} = g^{-1} g_{kl} $$

is the matrix inverse to q^{kl}.

MATTER FIELD

We shall consider here only the simplest example of a matter field. The action operator of a zero spin field in a prescribed metric field $g_{\mu\nu}$ can be written as

$$ W = \int (dx)[\phi^\mu\partial_\mu\phi + \tfrac{1}{2}\phi^\mu(-g)^{-1/2}g_{\mu\nu}\phi^\nu - \tfrac{1}{2}m^2(-g)^{1/2}\phi^2], $$

where ϕ^μ is a vector density. The constraint equation implied by variation of ϕ^k is

$$ 0 = \partial_k\phi + (-g)^{-1/2} g_{k\nu}\phi^\nu, $$

or equivalently, in the time gauge,

$$ -e_{(l)}{}^k\partial_k\phi = e_{(0)}{}^0 g^{-1/2} e_{\nu(l)}\phi^\nu. $$

The square of this local vector equation gives the relation

$$ \partial_k\phi q^{kl}\partial_l\phi = (e_{(0)}{}^0)^2\phi^\mu g_{\mu\nu}\phi^\nu + (\phi^0)^2. $$

An alternative combination, obtained by multiplication with $e^{k(l)}\partial_k\phi$, is

$$ -e_0{}^{(0)} g^{-1/2}\partial_k\phi q^{kl}\partial_l\phi = \phi^k\partial_k\phi - e_0{}^{(0)} e_{(0)}{}^k\phi^0\partial_k\phi. $$

The resulting form of the action operator, from which ϕ^k has been eliminated, is

$$ W = \int (dx)[\phi^0\partial_0\phi - e_0{}^{(0)} e_{(0)}{}^k T_k - e_0{}^{(0)} g^{-1/2} T^0], $$

where

$$ T_k = -\phi^0\partial_k\phi, $$

and

$$ T^0 = \tfrac{1}{2}[(\phi^0)^2 + \partial_k\phi q^{kl}\partial_l\phi + q^{1/2}m^2\phi^2]. $$

The equal-time commutation properties of these operators follow easily from the canonical commutation relations obeyed by ϕ and ϕ^0. Thus,

$$-i[T^0(x),T^0(x')] = -[q^{kl}(x)T_l(x)+q^{kl}(x')T_l(x')]\partial_k\delta(\mathbf{x}-\mathbf{x}'),$$

and

$$-i[T_k(x),T_l(x')] = -T_l(x)\partial_k\delta(\mathbf{x}-\mathbf{x}')-T_k(x')\partial_l\delta(\mathbf{x}-\mathbf{x}').$$

COORDINATE CONDITIONS

The action operator of the combined gravitational and matter field system is

$$W=\int(dx)[\Pi_{kl}\partial_0 q^{kl}+\phi^0\partial_0\phi-e_0{}^{(0)}e_{(0)}{}^k(\tau_k+T_k) - e_0{}^{(0)}g^{-1/2}(\tau^0+T^0)].$$

The constraint equations supplied by variation of $e_{(0)}{}^k$ and $e_0{}^{(0)}$ are

$$\tau_k+T_k=0,\quad \tau^0+T^0=0.$$

Alternative forms are

$$2(\partial_l\Pi_{kl}-\partial_k\Pi_{ll})=t_k+T_k=\theta_k,$$

and

$$-\partial_k\partial_l q^{kl}=2\kappa(t^0+T^0)=2\kappa\theta^0,$$

where

$$t_k=-\Pi_{lm}\partial_k q^{lm}+\partial_k[2\Pi_{lm}(q^{lm}-\delta_{lm})] - \partial_l[2\Pi_{km}(q^{lm}-\delta_{lm})],$$

and

$$t^0=(1/2\kappa)Q+2\kappa\Pi_{kl}(q^{kn}q^{lm}-q^{kl}q^{mn})\Pi_{mn}.$$

This version presents the constraint equations as implicit determinations of certain linear differential functions of the fields q^{kl} and Π_{kl}.

The same field combinations will occur in the time derivative term of the action operator if one writes

$$q^{kl}=q^{klT}+\tfrac{1}{2}(\partial_k q_l+\partial_l q_k)-\delta_{kl}\partial_m q_m+\partial_k\partial_l q,$$

and

$$\Pi_{kl}=\Pi_{kl}{}^T+\tfrac{1}{2}(\partial_k\Pi_l+\partial_l\Pi_k)-\delta_{kl}\partial_m\Pi_m+\partial_k\partial_l\Pi,$$

where q^{klT}, for example, the transverse-traceless part of q^{kl}, is such that

$$\partial_k q^{klT}=0,\quad q^{kkT}=0.$$

The two independent components of this field combine with q_k and q to represent the six-component field q^{kl}. These representations are such that

$$\int(d\mathbf{x})\Pi_{kl}\partial_0 q^{kl}=\int(d\mathbf{x})[\Pi_{kl}{}^T\partial_0 q^{klT} - (\tfrac{3}{2}\partial_k\partial_l\Pi_l+\tfrac{1}{2}\nabla^2\Pi_k)\partial_0 q_k+\Pi\partial_0(\nabla^2)^2 q],$$

under the conditions that validate the partial integrations, and here

$$\tfrac{3}{2}\partial_k\partial_l\Pi_l+\tfrac{1}{2}\nabla^2\Pi_k=\partial_l\Pi_{kl}-\partial_k\Pi_{ll}=\tfrac{1}{2}\theta_k,$$
$$(\nabla^2)^2 q=\partial_k\partial_l q^{kl}=-2\kappa\theta^0.$$

The action operator can now be reduced to

$$W=\int(dx)[\Pi_{kl}{}^T\partial_0 q^{klT}+\phi^0\partial_0\phi +\theta_k\partial_0(-\tfrac{1}{2}q_k)-\theta^0\partial_0(-2\kappa\Pi)],$$

which also exploits the freedom to add boundary terms. The operators θ_k and θ^0 are to be constructed from ϕ^0, ϕ and $\Pi_{kl}{}^T$, q^{klT}, together with q_k and Π. The pairs of fields are evidently canonical dynamical variables while q_k and Π are numerical transformation parameters. The action operator is formed additively from operators describing infinitesimal increments of x^0,

$$W_{dx^0}=\int(d\mathbf{x})[\Pi_{kl}{}^T dq^{klT}+\phi^0 d\phi +\theta_k d(-\tfrac{1}{2}q_k)-\theta^0 d(-2\kappa\Pi)].$$

The infinite-dimensional parametric transformation given here can be identified with a local description of the physical space-time evolution of the system. Thus, $d(-\frac{1}{2}q_k)$ and $d(-2\kappa\Pi)$ are interpreted as infinitesimal local space and time coordinate displacements, while θ_k and θ^0 appear as momentum and energy densities, respectively. With this physical identification of coordinate parameters, we can proceed to restrict the mathematical freedom of coordinate transformation in order to exhibit a physically distinguished class of coordinate system.

Under what circumstances are the q^{kl} not explicit functions of the space coordinate displacement parameters? The condition is

$$\tfrac{1}{2}(\partial_k dq_l+\partial_l dq_k)-\delta_{kl}\partial_m dq_m=0,$$

or equivalently

$$\partial_k dq_l+\partial_l dq_k=0,$$

which also implies that

$$\nabla^2 dq_k=0.$$

As a consequence of these restrictions, $-\frac{1}{2}dq_k$ can only be the linear space coordinate function

$$-\tfrac{1}{2}dq_k=d\epsilon_k(x^0)-d\omega_{kl}(x^0)x_l,$$
$$d\omega_{kl}+d\omega_{lk}=0,$$

which describes a rigid translation and rotation of the coordinate system. The associated generating operators are the total linear and angular momentum

$$P_k=\int(d\mathbf{x})\theta_k,$$

$$J_{kl}=\int(d\mathbf{x})(x_k\theta_l-x_l\theta_k).$$

In a similar way, Π_{kl} will not be an explicit function of the time displacement parameter if

$$\partial_k \partial_l d\Pi = 0,$$

so that $d\Pi$ is a linear space coordinate function,

$$-2\kappa d\Pi = d\epsilon^0(x^0) + d\omega_{0k}(x^0)x_k.$$

The corresponding generating operators are the total energy

$$P^0 = \int (d\mathbf{x})\theta^0,$$

and the Lorentz transformation generator

$$J^0{}_k - x^0 P_k = -\int (d\mathbf{x}) x_k \theta^0.$$

A given member of this distinguished class of Lorentz transformation equivalent coordinate systems is characterized by the coordinate conditions

$$-\tfrac{1}{2} q_k = x_k, \quad -2\kappa\Pi = x^0.$$

In such a coordinate system, the field operators simplify to

$$q^{kl} = q^{klT} + \partial_k \partial_l (q + \tfrac{3}{2}\mathbf{x}^2) + \delta_{kl},$$

and

$$\Pi_{kl} = \Pi_{kl}{}^T + \tfrac{1}{2}(\partial_k \Pi_l + \partial_l \Pi_k) - \delta_{kl}\partial_m \Pi_m.$$

A spatial boundary condition thereby indicated for points far outside regions occupied by energy,

$$|\mathbf{x}| \to \infty, \quad \partial_k \partial_l (q + \tfrac{3}{2}\mathbf{x}^2) \to 0,$$

is compatible with the fourth order differential equation obeyed by q or $q + \frac{3}{2}\mathbf{x}^2$.

The linear and angular momentum operators involve only the canonical variables in the explicit forms

$$P_k = \int (d\mathbf{x}) [-\phi^0 \partial_k \phi - \Pi_{lm}{}^T \partial_k q^{lmT}],$$

and

$$J_{kl} = \int (d\mathbf{x}) [-\phi^0 (x_k \partial_l - x_l \partial_k)\phi - \Pi_{mn}{}^T (x_k \partial_l - x_l \partial_k) q^{mnT} + 2\Pi_{lm}{}^T q^{kmT} - 2\Pi_{km}{}^T q^{lmT}].$$

All the anticipated commutation properties of these operators can be derived from the equal-time canonical commutation relations:

$$-i[\phi(x), \phi^0(x')] = \delta(\mathbf{x} - \mathbf{x}'),$$

$$-i[q^{klT}(x), \Pi_{mn}{}^T(x')] = [\delta_{mn}{}^{kl} \delta(\mathbf{x} - \mathbf{x}')]^T,$$

$$\delta_{mn}{}^{kl} = \tfrac{1}{2}(\delta_m{}^k \delta_n{}^l + \delta_n{}^k \delta_m{}^l),$$

including

$$[q^{lmT}(x), P_k] = -i\partial_k q^{lmT}(x),$$

and

$$[q^{mnT}(x), J_{kl}] = -i(x_k \partial_l - x_l \partial_k) q^{mnT}(x) + i(\delta_l{}^n q^{kmT} + \delta_l{}^m q^{knT} - \delta_k{}^n q^{lmT} - \delta_k{}^m q^{lnT}).$$

These observations show that the quantum-mechanical formalism associated with the canonical commutation relations satisfies the requirement of invariance under three-dimensional translations and rotations. The question of Lorentz invariance depends upon integral aspects of the energy density equal time commutator. It is at this vital point that the gravitational field differs from all other physical systems, for there is no explicit formula for θ^0 in terms of the fundamental variables but only an implicit determination by means of the constraint equations. While such a lack of explicitness in a classical theory would raise computational difficulties, in a quantum theory it could also be a formidable barrier to verifying the consistency of the formalism.

We shall consider aspects of this basic problem in a separate paper.