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Partial-Wave Bethe-Salpeter Equation

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The Bethe-Salpeter equation for higher wave bound states of two scalar particles is investigated in the ladder approximation. It is shown that all solutions have the Deser-Gilbert-Sudarshan-Ida integral representation and that they behave like $o[(p^2)^{-l-2}]$ as $p^2 \to \infty$ apart from a solid harmonic. The angular momentum l is continued to complex values, and it is proved that the wave functions are essentially holomorphic with respect to l in Re $l > -\frac{1}{4}$. The equation for the Regge trajectories is also discussed.

I. INTRODUCTION

THE relativistic bound state problem is usually dealt with by the Bethe-Salpeter equation. For solving it there are the following three approaches, which are mutually complementary.

The first approach, which was first introduced by Wick,[1] is to transform the relative momentum into an Euclidean vector. By this technique, one can avoid the singularity of a propagator and make use of many mathematical theorems.

The second one is to use the so-called Deser-Gilbert-Sudarshan-Ida integral representation.[2] A special case of this representation was used by Wick[1] and Cutkosky.[3] A general consideration for S-wave solutions was made by Wanders.[4] Recently, Ida and Maki[5] have proved that all S-wave solutions have this representation. On the other hand, Sato[6] has shown that the Fredholm theory is applicable for the weight functions in the cases of S and P waves. All these considerations are naturally restricted to the ladder approximation. This approach is useful for investigating the analyticity of the wave functions.

The third method is to utilize the fact that the invariant Bethe-Salpeter amplitude has a double dispersion representation when the total momentum is continued to a space-like region. Recently, the present author[7] has shown that even the exact Bethe-Salpeter equation can be solved in an elegant way by this method.

The purpose of the present paper is to investigate higher wave solutions. Unfortunately, the double dispersion approach does not seem to be suitable to this purpose. Hence, we use the second approach and confine ourselves to considering the ladder approximation. In Sec. II the Bethe-Salpeter equation is decomposed into partial-wave equations. In Sec. III we introduce the DGSI integral representation and derive an integral equation for the weight function. In Sec. IV it is proved that any partial-wave solution can always be represented as the form introduced in Sec. III. In the final section the angular momentum l is considered as a complex variable. The Fredholm solutions of the weight function will be obtained by using Sato's method,[6] and the wave function is shown to be holomorphic for Re $l > -\frac{1}{4}$ apart from some multiplicative factors. A connection with the Regge trajectory[8] is also discussed. In the Appendix the asymptotic behavior of the weight function is investigated.

[1] G. C. Wick, Phys. Rev. **96**, 1124 (1954). It should be remarked that for a *multiple* Feynman integral Wick's *simultaneous* rotation of integration paths of energy variables is not a mathematically justifiable notion. If one applies Cauchy's theorem correctly, one will generally meet complex singularities.

[2] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959). M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960).

[3] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

[4] G. Wanders, Helv. Phys. Acta **30**, 417 (1957).

[5] M. Ida and K. Maki, Progr. Theoret. Phys. (Kyoto) **26**, 470 (1961).

[6] I. Sato, J. Math. Phys. **4**, 24 (1963).

[7] N. Nakanishi (unpublished). See also N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **24**, 1275 (1960), and reference 14.

II. PARTIAL-WAVE DECOMPOSITION

We consider the Bethe-Salpeter equation in ladder approximation for two scalar particles having masses m_1 and m_2 which exchange a scalar meson μ:

$$[m_1^2-(p+k)^2][m_2^2-(p-k)^2]f(p) = \frac{\lambda}{\pi^2 i}\int d^4q \frac{f(q)}{\mu^2-(p-q)^2-i\epsilon}, \quad (2.1)$$

where p is the relative momentum, $2k$ the total momentum, λ the squared coupling constant, and $f(p)$ the wave function. For simplicity, we always take the rest system $k=(k_0,0,0,0)$. We assume the stability condition

$$(m_1+m_2)^2 > u \equiv 4k^2 > 0. \quad (2.2)$$

According to the addition theorem of the Legendre polynomial P_l,

$$P_l(\cos\omega) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi') \quad (2.3)$$

with

$$\cos\omega \equiv \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'), \quad (2.4)$$

and a formula

$$(\beta-\cos\omega)^{-1} = \sum_{l=0}^{\infty}(2l+1)Q_l(\beta)P_l(\cos\omega) \quad (2.5)$$

with

$$Q_l(\beta) \equiv \frac{1}{2}\int_{-1}^{1} d\zeta \frac{P_l(\zeta)}{\beta-\zeta}, \quad (2.6)$$

[8] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).

the propagator in (2.1) can be decomposed into

$$[\mu^2-(p-q)^2-i\epsilon]^{-1}=2\pi|\mathbf{p}|^{-1}|\mathbf{q}|^{-1}\sum_{l=0}^{\infty}Q_l(\beta)\times\sum_{m=-l}^{l}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi'),\quad(2.7)$$

where

$$\beta\equiv\frac{\mu^2+\mathbf{p}^2+\mathbf{q}^2-(p_0-q_0)^2-i\epsilon}{2|\mathbf{p}|\,|\mathbf{q}|}\quad(2.8)$$

and (θ,ϕ) and (θ',ϕ') are the polar angles of $\mathbf{p}$ and $\mathbf{q}$, respectively. If we put

$$f(p)=|\mathbf{p}|^{-1}\psi_l(|\mathbf{p}|,p_0)Y_{lm}(\theta,\phi),\quad(2.9)$$

(2.1) becomes

$$[m_1^2+\mathbf{p}^2-(p_0+k_0)^2][m_2^2+\mathbf{p}^2-(p_0-k_0)^2]\psi_l(|\mathbf{p}|,p_0)=\frac{2\lambda}{\pi i}\int_0^\infty d|\mathbf{q}|\int_{-\infty}^{\infty}dq_0\,Q_l(\beta)\psi_l(|\mathbf{q}|,q_0)\quad(2.10)$$

with (2.8). A similar equation was given by Lee and Sawyer.[9]

The trace of the kernel of (2.10) is

$$\sigma_l\equiv\frac{2}{\pi i}\int_0^\infty d|\mathbf{p}|\int_{-\infty}^{\infty}dp_0\times\frac{Q_l(1+\mu^2/2\mathbf{p}^2)}{[m_1^2+\mathbf{p}^2-(p_0+k_0)^2-i\epsilon][m_2^2+\mathbf{p}^2-(p_0-k_0)^2-i\epsilon]}$$

$$=\frac{1}{2}\int_{-1}^{1}dz\int_{-1}^{1}d\zeta\,P_l(\zeta)\int_0^\infty d|\mathbf{p}|\times\frac{\mathbf{p}^2}{[\mathbf{p}^2+\rho(z)]^{3/2}[\mu^2+2\mathbf{p}^2(1-\zeta)]},\quad(2.11)$$

where

$$\rho(z)\equiv\tfrac{1}{2}(1+z)m_1^2+\tfrac{1}{2}(1-z)m_2^2-\tfrac{1}{4}(1-z^2)u\geqq\rho_0>0.\quad(2.12)$$

III. INTEGRAL REPRESENTATION

As a simple extension of the integral representation in the S-wave case,[4,5] we assume that higher wave solutions are represented as

$$f(p)=\mathcal{Y}_{lm}(\mathbf{p})\int_{-1}^{1}dz\int_{-\infty}^{\infty}d\alpha\times\frac{\varphi_l^{[n]}(z,\alpha)}{[\alpha-\frac{1}{2}(1+z)(s-m_1^2)-\frac{1}{2}(1-z)(t-m_2^2)-i\epsilon]^{n+2}},\quad(3.1)$$

where $\mathcal{Y}_{lm}$ stands for a solid harmonic, i.e.,

$$\mathcal{Y}_{lm}(\mathbf{p})\equiv|\mathbf{p}|^lY_{lm}(\theta,\phi),\quad(3.2)$$

and

$$s\equiv(p+k)^2,\quad t\equiv(p-k)^2.\quad(3.3)$$

The integer n must be chosen so as to make the Feynman integral convergent, namely,

$$n+1>l/2.\quad(3.4)$$

In order that (3.1) is meaningful, we assume

$$\lim_{\alpha\to\infty}\varphi_l^{[n]}(z,\alpha)/\alpha^{n+1}=0,\quad(3.5)$$

$$\varphi_l^{[n]}(z,-\infty)=0.\quad(3.6)$$

Then a partial integration of (3.1) leads to

$$\varphi_l^{[n-1]}(z,\alpha)=(n+1)^{-1}(\partial/\partial\alpha)\varphi_l^{[n]}(z,\alpha).\quad(3.7)$$

Now, it is easily shown that for an arbitrary function F one has a formula[3]

$$\int d\mathbf{q}\,F(\mathbf{q}^2)\mathcal{Y}_{lm}(\mathbf{q}+\mathbf{p})=\mathcal{Y}_{lm}(\mathbf{p})\int d\mathbf{q}\,F(\mathbf{q}^2),\quad(3.8)$$

provided that both integrals are convergent. Substituting (3.1) in (2.1) and using (3.8) after Feynman parametrization, we obtain

$$f(p)=\frac{n+2}{2}\mathcal{Y}_{lm}(\mathbf{p})\lambda\int_{-1}^{1}dz'\int_{-\infty}^{\infty}d\alpha'\,\varphi_l^{[n]}(z',\alpha')\times\int_{-1}^{1}dz\int_{-\infty}^{\infty}d\alpha\int_0^1dx\,x^{l-n-1}[g(\alpha',z',x)]^{-n-1}\times\alpha^n[\theta(\alpha)-\theta(\alpha-R(z,z')g(\alpha',z',x))]\times[\alpha-\tfrac{1}{2}(1+z)(s-m_1^2)-\tfrac{1}{2}(1-z)(t-m_2^2)-i\epsilon]^{-n-3},\quad(3.9)$$

where

$$g(\alpha',z',x)\equiv x^{-1}[\alpha'+(1-x)\rho(z')]+(1-x)^{-1}\mu^2,\quad(3.10)$$

$$R(z,z')\equiv(1\mp z)/(1\mp z')\quad\text{for}\quad z\gtrless z'.\quad(3.11)$$

Comparing (3.9) with (3.1), according to the uniqueness theorem of the DGSI integral representation,[10] we have

$$\varphi_l^{[n]}(z,\alpha)=\lambda\int_{-1}^{1}dz'\int_{-\infty}^{\infty}d\alpha'\times K_l^{[n]}(z,\alpha;z',\alpha')\varphi_l^{[n]}(z',\alpha')\quad(3.12)$$

with

$$K_l^{[n]}(z,\alpha;z',\alpha')\equiv\frac{1}{2}\int_0^1dx\,x^{l-n-1}[g(\alpha',z',x)]^{-n-1}\times(\partial/\partial\alpha)\{\alpha^n[\theta(\alpha)-\theta(\alpha-R(z,z')g(\alpha',z',x))]\}.\quad(3.13)$$

By solving (3.12) one gets solutions of (2.1) through (3.1).

[9] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962). Their proof of the convergence of the series is mathematically insufficient because they used Wick's simultaneous rotation of integration paths (see footnote 1) and made term-by-term continuation (rotation) before establishing the convergence of the series. One should note that the scattering Green function is essentially different from the Bethe-Salpeter amplitude in the analyticity on the p_0 plane.

[10] N. Nakanishi, Phys. Rev. **127**, 1380 (1962).

The following properties can be derived from (3.12) and (3.13).

(1) $$\varphi_l^{[n]}(z,\alpha)=0 \quad \text{for} \quad \alpha<0. \tag{3.14}$$

Hence, we hereafter consider $\alpha \geqq 0$ only.

(2) $$\lim_{\alpha\to\infty} \varphi_l^{[n]}(z,\alpha)/\alpha^n=0 \quad \text{for} \quad n\geqq 0. \tag{3.15}$$

This property is proved in Appendix by using the assumption (3.5).

(3) $$(\partial/\partial\alpha)K_l^{[n]}(z,\alpha;z',\alpha') = -(\partial/\partial\alpha')K_l^{[n-1]}(z,\alpha;z',\alpha') \quad \text{for} \quad n\geqq 1. \tag{3.16}$$

It should be remarked that (3.16) is no longer valid for $n=0$.

From (3.12), (3.7), and (3.16) we obtain

$$\varphi_l^{[n-1]}(z,\alpha)=-\frac{\lambda}{n+1}\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha' \times\left[\frac{\partial}{\partial\alpha'}K_l^{[n-1]}(z,\alpha;z',\alpha')\right]\varphi_l^{[n]}(z',\alpha'). \tag{3.17}$$

By a partial integration (3.17) becomes the same equation as (3.12) except for the superscripts, which are now replaced by $n-1$. By repetition of the above procedure (3.12) reduces to

$$\varphi_l^{[0]}(z,\alpha)=\lambda\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha' \times K_l^{[0]}(z,\alpha;z',\alpha')\varphi_l^{[0]}(z',\alpha'), \tag{3.18}$$

where

$$K_l^{[0]}(z,\alpha;z',\alpha')=\frac{1}{2}\int_0^1 dx\frac{x^{l-1}}{g(\alpha',z',x)} \times[\delta(\alpha)-\delta(\alpha-R(z,z')g(\alpha',z',x))]. \tag{3.19}$$

If $\varphi_l^{[0]}$ is found, $\varphi_l^{[n]}$ is given by

$$\varphi_l^{[n]}(z,\alpha)=(n+1)!\int_{0-}^{\alpha}d\alpha\cdots\int_{0-}^{\alpha}d\alpha\ \varphi_l^{[0]}(z,\alpha) \tag{3.20}$$

(where there are n integrals on the right-hand side), which is always a solution of (3.12).

The trace of $K_l^{[n]}(z,\alpha;z',\alpha')$, which is naturally independent of n, is

$$\tilde{\sigma}_l\equiv\frac{1}{2}\int_{-1}^{1}dz\int_0^1 dx\frac{x^l(1-x)}{(1-x)\mu^2+x^2\rho(z)}, \tag{3.21}$$

or

$$\tilde{\sigma}_l=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3 \times\frac{x_3{}^l\delta(1-x_1-x_2-x_3)}{x_1m_1{}^2+x_2m_2{}^2+x_3\mu^2-x_1x_3m_1{}^2-x_2x_3m_2{}^2-x_1x_2u} \tag{3.22}$$

in Feynman parametric form.

Finally, we notice the following important result:

$$\lim_{\alpha\to\infty} \varphi_l^{[n]}(z,\alpha)=0 \quad \text{for} \quad 0\leqq n\leqq l+1. \tag{3.23}$$

Its proof is given in Appendix. In particular, for $n=l+1$ we have

$$\int_{0-}^{\infty}d\alpha\ \varphi_l^{[l]}(z,\alpha)=0, \tag{3.24}$$

which implies that the integral in (3.1) vanishes faster than $(p^2)^{-l-2}$ as $p^2\to\infty$. This assures the convergence of Sato's normalization integral.[6]

IV. EQUALITY OF THE TRACES σ_l AND $\tilde{\sigma}_l$

In this section, we prove, by Ida and Maki's method,[5] that any solution of (2.10) has the representation (3.1).

Let λ_{li} be eigenvalues of (2.10). Then

$$\sigma_l=\textstyle\sum_i \lambda_{li}{}^{-1}. \tag{4.1}$$

The totality of the eigenvalues of (3.12) is naturally a subset of $\{\lambda_{li}\}$. When $\lambda_{li}>0$, therefore, all solutions of (2.10) are represented as in (3.1) if and only if

$$\sigma_l=\tilde{\sigma}_l. \tag{4.2}$$

The positive definiteness of λ_{li} is proved only in the case $m_1=m_2$.[11] But we assume that the same is true also for $m_1\neq m_2$. In the following we demonstrate (4.2).

Comparing (2.11) with (3.21), we see that (4.2) immediately follows if

$$\eta_l(\nu)=\tilde{\eta}_l(\nu), \quad \nu\equiv\mu^2/\rho(z) \tag{4.3}$$

is proved, where

$$\eta_l(\nu)\equiv\int_{-1}^{1}d\zeta\ P_l(\zeta) \times\int_0^\infty d|\mathbf{p}|\frac{\mathbf{p}^2}{(\mathbf{p}^2+1)^{3/2}[\nu+2\mathbf{p}^2(1-\zeta)]}, \tag{4.4}$$

$$\tilde{\eta}_l(\nu)\equiv\int_0^1 dx\frac{x^l(1-x)}{(1-x)\nu+x^2}. \tag{4.5}$$

Both $\eta_l(\nu)$ and $\tilde{\eta}_l(\nu)$ satisfy dispersion relations

$$\eta_l(\nu)=\int_0^\infty d\nu'\frac{\xi_l(\nu')}{\nu'+\nu}, \tag{4.6}$$

$$\tilde{\eta}_l(\nu)=\int_0^\infty d\nu'\frac{\tilde{\xi}_l(\nu')}{\nu'+\nu}, \tag{4.7}$$

where

$$\xi_l(\nu')\equiv\int_{-1}^{1}d\zeta\ P_l(\zeta) \times\int_0^\infty d|\mathbf{p}|\frac{\mathbf{p}^2}{(\mathbf{p}^2+1)^{3/2}}\delta(\nu'-2\mathbf{p}^2(1-\zeta)), \tag{4.8}$$

[11] See reference 1 and 5. For demonstration of positiveness of the kernel, use the technique given in a footnote of N. Nakanishi, Brookhaven National Laboratory Report (unpublished).

$$\tilde{\xi}_l(\nu')\equiv\int_0^1 dx\, x^l(1-x)\delta((1-x)\nu'-x^2). \qquad (4.9)$$

Hence (4.3) is equivalent to

$$\xi_l(\nu')=\tilde{\xi}_l(\nu'). \qquad (4.10)$$

From (4.8) we have

$$\xi_l(\nu')=\frac{\nu'^{1/2}}{2}\int_{-1}^1 d\zeta\,\frac{P_l(\zeta)}{[\nu'+2(1-\zeta)]^{3/2}}. \qquad (4.11)$$

Using the generating function formula of the Legendre polynomial,

$$(1-2h\zeta+h^2)^{-1/2}=\textstyle\sum_{l=0}^{\infty} h^l P_l(\zeta) \quad \text{for} \quad |h|<1, \qquad (4.12)$$

we obtain

$$\sum_{l=0}^{\infty} h^l \xi_l(\nu') = -\nu'^{1/2}\frac{\partial}{\partial\nu'}\int_{-1}^1 d\zeta\,[(1-2h\zeta+h^2)(\nu'+2-2\zeta)]^{-1/2} = \frac{(\nu'+4)^{1/2}(1-h)-\nu'^{1/2}(1+h)}{2(\nu'+4)^{1/2}[(1-h)^2-h\nu']}. \qquad (4.13)$$

On the other hand, (4.9) leads to

$$\tilde{\xi}_l(\nu')=\frac{[\nu'+2-\nu'^{1/2}(\nu'+4)^{1/2}]^l[(\nu'+4)^{1/2}-\nu'^{1/2}]}{2^{l+1}(\nu'+4)^{1/2}}. \qquad (4.14)$$

Hence

$$\sum_{l=0}^{\infty} h^l \tilde{\xi}_l(\nu') = \frac{(\nu'+4)^{1/2}-\nu'^{1/2}}{2(\nu'+4)^{1/2}\{1-\frac{1}{2}h[\nu'+2-\nu'^{1/2}(\nu'+4)^{1/2}]\}}. \qquad (4.15)$$

We can easily check that (4.15) is identical with (4.13) by rationalizing the denominator of (4.15). Thus we have established the equality (4.2). Hence, in general, both traces of the nth iterated kernels coincide with each other because they are equal to $\sum_i \lambda_{li}^{-n}$.

V. FREDHOLM SOLUTIONS

In this section we shall present Fredholm solutions of (3.18) according to Sato's method.[6] The angular momentum l may now be complex. Such analytic continuation is unique if one requires that $\varphi_l^{[0]}(z,\alpha)$ vanishes for $l\to\infty$ in all directions in the right half-plane. We assume $\mu\neq 0$.

We put

$$a_l\equiv\frac{\lambda}{2}\int_{-1}^1 dz'\int_0^\infty d\alpha'\int_0^1 dx\,\frac{x^{l-1}}{g(\alpha',z';x)}\varphi_l^{[0]}(z',\alpha'). \qquad (5.1)$$

For continuation we replace x^l by Pf. x^l if necessary.[12] Our Eq. (3.18) is now rewritten as

$$\varphi_l^{[0]}(z,\alpha)=a_l\delta(\alpha)-\lambda\int_{-1}^1 dz' \times\int_0^\infty d\alpha'\,H_l(z,\alpha;z',\alpha')\varphi_l^{[0]}(z',\alpha'), \qquad (5.2)$$

with

$$H_l(z,\alpha;z',\alpha') \equiv\frac{R(z,z')}{2\alpha}\int_0^1 dx\,x^{l-1}\delta(\alpha-R(z,z')g(\alpha',z',x)). \qquad (5.3)$$

Carrying out the integration, we have

$$H_l(z,\alpha;z',\alpha')\equiv\frac{\omega\theta(\gamma-\alpha'-\mu^2-2\mu(\alpha'+\rho)^{1/2})}{2\alpha[(\gamma-\alpha'-\mu^2)^2-4\mu^2(\alpha'+\rho)]^{1/2}}. \qquad (5.4)$$

Here

$$\gamma\equiv\alpha/R(z,z'),$$
$$\rho\equiv\rho(z'),$$
$$\omega\equiv x_1^l(1-x_1)+x_2^l(1-x_2), \qquad (5.5)$$

and x_1 and x_2 are two roots of an equation

$$(\gamma+\rho)x^2+(\mu^2-\gamma-\alpha'-2\rho)x+\alpha'+\rho=0 \qquad (5.6)$$

and satisfy

$$0<x_1\leqq x_2<1. \qquad (5.7)$$

Hence for $\mathrm{Re}\,l\geqq 0$ we have

$$|\omega|<2, \qquad (5.8)$$

and for $\mathrm{Re}\,l<0$

$$|\omega|<2|(\gamma+\rho)/(\alpha'+\rho)|^{-\mathrm{Re}\,l} \qquad (5.9)$$

on account of $|x_1|>|x_1x_2|$. In a quite analogous way to Sato's calculation we can prove that the second iterated kernel $H_l^2(z,\alpha;z',\alpha')$ can be transformed into a bounded kernel belonging to a finite region if[13]

$$\mathrm{Re}\,l>-\tfrac{1}{4}. \qquad (5.10)$$

The resolvent is, therefore, given by a quotient

$$\lambda^2 D(z,\alpha;z',\alpha';l;\lambda^2)/D_0(l;\lambda^2), \qquad (5.11)$$

and the analyticity of D and D_0 is determined by iterated kernels. When $a_l\neq 0$, the solutions are written as

$$\varphi_l^{[0]}(z,\alpha)=a_l\int_{-1}^1 dz'\int_{0-}^\infty d\alpha'\,[\delta(z-z')\delta(\alpha-\alpha') +\lambda^2 D(z,\alpha;z',\alpha';l;\lambda^2)/D_0(l;\lambda^2)] \times\left[\delta(\alpha')-\lambda\int_{-1}^1 dz''\,H_l(z',\alpha';z'',0)\right]. \qquad (5.12)$$

[12] L. Schwartz, *Théorie des distributions* (Hermann and Cie, Paris, 1950), Chap. II. Pf. x^l equals x^l $\mathrm{Re}\,l>-1$, and for $\mathrm{Re}\,l\leqq -1$ it means to take an appropriate finite part when integrated over x.

[13] The number $-\frac{1}{4}$ comes out essentially from the factor $\alpha^{-5/4}$ in the inequality (A2) of Sato's paper (in his notation t is used instead of α).

If $a_l=0$, we must have $D_0(l;\lambda^2)=0$ in order that (5.2) has solutions. Hence, if a_l is chosen so as to be proportional to $D_0(l;\lambda^2)$, (5.12) can be used in any case. Thus, $\varphi_l^{[0]}$ is bounded everywhere except for $\alpha=0$ and holomorphic with respect to l in the domain (5.10). From (3.1) we see that the Bethe-Salpeter wave function $f(p)$ is holomorphic in

$$(s,t)\epsilon D^a(2),\quad \mathrm{Re}l>-\tfrac{1}{4}, \tag{5.13}$$

apart from the solid harmonic and a normalization constant, where $D^a(2)$ is a notation defined in reference 10.

Now, the eigenvalues of λ are determined by substituting (5.12) in (5.1). If λ is fixed and l is considered as a variable, then this equation gives the so-called Regge trajectories.[8] When λ is small, as the lowest order approximation to the Regge trajectory of the ground state we have

$$1=\lambda\tilde{\sigma}_l(u), \tag{5.14}$$

where $\tilde{\sigma}_l(u)$ is given by (3.22). This equation is essentially equivalent to the approximate trajectory equation of Bertocchi, Fubini, and Tonin.[14] For real l, (5.14) implies

$$dl/du>0, \tag{5.15}$$

where, of course, $u<(m_1+m_2)^2$ is assumed. Finally, we remark that (5.14) is also identical with the approximate equation given by Lee and Sawyer[9] because of the identity (4.2).

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APPENDIX

Here we investigate the asymptotic behavior of $\varphi_l^{[n]}(z,\alpha)$. From the result given in Sec. V, $\varphi_l^{[n]}(z,\alpha)$ will not contain such a factor as $[\ln\alpha]^{-1}$ asymptotically. Accordingly, we may understand that

$$\lim_{\alpha\to\infty}\alpha^{-m}\varphi_l^{[n]}(z,\alpha)=0 \tag{A1}$$

is equivalent to the statement that there exist positive numbers α_0, M, and ϵ ($<\frac{1}{2}$) such that

$$|\varphi_l^{[n]}(z,\alpha)|<M\alpha^{m-\epsilon}\quad\text{for}\quad\alpha>\alpha_0. \tag{A2}$$

First, we will show

$$\lim_{\alpha\to\infty}\varphi_l^{[0]}(z,\alpha)=0 \tag{A3}$$

under the assumption [see (3.5)]

$$|\varphi_l^{[0]}(z,\alpha)|<M\alpha^{1-2\epsilon}\quad\text{for}\quad\alpha>\alpha_0. \tag{A4}$$

For $\alpha>0$, by (5.2) together with (5.4) we have

$$|\varphi_l^{[0]}(z,\alpha)|\leqq\lambda\alpha^{-\epsilon}\int_{-1}^{1}\frac{dz'}{R^{1-\epsilon}}\int_0^\infty d\alpha'\frac{|\varphi_l^{[0]}(z',\alpha')|}{(\alpha'+\mu^2)^{1-\epsilon}}\times\frac{\theta(\gamma-\alpha'-\mu^2-2\mu(\alpha'+\rho)^{1/2})}{[(\gamma-\alpha'-\mu^2)^2-4\mu^2(\alpha'+\rho)]^{1/2}}, \tag{A5}$$

where $R\equiv R(z,z')$. Because of (A4) the integrals are bounded, and hence $\varphi_l^{[0]}(z,\alpha)$ vanishes as $\alpha\to\infty$.[15] From (A3) and (3.20) we obtain (3.15), i.e.,[16]

$$|\varphi_l^{[n]}(z,\alpha)|<M\alpha^{n-2\epsilon}\quad\text{for}\quad\alpha>\alpha_0. \tag{A6}$$

Now, we will prove (3.23). Starting from (A6), we shall inductively show that

$$|\alpha^{-k}\varphi_l^{[n]}(z,\alpha)|<M\alpha^{-2\epsilon}\ \text{for}\ \alpha>\alpha_0,\ 0\leqq k\leqq n\leqq l+1. \tag{A7}$$

Namely, we shall show that (A7) is valid for $k=m-1$ if it holds for $k=m\geqq1$.

For $\alpha>0$, (3.12) together with (3.13) leads to

$$|\alpha^{-m+1}\varphi_l^{[n]}(z,\alpha)|\leqq I_1(z,\alpha)+I_2(z,\alpha), \tag{A8}$$

where

$$I_1(z,\alpha)\equiv\frac{n\lambda}{2}\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha'\,|\varphi_l^{[n]}(z',\alpha')|\times\int_0^1 dx\,x^{l-n-1}g^{-n-1}\alpha^{n-m}\theta(g-\alpha),$$

$$I_2(z,\alpha)\equiv\frac{\lambda}{2}\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha'\,|\varphi_l^{[n]}(z',\alpha')|\times\int_0^1 dx\,x^{l-n-1}g^{-n-1}\alpha^{n-m+1}\delta(\alpha-Rg), \tag{A9}$$

with $g\equiv g(\alpha',z',x)$. Since

$$g^{-n-1}\alpha^{n-m}\theta(g-\alpha)\leqq g^{-m-1+\epsilon}\alpha^{-\epsilon}\theta(g-\alpha)\leqq x^{m+1-\epsilon}\alpha'^{-m-1+\epsilon}\alpha^{-\epsilon}, \tag{A10}$$

we have

$$I_1(z,\alpha)\leqq\frac{n\lambda}{2}\alpha^{-\epsilon}\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha'\,|\varphi_l^{[n]}(z',\alpha')|\alpha'^{-m-1+\epsilon}\times\int_0^1 dx\,x^{l+m-n-\epsilon}. \tag{A11}$$

Because of

$$l+m\geqq n, \tag{A12}$$

and the assumption (A7) with $k=m$, the integrals in (A11) are convergent and, hence, $I_1(z,\alpha)\to0$ as $\alpha\to\infty$.

[14] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962). See Eq. (4.13) of their paper [corresponding notations are as follows: $u\to t$, $m_1\to\mu$, $m_2\to\mu$, $\mu^2\to s_0$, $l\to\alpha(t)$].

[15] From (A5) we have only

$$|\varphi_l^{[0]}(z,\alpha)|\leqq M\alpha^{-\epsilon}/(1-z^2)^{1-\epsilon}\quad\text{for}\quad\alpha>\alpha_0,$$

but the singularities at $z=\pm1$ are spurious. Indeed, if we substitute the above inequality in (5.2), we get the desired result. But the above inequality is already sufficient for our applications.

[16] Of course, α_0, M, and ϵ may be different from those in (A4).

Likewise, since

$$g^{-n-1}\alpha^{n-m+1}\delta(\alpha-Rg) =\alpha^{1-\epsilon}g^{-m-1+\epsilon}R^{n-m+\epsilon}\delta(\alpha-Rg) \leqq\alpha^{1-\epsilon}x^{m+1-\epsilon}\alpha'^{-m-1+\epsilon}\delta(\alpha-Rg), \quad (A13)$$

we obtain

$$I_2(z,\alpha)\leqq\frac{\lambda}{2}\alpha^{-\epsilon}\int_{-1}^{1}dz'\int_{0-}^{\infty}d\alpha'\,|\varphi_l^{[n]}(z',\alpha')|\alpha'^{-m-1+\epsilon} \times J(z,\alpha;z',\alpha') \quad (A14)$$

with

$$J(z,\alpha;z',\alpha')\equiv\int_0^1 dx\,x^{l+m-n-\epsilon}\alpha\delta(\alpha-Rg) \leqq\frac{2\gamma\theta(\gamma-\alpha'-\mu^2-2\mu(\alpha'+\rho)^{1/2})}{[(\gamma-\alpha'-\mu^2)^2-4\mu^2(\alpha'+\rho)]^{1/2}} \quad (A15)$$

on account of (A12). Therefore, the α' integral is finite as $\alpha\rightarrow\infty$, and hence $I_2(z,\alpha)\rightarrow 0$. Thus we have established (A7).

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Backscatter from Inhomogeneous Media*

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A WKB approximation is used to calculate cross sections for the 180° scattering of scalar and vector waves by a class of spherically symmetric, repulsive potentials. These potentials are such that the corresponding index of refraction has a unique zero. The scalar problem is discussed in the framework of quantum mechanics, and the result is just the classical cross section. Electromagnetic backscatter from a dielectric is found to be three-quarters of the scalar approximation in the extreme geometrical-optics limit.

I. INTRODUCTION

INTEREST in radar cross sections has encouraged investigations on the backscatter of waves from inhomogeneous media. In general, this is a difficult problem to analyze. Exact solutions are rare, and the Born approximation[1] is worthless when the index of refraction differs significantly from unity. The Schiff approximation[2] is expected to have a wider range of validity, but its usefulness hinges on the evaluation of a difficult volume integral. In this paper, we consider the simplest spherically symmetric systems to which a "semiclassical" approximation is applicable. Specifically, the index of refraction of such a system is a continuous function of r, and it has a unique zero at r_0.

The scalar-wave problem is studied by investigating the equivalent problem of electron backscatter from repulsive potentials. The correspondence principle is derived for 180° scattering; that is, a WKB scattering amplitude is obtained which gives the correct classical cross section. The classical result is shown to have an upper limit of $\frac{1}{4}r_0^2$. In addition, the inverse square-law potential is examined in some detail, for the phase shifts are known exactly, and corrections to the classical result can be derived.

It is known[3,4] that the problem of electromagnetic scattering from a spherically symmetric dielectric is reducible to the solution of two scalar problems; i.e., two radial differential equations must be solved for two sets of phase shifts. For our purpose, the amplitude for vector backscatter is proportional to the difference of the corresponding scalar amplitudes. While difficulties arise because of the zero in the index of refraction, these scalar amplitudes can be replaced by WKB approximations analogous to the one introduced earlier. This approximation is valid in the extreme geometrical optics limit. Here expressions simplify, with the differential cross section for electromagnetic backscatter reducing to three-quarters of the result predicted on the basis of the scalar wave equation.

II. THE SCALAR PROBLEM

The time-independent scalar wave equation is

$$[\nabla^2+k^2n^2(r)]\psi(\mathbf{r})=0, \quad (2.1)$$

where $n(r)$ is the (spherically symmetric) index of refraction of the medium, and $2\pi/k$ is the wavelength of the incident wave. The asymptotic scattering solution of Eq. (2.1) is

$$\psi(\mathbf{r})\underset{r\rightarrow\infty}{\longrightarrow}e^{i\mathbf{k}\cdot\mathbf{r}}+\frac{e^{ikr}}{r}f(\theta), \quad |\mathbf{k}|=k, \quad (2.2)$$

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