Electromagnetic Corrections to Weak Interactions. The Beta Decays of the Muon, Neutron, and O^{14}†

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(Received 26 November 1962)

The momentum-dependent radiative corrections to the beta decays of the muon, the neutron, and O^{14} have been calculated to order α using the techniques of dispersion theory. The transition matrix elements can be expressed to this order (neglecting some effects of strong interactions) in terms of sets of vertex functions which satisfy once-subtracted dispersion relations. The absorptive parts of the vertex functions can be expressed to the appropriate order in terms of the vertex functions themselves and the amplitudes for electromagnetic scattering of the charged particles. It is a curious feature of the present calculation that the choice of the subtraction points is not arbitrary, but is determined uniquely by the requirement that such physically significant quantities as decay rates and the momentum spectra of the leptons should contain no infrared divergences when calculated including the contributions of processes in which soft photons are emitted [inner bremsstrahlung]. The subtraction constants play the role of renormalized weak coupling constants. The significance of this electromagnetic renormalization, and the connection between the choice of the subtraction point and the infrared divergence is examined in detail in the case of the muon. Two models for the beta decay of O^{14} have been considered. In one model, the nucleon involved in the transition is treated as a free particle insofar as the calculation of radiative corrections is concerned; in the other, the O^{14} and N^{14*} nuclei are treated as point particles, and the effects of the nuclear structure are ignored. The results obtained from the two models differ only slightly. Because of the appearance in the absorptive parts of the vertex functions of the form factors of the charged particles, evaluated for the particles on the mass shell, we are able to study analytically the effects on the transition amplitude of the finite *electromagnetic* structure of the nuclei. The effects of the finite spacial distribution of the decaying matter are treated using the usual multipole expansion of the nuclear matrix element. The leading electromagnetic structure correction is of the same form as the familiar $Z\alpha RW$ in the correction for finite nuclear structure (finite deBroglie wavelength effect), but is of a different origin, and leads to a near doubling of the total structure corrections. The known theoretical corrections to the decay rates for the $0+\rightarrow 0+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$ are summarized. Using the recent, very accurate data on the decays of the muon, O^{14} and Al^{26*}, we obtain the values $G_\mu=(1.436\pm0.001)\times10^{-49}$ erg cm^3, $G_\beta(O^{14})=(1.419\pm0.002)\times10^{-49}$ erg cm^3, and $G_\beta(Al^{26*})=(1.430\pm0.002)\times10^{-49}$ erg cm^3 for the *renormalized* vector coupling constants for these transitions. The less accurate data on the neutron yield $G_V=(1.356\pm0.068)\times10^{-49}$ erg cm^3. The results for G_μ and G_β are not directly comparable in the present theory, but the different values of G_β should be. The discrepancy of $(0.8\pm0.5)\%$ between the effective coupling constants for the decays of O^{14} and Al^{26*} could, therefore, be significant, and may yield information about the still uncertain Coulomb corrections to the nuclear matrix elements. If the validity of the cutoff-dependent results of perturbation theory is assumed, the renormalization constants can be evaluated, and one obtains $G_{\mu,\mathrm{bare}}=(1.431\pm0.001)\times10^{-49}$ erg cm^3 and $G_{\beta,\mathrm{bare}}(O^{14})=(1.404\pm0.002\pm0.007)\times10^{-49}$ erg cm^3. Those coupling constants differ by $1.9\pm0.2\%$, but because of uncertainties regarding the nuclear matrix element for O^{14}, the effects of strong interactions, and the possible existence of an intermediate vector meson which mediates the weak interactions, a direct comparison of these numbers may not be relevant to the possible universality of the Fermi interaction.

I. INTRODUCTION

THE hypothesis of Feynman and Gell-Mann,[1] that the vector component of the non-strangeness-changing weak interaction current is conserved, implies that the vector coupling constant G is unaffected by strong renormalizations.[2] If the Fermi interaction is universal, G should, therefore, have the same value for nuclear beta decay as for the decay of the μ meson. This conclusion is in reasonable agreement with experiment. Thus, from the recent, very precise data on the $0+\rightarrow 0+$ transition $O^{14}(\beta^+)N^{14*}$,[3] one obtains for G the value[4] $(1.4140\pm0.0022)\times10^{-49}$ erg cm^3; this differs

† Work performed under the auspices of the U. S. Atomic Energy Commission.

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[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); M. Gell-Mann, *ibid.* **111**, 362 (1958).

[2] J. Bernstein, M. Gell-Mann, and L. Michel, Nuovo Cimento **16**, 560 (1960).

[3] R. K. Bardin, C. A. Barnes, W. A. Fowler, and P. A. Seeger, Phys. Rev. **127**, 583 (1962); D. L. Hendrie and J. B. Gerhart, *ibid.* **121**, 846 (1961); J. W. Butler and R. O. Bondelid, *ibid.* **121**, 1770 (1961).

[4] Obtained from the uncorrected value $(1.4164\pm0.0022)\times10^{-49}$ erg cm^3 quoted by Bardin *et al.*, reference 3, by including the electronic and nuclear corrections to the f value given in Table I of the present paper. The value given by Bardin *et al.* is based on an end-point kinetic energy of 1812.6 ± 1.4 keV for the positron in the $O^{14}(\beta^+)N^{14*}$ transition, and a half-life for the decay obtained from a weighted average of the values obtained by Bardin *et al.* (71.00 ± 0.13 sec) and by Hendrie and Gerhart, reference 3 (70.91 ± 0.04 sec). The branching ratio to the ground state of N^{14} was taken as 0.006 ± 0.001 [R. Sherr, J. B. Gerhart, H. Horie, and W. F. Hornyak, Phys. Rev. **100**, 945 (1955)].

by only $1.0\pm0.2\%$ from the value $(1.4282\pm0.0011)\times10^{-49}$ erg cm^3 derived from the muon lifetime.[5] Since the weak current is not conserved in the presence of electromagnetic interactions, it is plausible to attribute the small discrepancy to electromagnetic corrections to the decay rates. These are of two types, the "radiative" corrections which are present for the decay of an isolated particle, and the nuclear electromagnetic corrections which involve, for example, the effects of the Coulomb field on the nuclear matrix element in the $O^{14}(\beta^+)N^{14*}$ transition. We are concerned in this paper only with the radiative corrections. These have been calculated in perturbation theory by a number of authors.[6–10] The results are finite in the case of the muon, and lead to a small change in the predicted decay rate,[8] $(\Delta\Gamma/\Gamma)_\mu = -0.0042$. However, the corrections obtained in the case of nuclear beta decay are ultraviolet divergent, and it is necessary to introduce a cutoff into the theory. The cutoff-dependent term in $(\Delta\Gamma/\Gamma)_\beta$, $(3\alpha/2\pi)\ln(\Lambda^2/m_pm_e)$, has customarily been treated by choosing Λ equal to the mass of the proton in the expectation that the proper inclusion of nuclear electromagnetic form factors would cut off the divergent contributions of virtual photons at momenta in this region. Berman and Sirlin[9] have recently investigated this assumption in detail, demonstrating that such a "natural" cutoff will indeed be present if the four form factors for a proton off the mass shell decrease sufficiently rapidly for infinite momentum transfers and infinite effective masses. Unfortunately, nothing is at present known about the off-mass-shell behavior of the form factors, and the validity of the cutoff theory is unclear. If it is nevertheless assumed that a cutoff $\Lambda\sim m_p$ is reasonable, the leading term reduces to $(3\alpha/2\pi)\ln(m_p/m_e)=0.026$; this represents the greater part of the electromagnetic correction to the decay of O^{14}, $(\Delta\Gamma/\Gamma)_\beta\sim0.017$.[8,9] When the radiative corrections are incorporated into the analysis, the discrepancy between the values of G derived from the decay rates of the muon and O^{14} is *increased*, to $2.0\pm0.2\%$ [$G_\mu=(1.4312\pm0.0022)\times10^{-49}$ erg cm^3, $G_\beta(O^{14})=(1.4020\pm0.0022)\times10^{-49}$ erg cm^3].[3,9] This discrepancy is perhaps not too serious: it may be reduced by corrections to the nuclear matrix element for the $O^{14}(\beta^+)N^{14*}$ associated with the Coulomb field of the nucleus,[11–14] and would be removed altogether should the weak interactions be mediated by a vector meson with a mass near that of the K meson.[15–17] It may, nevertheless, be of interest to examine the radiative corrections to the weak interactions from a different point of view. Such an attempt has been made by the authors using the techniques of dispersion relations; preliminary results were reported elsewhere.[18,19]

In the present paper, we wish to present the details of our previous work, as well as some additional results obtained very recently. The basic procedures are quite simple, and consist in the calculation of certain vertex functions for the weak interactions using dispersion relations. The absorptive parts of the vertex functions can be expressed, correct to order α, in terms of the weak vertex itself, and the amplitude for electromagnetic scattering of the charged particles. Not unexpectedly, it is found that the dispersion relations for those vertex functions which have Born terms require a subtraction, the subtraction constants playing the role of renormalized coupling constants. It is a curious feature of the present calculation that the choice of the subtraction point is not arbitrary, but is determined uniquely by the requirement that such physically meaningful quantities as decay rates and the momentum spectra of the leptons should contain no infrared divergences when calculated including the contributions of processes in which soft photons are emitted (inner bremsstrahlung). This point, which is closely connected with the appearance in the theory of renormalized rather than bare coupling constants, is examined in detail in the case of the muon. Because our results for the decay rates and the lepton spectra are perforce expressed in terms of the renormalized rather than the bare coupling constants, we are unable to discuss directly the universality of the Fermi interaction. This concept in its usual form requires that the bare coupling constants in the weak interaction Lagrangian be the

[5] Bardin *et al.*, reference 3, quote an uncorrected value of $(1.4282\pm0.0011)\times10^{-49}$ erg cm^3 for the weak coupling constant in the decay of the muon. This assumes a $V-A$ theory for the decay. The muon lifetime, taken as 2.210 ± 0.003 μsec, was obtained by averaging the result of R. A. Reiter, T. A. Romanowski, R. B. Sutton, and B. G. Chidley, Phys. Rev. Letters **5**, 22 (1960) [$\tau_\mu=2.211\pm0.003$ μsec] with that of V. L. Telegdi, R. A. Swanson, R. A. Lundy, and D. D. Yovanovitch, quoted by Reiter *et al.* [$\tau_\mu=2.208\pm0.004$ μsec]. The mass of the muon was taken as 206.77 m_e: J. Lathrop, R. A. Lundy, S. Penman, V. L. Telegdi, R. Winston, D. D. Yovanovitch, and A. J. Bearden, Nuovo Cimento **17**, 114 (1960) [$m_\mu=(206.76\pm0.03)m_e$]; S. Devons, G. Gidal, L. M. Lederman, and G. Shapiro, Phys. Rev. Letters **5**, 330 (1960) [$m_\mu=(206.78\pm0.03)m_e$]; G. Charpak, F. J. M. Farley, R. L. Garwin, T. Muller, J. C. Sens, V. L. Telegdi, and A. Zichichi, Phys. Rev. Letters **6**, 128 (1961) [$m_\mu=(206.77\pm0.01)m_e$].

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[17] R. A. Shaffer, Phys. Rev. **128**, 1452 (1962).

[18] L. Durand, III, L. F. Landovitz, and R. B. Marr, Phys. Rev. Letters **4**, 620 (1960). The difference between the dispersion and perturbation theoretic results for the muon spectrum is incorrect as given in Eq. (8) of the Letter; the correct result is given in Eq. (40) of the present paper. Table I of the Letter is also superseded by Table I of the present paper.

[19] Similar calculations have since been performed by Dr. Clifford Schumacher (private communication from Dr. Schumacher).

same for the muon and the neutron. Since there is no reason to expect the electromagnetic renormalizations to be equal for these particles, our results, as such, shed no light on the problem. Certain advantages are nevertheless to be found in this approach. It leads first to a clear separation of those electromagnetic corrections to the transition amplitudes which can be ascribed to a charge renormalization, hence, affect the decay rates and lepton spectra only by a scale factor, and the remaining momentum-dependent corrections which affect the spectral shape. The former do not appear in our results. The latter are present whichever mode of calculation is employed, and lead to significant changes in such quantities as the Michel parameter in the decay of the muon.[8] Calculation of the renormalization effects is required only for the discussion of universality, and it is only then that such questions as the behavior of the electromagnetic form factors of the nucleon off the mass shell are encountered. Although the form factors appear in the dispersion calculations in the absorptive parts of the vertex functions, these involve sums over real intermediate states, and one need only know the behavior of the form factors *on* the mass shell. If it is assumed that the form factors satisfy spectral representations, the vertex functions can be calculated analytically, and an expression obtained for the change in the transition amplitude caused by the finite electromagnetic structure of the nucleus. Although the leading term in this expression is of the same form as the familiar $Z\alpha WR$ term in the corrections for the finite nuclear structure of the decaying system[20] (the "finite de Broglie wavelength effect"), the origin of the two corrections is different. The inclusion of the electromagnetic effect leads to a near doubling of the total structure corrections.

The radiative corrections to the decays of the muon, neutron, and O^{14} are considered separately in Secs. II, III, and IV, respectively. In each case, the necessary approximations and the method of calculation are sketched, and complete results are given for the vertex functions, and the electron or positron spectrum in the decay. The results for the vertex functions are compared with those obtained in perturbation theory by Behrends Finkelstein, and Sirlin,[6] the comparison yielding perturbation theoretic values of the renormalization factors G_R/G. The details of the renormalization are examined most carefully in the case of the muon. In the calculation of the beta decay of the neutron, we have included in the electromagnetic corrections contributions which involve the anomalous magnetic moments of the proton and neutron, and which have not previously been considered in detail. The aforementioned electromagnetic structure corrections are important only for the beta decay of O^{14}, and are examined in Sec. IV C. Because of the length of the calculations, the results of each section are summarized at the end of that section; the over-all results are discussed in Sec. V. We would like to call particular attention to Table I, in which we have summarized all the well-established theoretical corrections to the decay rates of the neutron and the $0+\rightarrow 0+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$ of which we are aware.

[20] M. Morita, Phys. Rev. **113**, 1584 (1959); M. E. Rose and C. L. Perry, *ibid.* **90**, 479 (1953); and references contained therein.

II. ELECTROMAGNETIC CORRECTIONS TO THE DECAY OF THE MUON

A. General Formulation

In accordance with the usual V-A theory of the weak interactions, we will assume that the decay of the muon is generated by the weak Lagrangian,

$$\begin{aligned}\mathcal{L}_W(x)&=(G/\sqrt{2})\bar{\psi}_\nu(x)\gamma_\lambda(1+\gamma_5)\psi_\mu(x)\\&\qquad\times\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_\nu(x)+\text{H.c.}\\&=(G/\sqrt{2})\bar{\psi}_\nu(x)\gamma_\lambda(1+\gamma_5)\psi_\nu(x)\\&\qquad\times\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_\mu(x)+\text{H.c.}\end{aligned}\quad(1)$$

In the ensuing discussion, we will use the second form of the Lagrangian, which may be obtained from the first by a Fierz transformation on the spinor indices. This choice has the advantage that, to lowest order in G, but all orders in α, the neutrino covariant enters the transition matrix element for the decay of the muon as a simple factor,

$$\begin{aligned}\langle e\nu\bar{\nu}|S|\mu\rangle&=i\int dx\,\langle e\nu\bar{\nu}|\mathcal{L}_W(x)|\mu\rangle\\&=(2\pi)^4\delta^4(e+\nu+\bar{\nu}-\mu)(16e_0\mu_0\nu_0\bar{\nu}_0)^{-1/2}\\&\qquad\times\bar{u}(\nu)\gamma_\lambda(1+\gamma_5)v(\bar{\nu})F_\lambda(e,\mu).\end{aligned}\quad(2)$$

Electromagnetic corrections to the matrix element appear only in the vertex function $F_\lambda(e,\mu)$,

$$F_\lambda(e,\mu)=\langle e|J_\lambda(0)|\mu\rangle,\quad(3)$$

where

$$J_\lambda(0)=i(G/\sqrt{2})\bar{\psi}_e(0)\gamma_\lambda(1+\gamma_5)\psi_\mu(0).\quad(4)$$

For convenience, we have denoted the four-momenta of the various particles by the particle symbols, and have chosen the covariant normalization

$$\bar{u}(p)\gamma_\alpha u(p)=2ip_\alpha\quad(5)$$

for the free particle spinors. The general structure of the vertex function $F_\lambda(e,\mu)$ is easily determined using the properties of J_λ and the single-particle states under proper Lorentz transformations and the operation of Wigner time inversion,[21]

$$F_\lambda(e,\mu)=\bar{u}(e)\sum_{j=1}^{6}\Gamma_\lambda{}^j A_j(s)u(\mu)$$
$$=i\bar{u}(e)\{\gamma_\lambda[A_1(s)(1+\gamma_5)+A_2(s)(1-\gamma_5)]$$
$$+\sigma_{\lambda\nu}(\mu-e)_\nu[A_3(s)(1+\gamma_5)+A_4(s)(1-\gamma_5)]$$
$$+i(\mu-e)_\lambda[A_5(s)(1+\gamma_5)+A_6(s)(1-\gamma_5)]\}u(\mu). \quad (6)$$

The form factors $A_j(s)$ are functions of a single invariant parameter, conveniently chosen to be

$$s=-(\mu-e)^2=-(\nu+\bar{\nu})^2, \quad (7)$$

and are real for s in the physical region for the decay of the muon, $0\leq s\leq(m_\mu-m_e)^2$. As a consequence of the relation $(\mu-e)_\lambda=(\nu+\bar{\nu})_\lambda$ and the Dirac equations for the free neutrino spinors, the last two terms in $F_\lambda(e,\mu)$ do not contribute to the transition matrix element for the decay of the muon, Eq. (2), and we, therefore, restrict our attention to the first four terms.

The assumption of microscopic causality leads, by the usual heuristic arguments, to the conclusion that the form factors $A_j(s)$ are analytic functions of s in the complex s plane cut along the real axis from $s=(m_\lambda+m_e)^2$ to $s=\infty$, and are real for $s<(m_\lambda+m_e)^2$, s real. One, therefore, expects the form factors to satisfy the simple dispersion relations[22]

$$A_1(s)=A_1(s_0)+\frac{s-s_0}{\pi}\int_{(m_\mu+m_e)^2}^{\infty}\frac{\mathfrak{A}_1(s')ds'}{(s'-s_0)(s'-s)} \quad (8)$$

and

$$A_j(s)=\frac{1}{\pi}\int_{(m_\mu+m_e)^2}^{\infty}\frac{\mathfrak{A}_j(s')ds'}{s'-s},\quad j=2,\cdots 6. \quad (9)$$

where $\mathfrak{A}_j(s)=\mathrm{Im}A_j(s+i\epsilon)$, $\epsilon\rightarrow 0+$. We have made a single subtraction in the dispersion relation for $A_1(s)$; this will be shown to be necessary.

[21] The present notation and choice of vertex functions is somewhat different from that in reference 18. The use of γ_μ, $\sigma_{\mu\nu}(\mu-e)_\nu$, and $(\mu-e)_\mu$ as the covariants multiplying the A_j, while less convenient for the calculation of the electron spectrum and the spin correlation coefficient, has the advantage that the linear combinations (A_1+A_2) and (A_3+A_4) reduce to the familiar electron vertex functions in the equal mass limit, while the combination (A_5+A_6) vanishes. In particular, the new choice eliminates the covariant $(\mu+e)_\mu$ which acts like a current, and leads in the case of the electron to an incorrect (or unusual) definition of the renormalized electric charge. Correspondingly, the change in covariants leads to a redefinition of the renormalized weak charge in the present problem. The authors are not aware of any method by which the ambiguity in the choice of vertex functions can be eliminated. The present functions and those of reference 18 are related by: $a(s)=A_1(s)-m_eA_3(s)-m_\mu A_4(s)$, $b(s)=A_2(s)-m_\mu A_3(s)-m_eA_4(s)$, $c(s)=A_3(s)$, and $d(s)=A_4(s)$.

[22] The validity of the dispersion relations is easily proved for the triangle diagram of perturbation theory using the methods of R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. **111**, 1187 (1958).

B. Calculation of the Form Factors $A_j(s)$

The absorptive parts in Eqs. (8) and (9) may be determined by standard methods. We begin, not with the matrix element in Eq. (3), but with the related matrix element $\langle\bar{\mu}e^{\text{out}}|J_\lambda(0)|0\rangle$, which we write, following Lehmann, Symanzik, and Zimmermann,[23] in the form

$$\langle\bar{\mu}e^{\text{out}}|J_\lambda(0)|0\rangle$$
$$=i\int dx\,\bar{u}(e)e^{-ie\cdot x}\langle\bar{\mu}|\theta(x_0)[f_e(x),J_\lambda(0)]|0\rangle, \quad (10)$$

where

$$f_e(x)=(\gamma\cdot\partial+m_e)\psi_e(x). \quad (11)$$

We have omitted an equal time commutator which affects only the constant term in $A_1(s)$. The absorptive (or imaginary) parts of the functions $A_j(s)$ can be extracted using the observation that, under the operation of Wigner time inversion, $A_j(s)\rightarrow A_j{}^*(s)$, s real. Upon performing the necessary manipulations, the absorptive parts are obtained in terms of the familiar sum over intermediate states,

$$\bar{u}(e)\sum_j\Gamma_\lambda{}^j\mathfrak{A}_j(s)v(\bar{\mu})$$
$$=\tfrac{1}{2}(2\pi)^4\sum_\alpha\delta^4(\bar{\mu}+e-\alpha)\bar{u}(e)$$
$$\times\langle\bar{\mu}|f_e(0)|\alpha^{\text{out}}\rangle\langle\alpha^{\text{out}}|J_\lambda(0)|0\rangle. \quad (12)$$

For convenience, we have chosen the *out* states for our complete set. The fine-structure constant α appears to higher than the first power in the contributions from all intermediate states except those of the form $\bar{\mu}'+e'$. When only these states are retained in the evaluation of the absorptive part, the dispersion relations become a set of coupled integral equations for the form factors $A_j(s)$, accurate to order α, and can be solved by a single iteration. To the required accuracy, the amplitude for $\bar{\mu}-e$ scattering which appears in Eq. (12) may be evaluated in the first Born approximation. Thus,

$$\bar{u}(e)\langle\bar{\mu}|f_e(0)|\bar{\mu}'e'^{\text{out}}\rangle\rightarrow$$
$$e^2\bar{u}(e)\gamma_\beta u(e')\bar{v}(\bar{\mu}')\gamma_\beta v(\bar{\mu})[(e-e')^2+\lambda^2]^{-1}, \quad (13)$$

where we have introduced the usual fictitious photon mass λ in the photon propagator in order to circumvent later difficulties with infrared divergences. The vertex function $\langle\bar{\mu}e'^{\text{out}}|J_\lambda(0)|0\rangle$ is to be replaced by its leading term in powers of α. Since the conventional V-A theory without electromagnetic corrections corresponds to the choice $A_1=$ constant, $A_j=0$, $j=2,\cdots 6$, we write

$$\langle\bar{\mu}'e'^{\text{out}}|J_\lambda(0)|0\rangle\rightarrow iA_1(s_0)\bar{u}(e')\gamma_\lambda v(\mu'). \quad (14)$$

The functions $A_j(s)$, $j=2,\cdots 6$, and the dispersion integral in $A_1(s)$, are then clearly of order α relative to $A_1(s_0)$. Combining these results and denoting $A_1(s_0)$ by A_0 for simplicity, we obtain for the absorptive parts of

[23] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955); **6**, 319 (1957).

the functions $A_j(s)$, correct to order α,

$$\bar{u}(e)\sum_j \Gamma_\lambda{}^j \mathcal{A}_j(s)v(\bar{\mu})$$
$$=\tfrac{1}{2}ie^2A_0(2\pi)^4\sum\delta^4(\bar{\mu}+e-\bar{\mu}'-e')\theta(s-(m_\mu+m_e)^2)$$
$$\times[(e-e')^2+\lambda^2]^{-1}\bar{u}(e)\gamma_\beta u(e')\bar{u}(e')\gamma_\lambda$$
$$\times(1+\gamma_5)v(\bar{\mu}')\bar{v}(\bar{\mu}')\gamma_\beta v(\bar{\mu})$$
$$=iA_0(\alpha/8\pi)(p/\sqrt{s})\theta(s-(m_\mu+m_e)^2)$$
$$\times\int d\Omega'[(e-e')^2+\lambda^2]^{-1}\bar{u}(e)\gamma_\beta(-i\gamma\cdot e'+m_e)\gamma_\lambda$$
$$\times(1+\gamma_5)[-i\gamma\cdot(\bar{\mu}+e-e')-m_\mu]\gamma_\beta v(\bar{\mu}). \quad (15)$$

Here $s=-(\bar{\mu}+e)^2$, and p is the 3-momentum of either intermediate particle in their center-of-mass system,

$$2p\sqrt{s}=[(s-m_\mu{}^2-m_e{}^2)^2-(2m_\mu m_e)^2]^{1/2}. \quad (16)$$

After the remaining integrations over the directions of the electron 3-momentum in the c.m. system are performed, the result can be reduced to the standard form, and the functions $\mathcal{A}_j(s)$ extracted, through the use of the Dirac equations for the spinors $\bar{u}(e)$ and $v(\bar{\mu})$. The procedure is straightforward, and we give only the results of the rather lengthy calculation:

$$\mathcal{A}_1(s)=\tfrac{1}{2}\alpha A_0(2p\sqrt{s})^{-1}(s-m_\mu{}^2-m_e{}^2)\times\left[\ln\frac{4p^2+\lambda^2}{\lambda^2}-\frac{3}{2}\right]\theta(s-(m_\mu+m_e)^2); \quad (17)$$

$$\mathcal{A}_2(s)=\tfrac{1}{2}\alpha A_0m_\mu m_e(2p\sqrt{s})^{-1}\theta(s-(m_\mu+m_e)^2); \quad (18)$$

$$\mathcal{A}_3(s)=-\tfrac{1}{4}\alpha A_0(m_e/s)(2p\sqrt{s})^{-1}\times(s+m_\mu{}^2-m_e{}^2)\theta(s-(m_\mu+m_e)^2); \quad (19)$$

$$\mathcal{A}_4(s)=-\tfrac{1}{4}\alpha A_0(m_\mu/s)(2p\sqrt{s})^{-1}\times(s-m_\mu{}^2+m_e{}^2)\theta(s-(m_\mu+m_e)^2). \quad (20)$$

The form factors A_5 and A_6 do not contribute to the decay of the muon, and the corresponding absorptive parts have not been calculated. We have, furthermore, set λ^2 equal to zero wherever possible.

The form factors $A_j(s)$ are now easily calculated using the functions $\mathcal{A}_j(s)$ in the dispersion relations of Eqs. (8) and (9). We remark in particular that $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$, vanish as s^{-1} for $s\to\infty$; the dispersion integrals for A_2, A_3, and A_4 consequently converge without subtractions. In contrast, $\mathcal{A}_1(s)$ diverges logarithmically for $s\to\infty$,

$$\mathcal{A}_1(s)\to\tfrac{1}{2}\alpha A_0[\ln(s/\lambda^2)-\tfrac{3}{2}],\quad s\gg m_\mu{}^2, \quad (21)$$

and the dispersion relation for A_1 requires the indicated subtraction, at least when the calculation is restricted to terms of order α. This circumstance will be discussed in more detail later [Sec. II D]. The integrations are for the most part straightforward. It is convenient to express the results in terms of a dimensionless parameter t,

$$t=[m_\mu{}^2+m_e{}^2-s]/(2m_\mu m_e)=-(\mu\cdot e)/(m_\mu m_e),$$
$$1\leq t\leq(m_\mu{}^2+m_e{}^2)/(2m_\mu m_e). \quad (22)$$

We then obtain

$$A_1(t)=A_0\{1+\Delta(t,\lambda^2)-\Delta(t_0,\lambda^2)+(3\alpha/4\pi)[tR(t)-t_0R(t_0)]\}, \quad (23)$$

$$A_2(t)=(\alpha/4\pi)A_0R(t), \quad (24)$$

$$A_3(t)=-(\alpha/4\pi)A_0(m_\mu{}^2+m_e{}^2-2m_\mu m_et)^{-1}\times[(m_\mu-m_et)R(t)-m_e\ln(m_\mu/m_e)], \quad (25)$$

$$A_4(t)=-(\alpha/4\pi)A_0(m_\mu{}^2+m_e{}^2-2m_\mu m_et)^{-1}\times[(m_e-m_\mu t)R(t)+m_\mu\ln(m_\mu/m_e)], \quad (26)$$

where

$$R(t)=(t^2-1)^{-1/2}\ln[t+(t^2-1)^{1/2}] \quad (27)$$

and

$$\Delta(t,\lambda^2)=-\frac{\alpha t}{2\pi}\int_1^\infty\frac{dt'}{(t'+t)(t'^2-1)^{1/2}}\times\ln\left[\frac{4m_e{}^2m_\mu{}^2}{\lambda^2}\frac{t'^2-1}{m_\mu{}^2+m_e{}^2+2m_\mu m_et'}\right]. \quad (28)$$

The function $\Delta(t,\lambda^2)$ may be evaluated in terms of the Spence function $L(z)$,[24]

$$L(z)=-\int_0^z\frac{dt}{t}\ln|1-t|. \quad (29)$$

Upon rearranging the large number of Spence functions obtained initially, and dropping terms which are of order (m_e/m_μ) relative to unity in the physical region, one obtains for $\Delta(t,\lambda^2)$ the result

$$\Delta(t,\lambda^2)\to-(\alpha/2\pi)t(t^2-1)^{-1/2}\{\ln(m_e{}^2/\lambda^2)\ln u+(\pi^2/6)+2\ln(u^2-1)\ln u-3\ln^2u-\ln[(m_e/m_\mu)(1+u)]\times\ln[1-(m_e/m_\mu)u]-L(u^{-2})-L((m_e/m_\mu)u)\}, \quad (30)$$

where $u=t+(t^2-1)^{1/2}$, $1\leq u\leq(m_\mu/m_e)$.

C. The Infrared Divergence, Subtractions, and the Electron Spectrum

The electron spectrum in the decay of the muon, including those radiative corrections which arise from the exchange of virtual photons between the muon and electron, is readily calculated correct to order α using Eqs. (2) and (6). It is convenient to extract from the functions $A_j(t)$ the common factor A_0 which plays the role of the weak coupling constant in the present theory,

[24] K. Mitchell, Phil. Mag. **40**, 351 (1949). Our definition of $L(z)$ corresponds to that of Mitchell, and differs in sign from that used in references 6, 8, and 9. We note the following identities, which permit the reduction of the number of Spence functions in the final results, x real,

$$L(x)+L(1-x)=\tfrac{1}{6}\pi^2-\ln|x|\ln|1-x|,$$
$$L(x)+L(x^{-1})=\tfrac{1}{3}\pi^2-\tfrac{1}{2}\ln^2x,\quad x>0,$$
$$=-\tfrac{1}{6}\pi^2-\tfrac{1}{2}\ln^2|x|,\quad x<0,$$
$$L\left(\frac{x-1}{x+1}\right)-L\left(\frac{1-x}{1+x}\right)=-\pi^2/4+\ln x\ln\left(\frac{1-x}{1+x}\right)+L(x)-L(-x),\quad 0\leq x<1,$$
$$L(x)+L(-x)=\tfrac{1}{2}L(x^2).$$

and which is clearly present as a multiplicative factor to any order in α. Thus, we write

$$A_1(t)=A_0[1+a_1(t)], \tag{31}$$

$$A_j(t)=A_0 a_j(t), \quad j=2, 3, 4. \tag{32}$$

With this convention, the electron spectrum is given in the muon rest system by

$$dN_r(x)/dx=(A_0^2/48)\pi^{-3}m_\mu^5x^2[(3-2x)(1+2a_1)-24(m_e/m_\mu)x^{-1}(1-x)a_2-6(1-x)(m_ea_3+m_\mu a_4)], \tag{33}$$

where x is the customary variable $x=2e_0/m_\mu$, $0\lesssim x\lesssim 1$, and terms of order (m_e/m_μ) relative to unity have been omitted. To this must be added the electron spectrum $dN_\gamma(x)/dx$ for the decay accompanied by the emission of an unobserved photon. A consistent treatment of this process correct to order α yields results identical with those of perturbation theory, with the usual weak coupling constant G replaced by $\sqrt{2}A_0$. We shall, therefore, use for $dN_\gamma(x)/dx$ the results of Behrends, Finkelstein, and Sirlin,[6] as corrected by Berman[7] and by Kinoshita and Sirlin.[8]

Each of the partial spectra $dN_r(x)/dx$ and $dN_\gamma(x)/dx$ depends explicitly on the ficitious photon mass λ and diverges logarithmically for $\lambda\rightarrow 0$. It is, of course, expected that the total spectrum $dN/dx=[dN_r/dx+dN_\gamma/dx]$ will be independent of λ, as is the case in perturbation theory. The expected cancellation of the infrared divergent terms does not, in fact, take place for an arbitrary choice of the subtraction point s_0 (or t_0) in the dispersion relation for $A_1(t)$: Upon combining $dN_r(x)$ and $dN_\gamma(x)$, one obtains a term which involves the factor

$$A_0^2\{1+(\alpha/\pi)\ln(m_e^2/\lambda^2)[t_0R(t_0)-1]\}, \tag{34}$$

where $R(t)$ is defined in Eq. (27), and A_0 denotes as before $A_1(t_0,\lambda^2)$. For arbitrary t_0, a finite decay rate in the limit $\lambda\rightarrow 0$ can only be obtained to order α if the subtraction constant $A_1(t_0,\lambda^2)$ contains an infrared divergence sufficient to cancel that which appears explicitly. We note, however, that this unpleasant (and highly ambiguous) situation can be avoided by a judicious choice of t_0. The function $tR(t)$, defined by the dispersion integral

$$tR(t)=t\int_1^\infty \frac{dt'}{(t'+t)(t'^2-1)^{1/2}}, \tag{35}$$

is clearly real for t real, $-1<t<\infty$, and increases monotonically from $-\infty$ for $t\rightarrow -1$ to $+\infty$ for $t\rightarrow +\infty$. For the unique choice of the subtraction point $t_0=+1$, $t_0R(t_0)$ has the value $+1$, and the explicitly infrared divergent term in Eq. (34) vanishes. This circumstance makes it highly attractive to choose $t_0=1$ $[s_0=(m_\mu-m_e)^2]$, and to define the *renormalized* weak coupling constant for the decay of the muon as[25]

$$G_\mu=\sqrt{2}A_1(t)|_{t=1}. \tag{36}$$

The complete spectrum is now easily obtained. In the muon rest system, $t=(e_0/m_e)=\frac{1}{2}(m_\mu/m_e)x$; over almost the entire range, $t\gg 1$. We therefore give the results in this approximation. Thus, choosing $t_0=1$, and noting that

$$\Delta(1,\lambda^2)=-(\alpha/2\pi)[2+\ln(m_e^2/\lambda^2)], \tag{37}$$

we obtain

$$dN(x)/dx=(G_\mu^2/96)\pi^{-3}m_\mu^5x^2\times\{(3-2x)[1-(\alpha/\pi)(\omega-\tfrac{5}{2})]+(\alpha/2\pi)f(x)\}, \tag{38}$$

where $\omega=\ln(m_\mu/m_e)$ and $f(x)$ is a function defined by Kinoshita and Sirlin,[8]

$$f(x)=2(3-2x)\{\omega[\tfrac{3}{2}-2\ln x+2\ln(1-x)]+2L(x)-\tfrac{1}{3}\pi^2-2-2\ln^2x+3\ln x\ln(1-x)+\ln x-(1+x)x^{-1}\ln(1-x)\}+6(1-x)\ln x+\tfrac{1}{3}x^{-2}(1-x)\times[(\omega+\ln x)(5+17x-34x^2)-22x+34x^2]. \tag{39}$$

D. Comparison with Perturbation Theory: Renormalization of the Weak Vertex

The result for the electron spectrum given in Eq. (38) differs from that of Kinoshita and Sirlin[8] by the presence of a term proportional to the statistical spectrum,

$$-(\alpha/\pi)(\omega-\tfrac{5}{2})(G_\mu^2/96)\pi^{-3}m_\mu^5x^2(3-2x), \tag{40}$$

and by the appearance of the renormalized coupling constant G_μ rather than the bare coupling constant G. Thus, while perturbation theory yields for the muon decay rate the result[8]

$$\Gamma_\mu(\mathrm{KS})=(G^2/192)\pi^{-3}m_\mu^5\left[1-\frac{\alpha}{\pi}\left(\frac{\pi^2}{2}-\frac{25}{8}\right)\right]=(G^2/192)\pi^{-3}m_\mu^5[1-0.0042], \tag{41}$$

the present calculation yields

$$\Gamma_\mu=(G_\mu^2/192)\pi^{-3}m_\mu^5\left[1-\frac{\alpha}{\pi}\left(\omega+\frac{\pi^2}{2}-\frac{45}{8}\right)\right]=(G_\mu^2/192)\pi^{-3}m_\mu^5[1-0.0107]. \tag{42}$$

On the other hand, the effective value of the Michel parameter is unchanged to order α from its value in perturbation theory, $\rho_{\mathrm{eff}}=0.708$.[8]

The renormalized coupling constant $G_\mu/\sqrt{2}$ was defined in Eq. (36) as the value of the vertex function $A_1(t)$ at $t=1$ $[s=(m_\mu-m_e)^2]$. If this definition is applied to the vertex function as obtained in perturbation theory by Behrends, Finkelstein, and Sirlin,[6] one obtains

[25] The factor $\sqrt{2}$ is introduced to conform to the usual convention for a universal Fermi interaction.

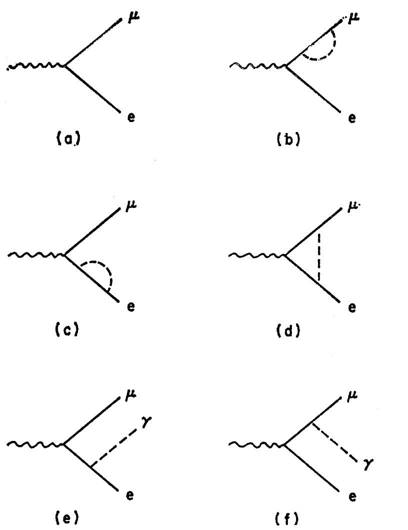

FIG. 1. Feynman diagrams which contribute to the radiative corrections to the decay of the muon.

a relation between the bare and renormalized coupling constants, given to order α in the limit $m_e/m_\mu \ll 1$, by

$$G_\mu/G = 1+(\alpha/2\pi)(\omega-\tfrac{5}{2}) = 1.0032. \quad (43)$$

With this identification, Eqs. (41) and (42) and the corresponding electron spectra are seen to be identical. It is interesting to investigate this relation in more detail.

The vertex function $F_\lambda(e,\mu)$, Eqs. (2)–(4), when calculated to order α in perturbation theory, involves contributions associated with the Feynman diagrams of Figs. 1(a)–(d). The properties of the corresponding matrix elements as functions of the variable $s=-(\mu-e)^2$ are easily derived using the methods of Karplus, Sommerfield, and Wichmann.[22] The diagram of Fig. 1(a) contributes only a constant term $(G/\sqrt{2})$ to the function $A_1(s)$, Eq. (6). This is changed to $(Z_{2e}Z_{2\mu})^{1/2}(G/\sqrt{2})$, with the wave-function renormalization constants calculated to order α, when the contributions of Figs. 1(b) and 1(c) are included. The triangle diagram of Fig. 1(d) contributes to all of the form factors $A_j(s)$, $j=1,\cdots 6$; the contributions are easily seen to be functions of s analytic in the entire finite s plane cut from $s=(m_\mu+m_e)^2$ to $s=\infty$.[22] The functions $A_j(s)$ for $j=2,\cdots 6$ vanish (in order α) at least as rapidly as $s^{-1}\ln s$ for $s\to\infty$ [cf. Eqs. (24)–(26)]. These functions, consequently, satisfy the unsubtracted dispersion relations which were assumed in Eq. (9). On the other hand, the contribution of the triangle diagram to $A_1(s)$ contains a constant term which acts as a vertex renormalization, infinite in Feynman perturbation theory, and additional terms which diverge as $\ln s$ and $\ln^2 s$ for $s\to\infty$. Because of the latter, a subtraction is necessary in the dispersion relation for $A_1(s)$ even if the constant term arising from the diagrams of Figs. 1(a)–1(d) is removed. The connection between the infrared divergence and the choice of the subtraction point is easily seen. $A_1(s)$ is obtained in perturbation theory in the form

$$A_1(s) = \text{const} + (\alpha/2\pi)[1-tR(t)]\ln(m_e^2/\lambda^2)+\cdots, \quad (44)$$

where t and $R(t)$ are defined in Eqs. (22) and (27). Writing a once-subtracted dispersion relation as in Eq. (8), $A_1(s)$ is reproduced in the form

$$A_1(s) = A_1(s_0,\lambda^2) + (\alpha/2\pi)[t_0R(t_0)-tR(t)]\ln(m_e^2/\lambda^2)+\cdots, \quad (45)$$

where $A_1(s_0,\lambda^2)$, which appears as the subtraction constant, is evidently of the form

$$A_1(s_0,\lambda^2) = \text{const}+(\alpha/2\pi) \times[1-t_0R(t_0)]\ln(m_e^2/\lambda^2)+\cdots. \quad (46)$$

However, from the dispersion-theoretic point of view, $A_1(s_0,\lambda^2)$ is a phenomenological parameter: one does not, for example, attempt to calculate the renormalization factors which enter the constant. The choice of the subtraction point, $t_0=1$, $s_0=(m_\mu-m_e)^2$, is then dictated by the requirement that, in the infrared divergent term in Eq. (45), $\ln(m_e^2/\lambda^2)$ appear with the proper coefficient. We, therefore, *define* the renormalized coupling constant G_μ [or more precisely, from the point of view of perturbation theory, we define the electromagnetic renormalization constant Z_{1W} for the weak vertex] by

$$G_\mu/\sqrt{2} = (Z_{2e}Z_{2\mu})^{1/2}Z_{1W}^{-1}(G/\sqrt{2}) = A_1(s_0) \quad (47)$$

with the indicated value of s_0. G_μ is then clearly independent of λ. It may be remarked that, while the Z's are each logarithmically divergent in Feynman perturbation theory, the product $(Z_{2e}Z_{2\mu})^{1/2}Z_{1W}^{-1}=G_\mu/G$ is finite to all orders in α;[9] the value of the ratio G_μ/G is given to order α, but with the neglect of terms in m_e/m_μ, in Eq. (43).[26] It is amusing to note that the same procedures used above, when applied to the electromagnetic vertex function of the electron, lead to the usual subtraction point $s_0=(e-e')^2=0$, and to the relation $Z_1=Z_2$ for the renormalization constants.

An analysis of the data on the lifetime of the muon[5] using Eq. (42) yields the value

$$G_\mu = (1.4358\pm0.0011)\times10^{-49}\ \text{erg cm}^3,$$

for the *renormalized* coupling constant. However, this result is not relevant to the hypothesis of the universality of the Fermi interaction and the conservation of the vector current in beta decay, which require that the *bare* coupling constant G for the decay of the muon be equal to the vector coupling constant G_V in the decay of the neutron. In order to test universality, one is forced, in the present approach, to calculate the renormalization G_μ/G. Using the results of perturbation theory[8] or Eq. (43), one obtains for the *bare* coupling

[26] It is interesting to note that the renormalized coupling constant, hence, the renormalized vertex function, has a logarithmic mass singularity for $m_e\to 0$, m_μ finite, which is not present in the unrenormalized quantities. This has been discussed by T. Kinoshita, J. Math. Phys. 3, 650 (1962), who pointed out this behavior to the authors.

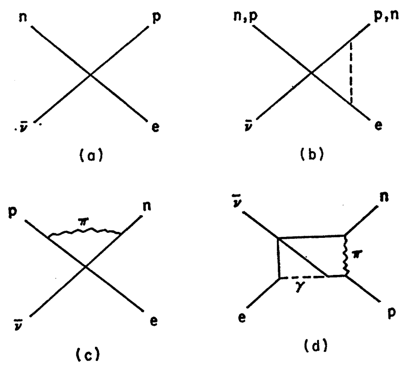

FIG. 2. Representative Feynman diagrams for the radiative corrections to the decay of the neutron.

constant

$$G=(1.4312\pm0.0011)\times10^{-49}\text{ erg cm}^3.$$

The situation is further complicated by the possibility that the weak interactions are mediated by vector bosons.[15] The electromagnetic corrections to the decay of the muon in this theory have recently been examined by Lee[16] using a theory of the electromagnetic interactions of charged vector mesons developed by Lee and Yang,[27] and by Shaffer[17] using a cutoff-dependent theory; but these developments are beyond the scope of the present paper.

III. ELECTROMAGNETIC CORRECTIONS TO THE DECAY OF THE NEUTRON

A. General Formulation of the Problem

The basic interaction Lagrangian for the V-A theory of beta decay has the form

$$\mathcal{L}_W(x)=(G/\sqrt{2})\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_\nu(x)\times\bar{\psi}_p(x)\gamma_\lambda(1+\gamma_5)\psi_n(x)+\text{H.c.},\quad(48)$$

where, with the assumption of a universal Fermi interaction, G is the same bare coupling constant as appears in Eq. (1). The transition matrix element for the decay of the neutron is affected not only by electromagnetic, but also by strong interactions, leading to an extremely complicated situation in which the neutron, proton, and electron can each interact with the others after leaving the weak vertex. A few of the possible diagrams are given in Fig. 2. Topologically, the diagrams are of three types. The first type, Fig. 2(a), corresponds to the renormalized Born approximation, with all particles emerging from a single point. In the second type, represented by Figs. 2(b) and 2(c), two of the particles emerge from single vertex, while the other two particles interact fully. The final type of diagram is typified by Fig. 2(d); in this case, the neutron, proton, and electron all interact after leaving the weak vertex. We concern ourselves only with the vertex type corrections to the basic diagram.

The effects on the beta decay matrix element of the strong interactions have been considered in detail by Goldberger and Treiman.[28] These lead in general to a renormalization of the basic interaction Lagrangian, and to the presence in the beta-decay matrix element of extra-induced pseudoscalar and weak magnetic terms. The influence of the latter, and of the momentum dependence of the matrix elements caused by the extended space-time structure of the complete vertex, is negligible for the decay of the neutron in the absence of electromagnetic corrections. As a seemingly reasonably approximation,[29] we therefore replace the bare Lagrangian of Eq. (48) by a renormalized Lagrangian, but shall ignore the other effects of strong interactions. According to the conserved vector current hypothesis,[1] there is in fact no renormalization of the vector component of the interaction in the absence of electromagnetic interactions.[2] However, such interactions are present, and we must, therefore, expect a renormalization $G\rightarrow G'$, the details of which depend upon both strong and electromagnetic effects. The calculation of this renormalization is beyond the scope of the present paper. On the other hand, the axial vector current is observed experimentally to be renormalized relative to the vector current by a factor $\rho=-(G_A/G_V)=1.25\pm0.06$,[30] with most of the renormalization presumably arising from strong interactions. We, therefore, take for our effective Lagrangian

$$\mathcal{L}_W(x)=(G'/\sqrt{2})\bar{\psi}_e(x)\gamma_\lambda(1+\gamma_5)\psi_\nu(x)\times\bar{\psi}_p(x)\gamma_\lambda(1+\rho'\gamma_5)\psi_n(x)+\text{H.c.},\quad(49)$$

and shall limit the discussion to the electromagnetic corrections to the transition amplitude associated with the electron-proton and electron-neutron interactions. Ignoring strong interactions, it is clear that the corrections associated with the two types of interactions are independent to order α except insofar as both can lead to renormalizations of the Born term, Fig. 2(a).[31] We, therefore, ignore the presence of the electron-neutron interaction for the time being, and discuss the appropriate changes later.

In the foregoing approximation, the transition ampli-

[27] T. D. Lee and C. N. Yang (to be published).

[28] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[29] This approximation should perhaps be checked because of the very large value of the induced pseudoscalar coupling, reference 28.

[30] The value $\rho=1.25\pm0.06$ was derived from the measured value of the neutron spin-electron momentum correlation coefficient, $A=-0.11\pm0.02$ [M. T. Burgy, V. E. Krohn, T. B. Novey, G. R. Ringo, and V. L. Telegdi, Phys. Rev. Letters **1**, 324 (1958)], and is independent of the value of the vector coupling constant. With the assumption of universality and the value of G obtained from the $0+\rightarrow0+$ transitions, one obtains from the half-life of the neutron the slightly smaller value $\rho=1.19\pm0.03$. The situation with respect to the determination of ρ from the decays of the neutron and complex nuclei has been summarized by O. C. Kistner and B. M. Rustad, Phys. Rev. **114**, 1329 (1959).

[31] The lowest order electromagnetic correction which involves both the proton and the neutron has the topology of Fig. 2(d), with the pion replaced by a photon. This diagram is of order α^2 relative to the Born term, Fig. 2(a).

tude for the decay of the neutron is given by

$$\langle pe\bar{\nu}|S|n\rangle = i\int dx\langle pe\bar{\nu}|\mathcal{L}_{W'}(x)|n\rangle$$
$$= i(2\pi)^4\delta^4(p+e+\bar{\nu}-n)(16e_0\bar{\nu}_0p_0n_0)^{-1/2}$$
$$\times[\bar{v}^c(\bar{\nu})(1+\gamma_5)u(n)G(e,p)$$
$$+i\bar{v}^c(\bar{\nu})(1+\gamma_5)\gamma_\lambda u(n)G_\lambda(e,p)]. \quad (50)$$

Here

$$G(e,p) = \langle pe|J(0)|0\rangle \quad (51)$$

with

$$J(0) = (G'/\sqrt{2})(1+\rho')\bar{\psi}_p(0)(1-\gamma_5)\psi_e{}^c(0), \quad (52)$$

while

$$G_\lambda(e,p) = \langle p,e|J_\lambda(0)|0\rangle \quad (53)$$

with

$$J_\lambda(0) = \tfrac{1}{2}i(G'/\sqrt{2})(1-\rho')\bar{\psi}_p(0)\gamma_\lambda(1-\gamma_5)\psi_e{}^c(0). \quad (54)$$

To bring the electron and proton field operators into the same covariants, we have used a Fierz transformation and the formal operation of charge conjugation,

$$\psi_e{}^c(x) = C\bar{\psi}_e{}^T(x), \quad \bar{\psi}_\nu{}^c(x) = -\psi_\nu{}^T(x)C^{-1}, \quad (55)$$

where C is an antisymmetric unitary matrix such that

$$C^{-1}\gamma_\mu C = -\gamma_\mu{}^T. \quad (56)$$

The general structure of the two vertex functions is easily determined. Thus, $G(e,p)$ can be expressed in terms of two form factors,

$$G(e,p) = \bar{u}(p)[G_1(s)(1-\gamma_5)+G_2(s)(1+\gamma_5)]u^c(e), \quad (57)$$

while $G_\lambda(e,p)$ has a form similar to that of the vertex function $F_\lambda(e,\mu)$ for the decay of the muon,

$$G_\lambda(e,p) = \bar{u}(p)\sum_{j=3}^{8}\Gamma_\lambda{}^jG_j(s)u^c(e)$$
$$= i\bar{u}(p)\{\gamma_\lambda[G_3(s)(1-\gamma_5)+G_4(s)(1+\gamma_5)]$$
$$+m_p{}^{-1}\sigma_{\lambda\nu}(p+e)_\nu[G_5(s)(1-\gamma_5)+G_6(s)(1+\gamma_5)]$$
$$+im_p{}^{-1}(p+e)_\lambda[G_7(s)(1-\gamma_5)$$
$$+G_8(s)(1+\gamma_5)]\}u^c(e). \quad (58)$$

The usual V-A theory without electromagnetic corrections, but renormalized with respect to the strong interactions, corresponds to

$$G_1 = (G'/\sqrt{2})(1+\rho'), \quad G_3 = \tfrac{1}{2}(G'/\sqrt{2})(1-\rho'),$$

and $G_j = 0$, $j \neq 1$, 3 [cf. Eq. (51)].

According to the usual heuristic arguments, the eight form factors $G_j(s)$, $j=1, \cdots 8$ are functions of the invariant parameter

$$s = -(e+p)^2, \quad (59)$$

analytic in the complex s plane cut from $s=(m_p+m_e)^2$ to $s=\infty$, and are expected to satisfy dispersion relations of the form

$$G_{1,3}(s) = G_{1,3}(s_0) + \frac{s-s_0}{\pi}\int_{(m_p+m_e)^2}^{\infty}\frac{\mathcal{G}_{1,3}(s')ds'}{(s'-s_0)(s'-s)} \quad (60)$$

and

$$G_j(s) = \frac{1}{\pi}\int_{(m_p+m_e)^2}^{\infty}\frac{\mathcal{G}_j(s')ds'}{s'-s}, \quad j=2, 4, \cdots 8. \quad (61)$$

Here $\mathcal{G}_j(s) = \mathrm{Im}G_j(s+i\epsilon)$, $\epsilon\to 0+$. The physical region for the decay of the neutron corresponds to the range $(m_p+m_e)^2 \leq s \leq m_n{}^2$, with s taken to approach the real axis from above. It may be noted that, as a consequence of the relation $(p+e)_\lambda = (n-\bar{\nu})_\lambda$ and the Dirac equations for the free neutron and neutrino spinors, the functions G_1, G_2 and G_7, G_8 contribute to the decay of the neutron only in the combinations (G_1+G_7) and (G_2+G_8). We, nevertheless, consider the functions separately.

B. Calculation of the Form Factors $G_j(s)$

The absorptive parts $\mathcal{G}_j(s)$ of the functions $G_j(s)$ are given by the usual sums over intermediate states:

$$\bar{u}(p)[\mathcal{G}_1(s)(1-\gamma_5)+\mathcal{G}_2(s)(1+\gamma_5)]u^c(e)$$
$$= \tfrac{1}{2}(2\pi)^4\sum_\alpha \delta^4(p+e-\alpha)\bar{u}(p)$$
$$\times\langle e|f_p(0)|\alpha^{\text{out}}\rangle\langle\alpha^{\text{out}}|J(0)|0\rangle, \quad (62)$$

and

$$\bar{u}(p)\sum_j \Gamma_\lambda{}^j\mathcal{G}_j(s)u^c(e) = \tfrac{1}{2}(2\pi)^4\sum_\alpha \delta^4(p+e-\alpha)\bar{u}(p)$$
$$\times\langle e|f_p(0)|\alpha^{\text{out}}\rangle\langle\alpha^{\text{out}}|J_\lambda(0)|0\rangle. \quad (63)$$

We shall at this point follow the procedure adopted in the case of the muon, and retain only the contributions of order α to the absorptive parts. However, in contrast to that case, such contributions are associated not only with the lightest intermediate state, that of an electron and a proton, but with all intermediate states obtained by adding pions and other strongly interacting particles. The lightest such state involves a single pion, and consequently contributes to the $\mathcal{G}_j(s)$ only for $s>(m_p+m_e+m_\pi)^2$. This range of s is remote from the physical region, $(m_p+m_e)^2\leq s\leq m_n{}^2$, and an examination of the dispersion integrals indicates that the pionic contributions to the $G_j(s)$ are (at least nominally) of order m_e/m_π relative to the leading terms. We, therefore, confine our attention to the single intermediate state which consists of a proton plus an electron. The dispersion relations for the form factors then become a set of coupled integral equations, correct to order α, which can be solved by a single iteration. The amplitude for e-p scattering which appears in Eqs. (62) and (63) is given to the requisite accuracy by the first Born approximation,

$$\bar{u}(p)\langle e|f_p(0)|p'e'^{\text{out}}\rangle \to e^2\bar{u}^c(e')\gamma_\beta u^c(e)$$
$$\times\bar{u}(p)[\gamma_\beta F_{1p}((p'-p)^2)+(\kappa_p/2m_p)F_{2p}((p'-p)^2)$$
$$\times\sigma_{\beta\nu}(p'-p)_\nu]u(p')[(p'-p)^2+\lambda^2]^{-1}, \quad (64)$$

where F_{1p} and F_{2p} are the usual Dirac and Pauli electromagnetic form factors for the proton. The weak vertex functions are replaced by their leading terms,

$$\langle p'e^{\text{out}}|J(0)|0\rangle \to G_1(s_0)\bar{u}(p')(1-\gamma_5)u^c(e'), \quad (65)$$

$$\langle p'e'^{\text{out}}|J_\lambda(0)|0\rangle \to iG_3(s_0)\bar{u}(p')\gamma_\lambda(1-\gamma_5)u^c(e'). \quad (66)$$

The functions $\mathcal{G}_j(s)$ are obtained in this approximation by substituting Eqs. (64)–(66) in Eqs. (62) and (63). The results may be reduced to the standard form, and the $\mathcal{G}_j(s)$ identified, after the indicated spin sum and the integration over the azimuthal direction of the relative momentum in the electron-proton center-of-mass system are performed. The final integration over the polar direction of the relative momentum can also be performed if we introduce the usual integral representations for the proton form factors,[32]

$$F_{1p}(q^2)=1-\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{f_{1p}(x)}{x(x+q^2)}dx, \quad (67)$$

and

$$F_{2p}(q^2)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{f_{2p}(x)}{x+q^2}dx. \quad (68)$$

The resulting expressions are rather complicated. Thus, for $\mathcal{G}_1(s)$ and $\mathcal{G}_2(s)$ we obtain

$$\begin{aligned}\mathcal{G}_1(s)=&-\tfrac{1}{4}\alpha G_1(s_0)(2p\sqrt{s})^{-1}\\ &\times\theta(s-(m_p+m_e)^2)\int_0^{4p^2}dq^2\\ &\times\{[-2(s-M^2)(q^2+\lambda^2)^{-1}+(2p^2s)^{-1}(M^2s-m^4)]\\ &\times F_{1p}(q^2)+2\kappa_pF_{2p}(q^2)\}, \quad (69)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_2(s)=&-\tfrac{1}{4}\alpha G_1(s_0)(2p\sqrt{s})^{-1}(4p^2)^{-1}\\ &\times\theta(s-(m_p+m_e)^2)\int_0^{4p^2}dq^2\\ &\times[4m_em_pF_{1p}(q^2)+4p^2\kappa_p(m_e/m_p)F_{2p}(q^2)], \quad (70)\end{aligned}$$

where $M^2=m_p^2+m_e^2$, $m^2=m_p^2-m_e^2$, and

$$2p\sqrt{s}=[(s-M^2)^2-4m_p^2m_e^2]^{1/2}. \quad (71)$$

These functions have several interesting features which are characteristic also of the $\mathcal{G}_j(s)$ for $j=3,\cdots 8$. We note first the leading term in $\mathcal{G}_1(s)$,

$$\begin{aligned}(s-M^2)(2p\sqrt{s})^{-1}&\int_0^{4p^2}dq^2(q^2+\lambda^2)^{-1}\\ \times F_{1p}(q^2)\to&\int_0^{s}dq^2(q^2+\lambda^2)^{-1}F_{1p}(q^2),\quad s\to\infty. \quad (72)\end{aligned}$$

[32] The assumption of a subtracted dispersion relation for F_{1p} is not necessary, but leads to a convenient separation of structure-independent and structure-dependent terms. The asymptotic form of F_{1p} for $q^2\to\infty$ is not of crucial importance, viz., the remarks in connection with Eq. (72).

A similar term appears in $\mathcal{G}_3(s)$ [and, in the case of the muon, in $\mathcal{A}_1(s)$, but with $F_{1p}(q^2)$ replaced by unity]. For $s\to\infty$, this expression approaches the indicated limiting value, constant and independent of s if $F_{1p}(q^2)$ vanishes sufficiently rapidly for $q^2\to\infty$. In particular, the result does not vanish for $s\to\infty$, barring a purely fortuitous cancellation, and single subtractions are required in the dispersion relations for $G_1(s)$ and $G_3(s)$, Eqs. (60). On the other hand, the Pauli form factor $F_{2p}(q^2)$ plays an essential role in securing the convergence of the dispersion integrals for the remaining functions. This is clearly evident from the second term in Eq. (70),

$$(2p\sqrt{s})^{-1}\int_0^{4p^2}F_{2p}(q^2)dq^2\to s^{-1}\int_0^{s}F_{2p}(q^2)dq^2,\quad s\to\infty. \quad (73)$$

If $F_{2p}(q^2)$ vanishes sufficiently strongly for $q^2\to\infty$, say as $(q^2)^{-\alpha}$, $\alpha>0$, the foregoing expression vanishes for $s\to\infty$, and no subtractions are necessary in the dispersion relation for $G_2(s)$, Eq. (61). At present, it is known that $F_{2p}(q^2)$ decreases rapidly for small values of q^2, and is nearly zero for $q^2\sim 25\text{F}^{-2}$.[33] We shall assume that this rapid convergence toward zero persists for larger values of q^2, and will use as an approximate form for $F_{2p}(q^2)$ the Hofstadter form factor[33]

$$F_{2p}(q^2)\sim[1+(q^2a^2/12)]^{-2},\quad a\sim 0.8-1.0\text{F}. \quad (74)$$

The unsubtracted dispersion integrals for the $G_j(s)$, $j\neq 1$, 3, are then convergent, and, in fact, converge quite strongly. We note finally that the expressions for $\mathcal{G}_1(s)$ and $\mathcal{G}_2(s)$ in Eqs. (69) and (70) involve a number of terms which may be omitted, being of order (m_e/m_p) or higher relative to the leading terms. These are easily identified by changing from s to the dimensionless variable

$$t=-p\cdot e/m_pm_e=(s-m_p^2-m_e^2)/(2m_pm_e). \quad (75)$$

The integrations in Eqs. (60) and (61) cover the range $1\leq t'<\infty$. However, the main contributions to the dispersion integrals arise from values of t' close to the physical region, $1\leq t\leq(m_n^2-m_p^2)/(2m_pm_e)\sim 2.53$. Thus, for example, a factor $(s-M^2)$ is of order $2m_pm_e$ while a factor $4p^2$ is of order $4m_e^2$. From this remark, and the observation that the most important values of the spectral variable x in Eq. (67) are apparently associated with the ρ meson $[x=(750\text{ MeV})^2]$ and the ω meson $[x=(785\text{ MeV})^2]$,[33,34] it is clear that the functions $G_j(s)$ are insensitive to the precise behavior of $F_{1p}(q^2)$. It is consequently a good approximation to replace $F_{1p}(q^2)$ by its value for $q^2=0$, unity; the resulting simplifications in the functions $\mathcal{G}_j(s)$ are considerable. It is also possible

[33] F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. **124**, 1623 (1961), and references contained therein.

[34] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961). B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, *ibid.* **7**, 178 (1961).

to replace $F_{2p}(q^2)$ by unity in those expressions which yield convergent dispersion integrals in this approximation. While the full momentum dependence of $F_{2p}(q^2)$ must be retained elsewhere to obtain convergent results, the resulting contributions to the G_j are for the most part negligible.

On the basis of the foregoing discussion, replacing $F_{1p}(q^2)$ and $F_{2p}(q^2)$ where possible by unity, and omitting terms of order m_e/m_p, we obtain for the absorptive parts of the functions G_j, expressed in terms of the variable t,

$$\mathcal{G}_1(t) \to \tfrac{1}{2}\alpha G_1(s_0)t(t^2-1)^{-1/2}\theta(t-1) \times\{\ln[(4m_e^2/\lambda^2)(t^2-1)]-1\}, \quad (76)$$

$$\mathcal{G}_2(t) \to -\tfrac{1}{2}\alpha G_1(s_0)(t^2-1)^{-1/2}\theta(t-1), \quad (77)$$

$$\mathcal{G}_3(t) \to \tfrac{1}{2}\alpha G_3(s_0)t(t^2-1)^{-1/2}\theta(t-1) \times\{\ln[(m_e^2/\lambda^2)(t^2-1)]-\tfrac{3}{2}+\tfrac{1}{4}\kappa_p\}, \quad (78)$$

$$\mathcal{G}_4(t) \to \tfrac{1}{4}\alpha G_3(s_0)(1+\tfrac{1}{2}\kappa_p)(t^2-1)^{-1/2}\theta(t-1), \quad (79)$$

$$\mathcal{G}_5(t) \to \tfrac{1}{4}\alpha G_3(s_0)(t^2-1)^{-1/2}\theta(t-1) \times\left\{(1-\tfrac{1}{2}\kappa_p)t[1+(2m_e/m_p)t]^{-1} -\kappa_p(2p^2)^{-1}\int_0^{4p^2}(q^2-4p^2)F_{2p}(q^2)dq^2\right\}, \quad (80)$$

$$\mathcal{G}_6(t) \to \tfrac{1}{4}\alpha G_3(s_0)(1-\tfrac{1}{2}\kappa_p)(t^2-1)^{-1/2}\theta(t-1), \quad (81)$$

$$\mathcal{G}_7(t) \to -\tfrac{1}{4}\alpha G_3(s_0)(1-\tfrac{1}{2}\kappa_p)t(t^2-1)^{-1/2} \times[1+(2m_e/m_p)t]^{-1}\theta(t-1), \quad (82)$$

$$\mathcal{G}_8(t) \to \tfrac{1}{4}\alpha G_3(s_0)(3+\tfrac{1}{2}\kappa_p)(t^2-1)^{-1/2}\theta(t-1). \quad (83)$$

The neutron decay rate and the electron spectrum in the decay depend to order α only on the real parts of the functions $G_j(t)$. Using the dispersion relations in Eqs. (60) and (61), these are found to be

$$\mathrm{Re}G_1(t)=G_1(s_0)\{1+D(t,\lambda^2)+(\alpha/2\pi)[tR(t)-1]\}, \quad (84)$$

$$\mathrm{Re}G_2(t)=(\alpha/2\pi)G_1(s_0)R(t), \quad (85)$$

$$\mathrm{Re}G_3(t)=G_3(s_0)\{1+D(t,\lambda^2) +(\alpha/8\pi)(6-\kappa_p)[tR(t)-1]\}, \quad (86)$$

$$\mathrm{Re}G_4(t)=-(\alpha/8\pi)G_3(s_0)(2+\kappa_p)R(t), \quad (87)$$

$$\mathrm{Re}G_5(t)=(\alpha/8\pi)G_3(s_0)\{(2-\kappa_p)[\ln(m_p/m_e)-tR(t)] \times[1+(2m_e/m_p)t]^{-1}+1.36\kappa_p\}, \quad (88)$$

$$\mathrm{Re}G_6(t)=-(\alpha/8\pi)G_3(s_0)(2-\kappa_p)R(t), \quad (89)$$

$$\mathrm{Re}G_7(t)=-(\alpha/8\pi)G_3(s_0)(2-\kappa_p) \times[\ln(m_p/m_e)-tR(t)][1+(m_e/m_p)t]^{-1}, \quad (90)$$

$$\mathrm{Re}G_8(t)=-(\alpha/8\pi)G_3(s_0)(6+\kappa_p)R(t), \quad (91)$$

where $R(t)$ is the function defined in Eq. (27). The constant term $1.36\times\kappa_p$ in $G_5(t)$ arises from the form-factor-dependent term in $\mathcal{G}_5(t)$, the numerical factor corresponding to the use of the Hofstadter form, Eq. (74), for $F_{2p}(q^2)$, choosing $a=0.8$F. The result is valid only for small values of t, $t\ll m_p/m_e$, and does not have the proper limiting behavior for $t\to\infty$. The function $D(t,\lambda^2)$ is given by

$$\begin{aligned}D(t,\lambda^2)&=(\pi\alpha/2)t(t^2-1)^{-1/2}+D'(t,\lambda^2),\\ &=(\pi\alpha/2)t(t^2-1)^{-1/2}\\ &\quad+(\alpha/2\pi)[2+(1-tR(t))\ln(m_e^2/\lambda^2)]\\ &\quad+(\alpha/2\pi)t(t^2-1)^{-1/2}[3\ln^2u-2\ln u\ln(u^2-1)\\ &\quad+L(u^{-2})-(\pi^2/6)].\end{aligned} \quad (92)$$

Here $L(z)$ is again the Spence function,[24] defined in Eq. (29), and $u=t+(t^2-1)^{1/2}$. The subtraction point in the dispersion relations for G_1 and G_3 has been chosen as $t_0=-1$, $s_0=(m_p-m_e)^2$, to insure the explicit cancellation of the λ-dependent terms in the electron spectrum when inner bremsstrahlung processes are included. The arguments are essentially the same as were given in the case of the muon. The first term in $D(t,\lambda^2)$ diverges for $t\to 1$. Since $t=e_0/m_e$ in the proton rest system, this term is of the form $(\pi\alpha/2v_e)$, where v_e is the electron velocity. It may consequently be identified with the term of order α in an expansion of the usual Fermi factor $F^-(Z,t)$ in the beta-decay matrix element, and will henceforth be omitted from the "radiative" corrections.

C. Contributions of the Electron-Neutron Interaction

As noted previously, the electromagnetic interactions between the electron and the neutron also lead to modifications of the beta decay matrix element, which, ignoring the electron-proton interaction, may be written in the form

$$\langle pe\bar\nu|S|n\rangle=i(2\pi)^4\delta^4(p+e+\bar\nu-n)(16e_0\bar\nu_0p_0n_0)^{-1/2} \times[\bar u(p)(1+\gamma_5)v(\bar\nu)G'(e,n) +i\bar u(p)\gamma_\lambda(1+\gamma_5)v(\bar\nu)G_\lambda'(e,n)], \quad (93)$$

where

$$G'(e,n)=\langle e|(G'/\sqrt2)(1+\rho')\bar\psi_e(0)(1-\gamma_5)\psi_n(0)|n\rangle =\bar u(e)[G_1'(\bar s)(1-\gamma_5)+G_2'(\bar s)(1+\gamma_5)]u(n), \quad (94)$$

and

$$\begin{aligned}G_\lambda'(e,n)&=\langle e|i\tfrac{1}{2}(G'/\sqrt2)(1-\rho')\bar\psi_e(0)(1-\gamma_5)\gamma_\lambda\psi_n(0)|n\rangle\\ &=i\bar u(e)\{\gamma_\lambda[G_3'(\bar s)(1+\gamma_5)+G_4'(\bar s)(1-\gamma_5)]\\ &\quad+m_n^{-1}\sigma_{\lambda\nu}(e-n)_\nu[G_5'(\bar s)(1-\gamma_5)+G_6'(\bar s)(1+\gamma_5)]\\ &\quad+im_n^{-1}(e-n)_\lambda[G_7'(\bar s)(1-\gamma_5)\\ &\quad+G_8'(\bar s)(1+\gamma_5)]\}u(n).\end{aligned} \quad (95)$$

The form factors G_j' are functions of the variable

$$\bar s=-(n-e)^2 \quad (96)$$

analytic in the complex $\bar s$-plane cut from $\bar s=(m_n+m_e)^2$ to $\bar s=\infty$, and are assumed to obey dispersion relations of the same type as the functions $G_j(s)$, Eqs. (60) and (61). As was remarked earlier, the functions $G_j(s)$ and $G_j'(\bar{s})$ for $j\neq 1$, 3 are independent. On the other hand, the functions with $j=1$ or 3 multiply the same covariant in the transition amplitude, and involve as a common term the renormalized Born approximation. We will, therefore, replace the functions $G_{1,3}(s)$ and $G_{1,3}'(\bar{s})$ by functions of two variables, $G_{1,3}(s,\bar{s})$ which are of the form

$$G_{1,3}(s,\bar{s})=G_{1,3}(s_0\bar{s}_0)+\hat{G}_{1,3}(s)+\hat{G}_{1,3}'(\bar{s}). \quad (97)$$

The common subtraction constants $G_{1,3}=G_{1,3}(s_0,\bar{s}_0)$ play the role of renormalized coupling constants in the present theory. With this convention, $G_1(s_0)$ and $G_3(s_0)$ are to be replaced in Eqs. (76)–(91) by G_1 and G_3; the functions $\mathrm{Re}\hat{G}_1(s)$ and $\mathrm{Re}\hat{G}_3(s)$ are then given by $[\mathrm{Re}G_{1,3}(s)-G_{1,3}]$.

The calculation of the functions $G_j'(\bar{s})$ differs only in minor details from that outlined above for the $G_j(s)$. In particular, it is necessary to introduce the aforementioned subtractions in $\bar{s}$ in the dispersion relations for $G_1(s,\bar{s})$ and $G_3(s,\bar{s})$ for precisely the same reasons as were mentioned following Eq. (72) in the case of the proton. However, because of the vanishing of the neutron charge, the Dirac form factor $F_{1n}(q^2)$ satisfies a dispersion relation of the form

$$F_{1n}(q^2)=-\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{f_{1n}(x)}{x(x+q^2)}dx, \quad (98)$$

and vanishes linearly with q^2 for $q^2\to 0$. $F_{1n}(q^2)$ is in fact observed experimentally to be much less than unity for values of q^2 less than about 20 F^{-2}.[35] The extra factor of q^2 in Eq. (98) has two significant effects. First, it eliminates from the $\bar{s}$-dependent terms in G_1 and G_3 the infrared divergence characteristic of the s-dependent terms, Eqs. (84) and (86). Correspondingly, the neutron does not contribute to the inner bremsstrahlung processes during the decay $[q^2=0]$, and the choice of the subtraction point $\bar{s}_0$ is not determined by the infrared divergence problem. Secondly, the integrals involving $F_{1n}(q^2)$ vanish at least as rapidly as p^2 for $p^2\to 0$ in all of the $\mathcal{G}_j'(\bar{s})$ [cf. Eq. (69)], and consequently lead to negligible contributions to the $G_j'(\bar{s})$. The only significant effects of the electron-neutron interaction are therefore associated with the anomalous magnetic moment κ_n of the neutron. A straightforward calculation leads to results for the $\mathcal{G}_j'(\bar{s})$ which differ from the corresponding κ_p-dependent parts of the $\mathcal{G}_j(s)$ only in the over-all signs for $j=3$, 4, 5, and 6, and in the replacement of the proton mass and anomalous magnetic moment by the same quantities for the neutron. Using the dispersion relations of Eqs. (60) and (61), one then obtains for the (real) functions G_j', expressed as functions of the variable

$$\bar{t}=-n\cdot e/m_nm_e=(m_n^2+m_e^2-\bar{s})/(2m_nm_e),$$
$$\hat{G}_1'(\bar{t})\sim G_2'(\bar{t})\sim O(m_e/m_n), \quad (99)$$

[35] C. DeVries, R. Hofstadter, and R. Herman, Phys. Rev. Letters **8**, 381 (1962).

$$\hat{G}_3'(\bar{t})=(\alpha/8\pi)G_3\kappa_n[\bar{t}R(\bar{t})-1], \quad (100)$$

$$G_4'(\bar{t})=-G_6'(\bar{t})=-G_8'(\bar{t})=-(\alpha/8\pi)G_3\kappa_nR(\bar{t}), \quad (101)$$

$$G_5'(\bar{t})=(\alpha/8\pi)G_3\kappa_n\{[\ln(m_n/m_e)-\bar{t}R(\bar{t})]\times[1-(2m_e/m_n)\bar{t}]^{-1}+1.36\}, \quad (102)$$

$$G_7'(\bar{t})=(\alpha/8\pi)G_3\kappa_n[\ln(m_n/m_e)-\bar{t}R(\bar{t})]\times[1-(2m_e/m_n)\bar{t}]^{-1}, \quad (103)$$

where $R(t)$ is defined in Eq. (27), and we have used the Hofstadter form factor in Eq. (74) to evaluate the form-factor-dependent term in $G_5'(\bar{t})$. The subtraction point in the dispersion relation for G_1 and G_3 has been chosen as $\bar{t}_0=1$, $\bar{s}_0=(m_n-m_e)^2$. This choice, although not necessary, is suggested by the similar choices $s_0=(m_p-m_e)^2$ and $s_0=(m_\mu-m_e)^2$ which were necessary for the proton terms and in the case of the muon.[36]

D. The Electron Spectrum in the Decay of the Neutron

The transition amplitude for the decay of the neutron may be reduced in the static limit for the nucleons to the form

$$\begin{aligned}\langle pe\bar{\nu}|S|n\rangle &=i(2\pi)^4\delta^4(p+e+\bar{\nu}-n)(16e_0\bar{\nu}_0p_0n_0)^{-1/2}\\ &\times\{\bar{v}^c(\bar{\nu})(1+\gamma_5)u(n)\bar{u}(p)\\ &\times[F_1(1-\gamma_5)+F_2(1+\gamma_5)]u^c(e)\\ &-\bar{v}^c(\bar{\nu})(1+\gamma_5)\gamma_\lambda u(n)\bar{u}(p)\\ &\times[F_3\gamma_\lambda(1-\gamma_5)+F_4\gamma_\lambda(1+\gamma_5)]u^c(e)\}, \quad (104)\end{aligned}$$

where the F's are linear combinations of the G's. A number of the G's in fact cancel out in the sums, and one obtains for $(m_e/m_p)t\ll 1$

$$F_1=(G_V/\sqrt{2})(1+\rho)\{1+D'(t,\lambda^2)+(\alpha/2\pi)[tR(t)-1]\}+i\mathcal{G}_1(t) \quad (105)$$

$$F_2=(\alpha/\pi)(G_V/\sqrt{2})\rho R(t)+i[\mathcal{G}_2(t)+\mathcal{G}_6(t)+\mathcal{G}_8(t)], \quad (106)$$

$$\begin{aligned}F_3=\tfrac{1}{2}(G_V/\sqrt{2})(1-\rho)\{1+D'(t,\lambda^2)+(\alpha/2\pi)[tR(t)-1]\\ -(\alpha/8\pi)(\kappa_p-\kappa_n-2)[\ln(m_p/m_e)-1]\}\\ +i[\hat{\mathcal{G}}_3(t)+\hat{\mathcal{G}}_3'(\bar{t})+\mathcal{G}_5(t)+\mathcal{G}_5'(\bar{t})], \quad (107)\end{aligned}$$

$$F_4=-(\alpha/4\pi)(G_V/\sqrt{2})(1-\rho)R(t)+i[\mathcal{G}_4(t)+\mathcal{G}_6(t)]. \quad (108)$$

[36] The ambiguity in the subtraction point affects only $\hat{G}_3(\bar{t})$ $[\hat{G}_1(\bar{t})\sim O(m_e/m_n)]$, and leads to the replacement of the factor $[\bar{t}R(\bar{t})-1]$ by $[\bar{t}R(\bar{t})-\bar{t}_0R(\bar{t}_0)]$. The change is not large unless the subtraction point is chosen near the inverse square root-type singularity of the second term at $\bar{t}_0=-1$. An alternative choice of subtraction point corresponding to the value $(m_n-m_p)^2$ for the momentum transfer variable $-(n-p)^2$ would also be attractive did it not lead to a value of $\bar{t}_0$ near this singularity. It is possible that the ambiguity would be resolved by the consideration of those higher order terms in the decay amplitude which are infrared divergent and depend on both s and $\bar{s}$. We have not attempted to carry out such an investigation. The analytic properties of the higher order functions are essentially unknown, but it is likely that these functions have complex singularities, hence, do not satisfy double-dispersion relations of any simple kind.

It is interesting to note that those magnetic moment terms which were essentially independent of the behavior of the form factors F_{2p} and F_{2n} for large q^2 appear only in F_3 and then in the isotopic vector combination $(\kappa_p-\kappa_n)$. The form-factor-dependent terms appear in the F's only in the isotopic scalar combination $(\kappa_pF_{2p}+\kappa_nF_{2n})$ which contributes negligibly to the results, and have consequently been omitted. The transition amplitude in Eq. (104) is, therefore, essentially independent of the details of the charge and magnetic moment distributions of the nucleons.

The subtraction constants $G_1(s_0,\bar{s}_0)$ and $G_3(s_0,\bar{s}_0)$, which play the role of renormalized coupling constants in the present theory, have been written in Eqs. (105)-(108), according to the usual conventions, as

$$G_1(s_0,\bar{s}_0)=(G_V/\sqrt{2})(1+\rho), \tag{109}$$

$$G_3(s_0,\bar{s}_0)=\tfrac{1}{2}(G_V/\sqrt{2})(1-\rho), \tag{110}$$

$$s_0=(m_p-m_e)^2,\quad \bar{s}_0=(m_n-m_e)^2. \tag{111}$$

Here G_V is the vector coupling constant, renormalized with respect both to strong and electromagnetic interactions, while $\rho=-G_A/G_V$ gives the relative renormalization of the axial vector and vector couplings.

A brief calculation yields for the electron spectrum in the decay of the neutron, corrected to order α for the exchange of virtual photons, the result

$$\begin{aligned}dN_r(t)=&\tfrac{1}{2}m_e^5\pi^{-3}G_V^2(t_m-t)^2(t^2-1)^{1/2}tF^-(1,t)dt\\ &\times\{(1+3\rho^2)[1+2D'(t,\lambda^2)\\ &+(\alpha/\pi)[(t^2-1)t^{-1}R(t)-1]]\\ &-(\alpha/8\pi)(\rho-1)(3\rho-1)(\kappa_p-\kappa_n-2)\\ &\times[\ln(m_p/m_e)-1]\}.\end{aligned} \tag{112}$$

Here $F^-(Z,t)$ is the usual Fermi factor which corrects for the presence of the attractive Coulomb interaction between the electron and proton. The remaining terms of order α represent the "radiative" corrections. To $dN_r(t)$ must be added the spectrum for the decay accompanied by the emission of photons,

$$\begin{aligned}dN_\gamma(t)=&(\alpha/2\pi)m_e^5\pi^{-3}G_V^2(t_m-t)^2(t^2-1)^{1/2}tF^-(1,t)dt\\ &\times\{[tR(t)-1][\ln(m_e^2/\lambda^2)+2\ln 2(t_m-t)\\ &+\tfrac{2}{3}(t_m-t)t^{-1}-2]+2+(1/12)(t_m-t)^2t^{-1}R(t)\\ &+t(t^2-1)^{-1/2}[3\ln^2u-2\ln u\ln(u^2-1)\\ &+L(u^{-2})-(\pi^2/6)]\},\end{aligned} \tag{113}$$

where, as usual $t=e_0/m_e$, $t_m=(m_n-m_p)/m_e$, and $u=[t+(t^2-1)^{1/2}]$. Because of our choice of the subtraction point s_0, the complete spectrum is independent of the photon mass λ,

$$\begin{aligned}dN(t)=&\tfrac{1}{2}m_e^5\pi^{-3}G_V^2(t_m-t)^2(t^2-1)^{1/2}tF^-(1,t)dt\\ &\times\{(1+3\rho^2)[1+(\alpha/\pi)m(t)]\\ &-(\alpha/8\pi)(\rho-1)(3\rho-1)n(t)\},\end{aligned} \tag{114}$$

where to order m_e/m_p,

$$\begin{aligned}m(t)=&4+t^{-1}R(t)[(1/12)(t_m-t)^2-1]\\ &+[tR(t)-1][2\ln 2(t_m-t)+\tfrac{2}{3}(t_m-t)t^{-1}-1]\\ &+2t(t^2-1)^{-1/2}[3\ln^2u-2\ln u\ln(u^2-1)\\ &+L(u^{-2})-(\pi^2/6)],\end{aligned} \tag{115}$$

and

$$n(t)=(\kappa_p-\kappa_n-2)[\ln(m_p/m_e)-1]. \tag{116}$$

It is useful to note that the variable u is related to the electron velocity v by

$$u^2=(1+v)/(1-v). \tag{117}$$

E. Comparison with Perturbation Theory

The electron-proton vertex functions $G_j(s)$ have been calculated in perturbation theory, neglecting the contributions of the anomalous magnetic moment of the proton, by Behrends, Finkelstein, and Sirlin.[6] Comparison with our results [Eqs. (84)–(91), with $\kappa_p=0$] indicates that the two methods are equivalent for the G_j with $j\neq 1$, 3. On the other hand, the vertex functions G_1 and G_3 differ by constant terms which may be ascribed to renormalization effects. If it is required that the dispersion and the perturbation-theoretic results be equivalent, one obtains the relations

$$\begin{aligned}G_V(1+\rho)=G'(1+\rho')\{1+(\alpha/2\pi)[\tfrac{3}{2}\ln(m_p/m_e)\\ -(5/4)+3\ln(\Lambda/m_p)]\},\end{aligned} \tag{118}$$

and

$$G_V(1-\rho)=G'(1-\rho')\{1+(\alpha/2\pi)[\ln(m_p/m_e)-\tfrac{5}{2}]\}. \tag{119}$$

Here G_V and ρG_V are the completely renormalized coupling constants defined in Eqs. (109) and (110), while G' and $\rho'G'$ are the coupling constants which appear in the effective Lagrangian in Eq. (49). (It should be recalled that G' is *not* equal to the bare coupling constant G which enters the universal Fermi interaction, but differs from that constant by a renormalization arising from the combination of electromagnetic and strong interactions between the neutron and proton.) The renormalization factor for G_3, Eq. (119), is finite and differs as expected from that encountered in the case of the muon only by the replacement of m_μ by m_p. The situation with respect to G_1 is quite different. This vertex function, when calculated in perturbation theory for a point proton, is ultraviolet divergent. Finite results have been secured only through the introduction of an ultraviolet cutoff Λ in the momentum integration. The significance of the perturbation theoretic results for G_1, and of the Λ-dependent renormalization factor in Eq. (118), is consequently not clear. It has been customary to choose Λ equal to the mass of the proton in the expectation that a proper inclusion of the electromagnetic structure of that particle would provide a natural cutoff at momenta in that region.[37] This point has been examined in detail by Berman and Sirlin,[9] who conclude that such a cutoff will in fact be present if the four electromagnetic form factors for a proton off the mass shell vanish for infinite momentum transfers and infinite effective masses. Unfortunately, nothing is as yet known about the off mass shell behavior of the form factors, and the perturbation theoretic results for G_1 remain ambiguous. The authors therefore prefer the results obtained using the dispersion theory approach, which involve no *ad hoc* assumptions about the behavior of the proton form factors off the mass shell, and are expressed in terms of completely renormalized quantities defined by the values of the decay amplitudes at specific (unphysical) points. This approach unfortunately precludes a direct test of the universal Fermi interaction through a comparison of the bare coupling constants in the decays of the neutron and the muon (nor is such a test possible in perturbation theory without a specific assumption about Λ, and a calculation of G'/G and ρ'/ρ), but yields, on the other hand, relatively unambiguous results for the electron spectrum, spin correlation parameters, and the neutron decay rate.[36]

It is clear from the foregoing that the functions $m(t)$ and $n(t)$ which appear in our result for the electron spectrum in the decay of the neutron, Eq. (114), differ from the perturbation theoretic results of Kinoshita and Sirlin[8] ($\rho=1$) and the more recent results of Berman and Sirlin[9] ($\rho\neq1$) only by the presence of the anomalous magnetic moment terms and the formally infinite renormalization effects. However, the resulting expressions for the neutron decay rate are quite different. An analytic calculation by Kinoshita and Sirlin (KS), using the approximations $t_m\gg1$, $\rho=1$, and $\Lambda=m_p$, gave the result[8]

$$\Gamma_n(\mathrm{KS})=\Gamma_0'\{1+(\alpha/2\pi)[3\ln(m_p/2E_m)-2.85]\}$$
$$=\Gamma_0'[1+0.017]. \qquad (120)$$

Here $E_m=m_n-m_p=1.293$ MeV, and Γ_0' is the uncorrected decay rate expressed in terms of G' and ρ',

$$\Gamma_0'=\tfrac{1}{2}m_e^5\pi^{-3}G'^2(1+3\rho'^2)f,\quad f=1.688. \qquad (121)$$

In the same approximation, the dispersion relation treatment of the neutron decay yields a decay rate

$$\Gamma_n=\Gamma_0\{1-(\alpha/2\pi)[3\ln(2E_m/m_e)+0.35]\}$$
$$=\Gamma_0(1-0.0060), \qquad (122)$$

where Γ_0 is given by Eq. (121) with G' and ρ' replaced by G_V and ρ. More precise numerical calculations by Berman and Sirlin including terms of order m_p/Λ omitted in Eq. (120) yielded for the result of perturbation theory $\Gamma_n(\mathrm{KS})=\Gamma_0'(1+0.018\pm0.005)$ for $1.8m_p>\Lambda>0.3m_p$ and $\rho=1.2$. On the other hand, numerical calculations based on Eqs. (114)–(116) show that the correction arising from $m(t)$ decreases Γ_n by 0.546%, while that arising from $n(t)$ decreases Γ_n by 0.043% for $\rho=1.2$, and by 0.053% for $\rho=1.25$.[30] Adopting the latter value of ρ, we obtain for the result of dispersion theory $\Gamma_n=\Gamma_0(1\text{–}0.0060)$. The radiative corrections thus *increase* the decay rate expressed in terms of the partially renormalized quantities G' and ρ' but *decrease* the rate expressed in terms of the completely renormalized quantities G_V and ρ. It is amusing to note that the radiative corrections to the muon and neutron decay rates differ by only 0.47% when these quantities are expressed in terms of the physically significant renormalized coupling constants.

We remark finally on the different manner in which the anomalous magnetic moments of the nucleons affect the radiative corrections in the dispersion- and perturbation-theoretic calculations. In the latter, these introduce additional logarithmic divergences in the electron-proton and electron-neutron vertex functions,[7] and presumably finite terms as well. These contributions to the vertex functions have not been calculated in detail. On the other hand, the magnetic moment contributions to the vertex functions as calculated in dispersion theory are finite, and the dispersion relations for the G_j require no subtractions because of these terms, provided the form factors $F_{2p}(q^2)$ and $F_{2n}(q^2)$ vanish sufficiently strongly for $q^2\to\infty$. This strongly suggests that a proper inclusion of the magnetic structure of the nucleons would eliminate the divergences encountered in perturbation theory, and lead to anomalous magnetic moment contributions to the vertex functions which vanish for s, $\bar{s}\to\infty$. It would be of interest to investigate this point in more detail. The over-all effect of the anomalous magnetic moment terms is rather small in the present treatment of the decay. The moments appear in the spectrum in the large isotopic vector combina-

[37] It could perhaps be argued that an equally plausible choice for Λ is the "natural cutoff" of 100–300 GeV for the weak interactions determined, for example, from the requirement that the cross section for the scattering process $\nu+\bar{\nu}\to\mu+\bar{e}$ not exceed the limit imposed by unitarity. The results would be quite different. Another possibility is to ascribe the cutoff to the pionic structure of the weak vertex. If the vector weak interaction current is assumed to be proportional to the $\pm$ components of the conserved vector isotropic spin current,[1] the pionic structure of the weak vertex must in fact be identical to the structure of the isovector electromagnetic vertex for the nucleon. That is, the weak vertex appears with a form factor $F_{1V}((p-n)^2)$, where F_{1V} is expressed in terms of the electromagnetic form factors for the nucleons by $F_{1V}=\frac{1}{2}(F_{1p}-F_{1n})\sim\frac{1}{2}F_{1p}$. The proton form factor [or more precisely, the form factors for a proton off the mass shell (reference 7)], therefore, appears twice in the electromagnetic corrections to the weak vertex function. Although a cutoff in the perturbation integral based on the structure of the nucleon is appealing, the requisite experimental information is not at hand, and the question as to whether the form factors have the required properties must be considered as open. The behavior of the form factors on the mass shell suggests that a cutoff at about the mass of the ρ meson may be reasonable (references 33, 34, and 35). However, F_{1p} decreases only slowly for large q^2, and there has been some speculation that it might approach a constant value for $q^2\to\infty$ [see, for example, R. G. Sachs, Phys. Rev. **126**, 2256 (1962)]. The authors are unable to shed any light on this problem. We note only that the use of a cutoff $\Lambda=m_p$ corresponding to a proton form factor $F_{1p}(q^2)=m_p^2(m_p^2+q^2)^{-1}$ leads in perturbation theory to an electromagnetic correction to Γ_n of $+1.2\%$ rather than the 1.8% quoted above. A detailed study of the dependence of the correction on the form and magnitude of the cutoff is given in reference 9.

tion only in the function $n(t)$, Eq. (116), and lead when considered alone to a 0.115% decrease in Γ_n.

IV. THE BETA DECAY OF O^{14}

A. Single Particle Model of the Decay of O^{14}

The very accurate data[3] obtained in recent years on pure Fermi transition $O^{14}(\beta^+)N^{14*}$ $(0+\to 0+)$ have been used to determine the value of the vector coupling constant in the V-A theory of nuclear beta decay. The nuclear matrix element is thought to differ from the value $\sqrt{2}$ characteristic of a transition between adjacent members of an isotopic triplet only because of effects associated with the nuclear Coulomb fields; these have been estimated by several authors,[11–14] and are probably small. However, one may expect additional radiative corrections of the type which we have considered to be present. These involve the interaction of the positron with the decaying nucleon, and the emission of inner bremsstrahlung during the decay. It is of interest in this connection to consider a model in which the decay is treated as that of a free nucleon insofar as the calculation of these radiative corrections is concerned, the remainder of the nucleus simply providing the proper selectrion rules and decay energy, and the Coulomb field in which the decay positron moves.[38] The positron spectrum is easily calculated in this model using our previous results. The appropriate vertex functions are obtained from those for the neutron decay by an analytic continuation in the variable s to the physical region for the proton decay, $s=-(p-\bar{e})^2$. Inspection of the dispersion integrals shows that the vertex functions G_j are real in this region, and are obtainable from the real parts of the functions G_j for the neutron decay, Eqs. (84)–(91), by the replacement of t by $-t$, $R(t)$ by $-R(t)$, and $D(t,\lambda^2)$ by $D'(t,\lambda^2)$, with $t=-p\cdot\bar{e}/m_pm_e$. Similar replacements are necessary in the functions G_j', Eqs. (99)–(103), and these are now complex. Upon making these changes and retaining only the Fermi component of the resulting transition matrix element, the positron spectrum, including the effects of inner bremsstrahlung, is found to be

$$dN(t)=\tfrac{1}{2}m_e{}^5\pi^{-3}G_V{}^2|M|^2(t_m-t)^2(t^2-1)^{1/2}tF^+(z,t)dt \times[1+(\alpha/\pi)m(t)+(\alpha/8\pi)(\rho-1)n(t)], \quad (123)$$

where M is the nuclear matrix element, $M\sim\sqrt{2}$, and $m(t)$ and $n(t)$ are the functions defined in Eqs. (115) and (116). The decay rate of O^{14} in this model is given by

$$\Gamma(O^{14})=\tfrac{1}{2}m_e{}^5\pi^{-3}G_V{}^2|M|^2f[1-0.0079], \quad (124)$$

where f, calculated without radiative corrections using the positron end-point energy $e_0=2.3236\pm0.0014$ MeV of Bardin *et al.*,[3,4] is equal to 42.97 ± 0.13. The factor in square brackets represents the effect of the radiative corrections, and was calculated by numerical integration.

[38] This model has been used in references 7, 8, and 9 to estimate the radiative corrections to the beta decay of O^{14}. The justification for such a model is discussed in detail in reference 10.

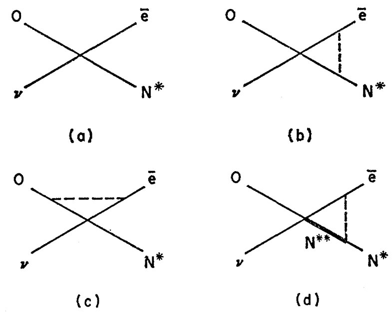

FIG. 3. Feynman diagrams for the radiative corrections to the beta decay of O^{14} in the "elementary particle" model of Sec. IVB.

B. Beta Decay of Structureless Nuclei

It is interesting to compare the foregoing result with that previously reported by the authors,[18] in which the radiative corrections were calculated treating the O^{14} and N^{14*} nuclei as elementary particles. The transition matrix element in this case is of the form

$$\langle\bar{e}\nu p_2|S|p_1\rangle =i(2\pi)^4\delta^4(e+\nu+p_2-p_1)(16\bar{e}_0\nu_0p_{10}p_{20})^{-1/2}\bar{u}(\nu)i\gamma_\alpha v(\bar{e}) \times[A(\bar{s},s)(p_1+p_2)_\alpha+B(s,\bar{s})(p_1-p_2)_\alpha], \quad (125)$$

where p_1 and p_2 denote the 4-momenta of the initial and final nuclei, $s=-(p_2+\bar{e})^2$, and $\bar{s}=-(p_1-\bar{e})^2$. The finite extension of the nucleus leads to an explicit dependence of the functions A and B on the momentum transfer variable Q^2,

$$Q^2=-(p_1-p_2)^2=m_1{}^2+m_2{}^2+m_e{}^2-s-\bar{s}.$$

However, this dependence may be adequately accounted for by the usual multipole expansion of the nuclear matrix element, and will be neglected. In the absence of electromagnetic corrections, $B=0$ and $A=\text{constant}\times M$, where M is the nuclear matrix element for the allowed Fermi transition.

The calculation of the functions A and B follows closely that described previously for the muon and the neutron. The contributions which are included in the iterative procedure to order α correspond to the diagrams of Figs. 3(b) and (c). The vertex function A is then given by

$$\mathrm{Re}A(s,\bar{s})=A(s_0,s_0)+\frac{\bar{s}-\bar{s}_0}{\pi}P\int_{(m_1+m_e)^2}^{\infty}\frac{\mathrm{Im}A_1(s')ds'}{(s'-\bar{s}_0)(s'-\bar{s})} +\frac{s-s_0}{\pi}P\int_{(m_2+m_e)^2}^{\infty}\frac{\mathrm{Im}A_2(s')ds'}{(s'-s_0)(s'-s)}, \quad (126)$$

where

$$\mathrm{Im}A_{1,2}(s)=\mp\tfrac{1}{2}Z_{1,2}\alpha A(s_0,\bar{s}_0)(2p\sqrt{s})^{-1} \times(s-m_{1,2}{}^2-m_e{}^2)\theta(s-(m_{1,2}+m_e)^2) \times\int_0^{4p^2}dq^2[(q^2+\lambda^2)^{-1}-(4p^2)^{-1}]F_{1,2}(q^2), \quad (127)$$

and we have dropped some terms of relative order

$m_e/m_{1,2}$. The functions $F_{1,2}(q^2)$ are the charge form factors of the initial and final nuclei, which, to an adequate degree of approximation, may be taken as equal. The function $B(\bar{s},s)$ does not contribute significantly to the decay, and will be ignored. We remark only that $F_{1,2}(q^2)$ must vanish for $q^2\to\infty$ if the dispersion integrals for $B(s,\bar{s})$ are to converge without a subtraction.

In the evaluation of $\mathrm{Im}A_{1,2}(s)$, we have included only the contributions of those electron-nucleon intermediate states in which the nucleus is not excited. The validity of this approximation depends on the rapidity of convergence of the dispersion integrals for $s'\gg s$, $\bar{s}'\gg\bar{s}$, and on the remoteness from the physical region of the singularities associated with those excited states of the nuclei which can contribute to the absorptive parts. We will consider in detail the possibility that low-lying excited states of N^{14} could contribute significantly to $\mathrm{Im}A_2(s)$ through diagrams such as that of Fig. 3(d). The only transitions from the 0+, $T=1$ state of O^{14} to the levels of N^{14} which can have large (allowed) matrix elements are the vector transitions to the 0+, $T=1$ levels, and the axial vector transitions to the 1+, $T=1$ levels. Other transitions involve either forbidden matrix elements, or a violation of the isotopic spin selection rules, and are strongly inhibited. Furthermore, were it not for the change in the Coulomb field of the nucleus between the initial and final states, the matrix element for an allowed transition to a possible high 0+, $T=1$ level of N^{14} (the first such level is at 8.62 MeV[39]) would involve the overlap between wave functions for different energy levels of the same system, and consequently would vanish identically. Because of the smallness of the Coulomb interaction for $Z=7$, the matrix element is probably of neglibible size in any case. The first possibility other than the desired transition to the 2.311-MeV level of N^{14} is an allowed axial vector transition to a possible 1^+ level at 8.99 MeV,[39] provided the isotopic spin of this level is $T=1$. However, the subsequent interaction of the nucleus with the positron, with the concomitant de-excitation to the 0+ level, proceeds via an $M1$ transition, with the consequence that the electromagnetic matrix element will be small relative to the $E0$ matrix element $ZeF_2(q^2)$ which enters Eq. (127). The conservation of isotopic spin and the selection rules for the weak and electromagnetic transitions, therefore, preclude the existence of significant contributions to $\mathrm{Im}A_2$, or, by similar arguments, to $\mathrm{Im}A_1$, associated with low-lying excited states of N^{14} or O^{14}. The arguments are less certain for large excitations, but the effects on $\mathrm{Re}A$ of small changes in $\mathrm{Im}A_{1,2}$ for large $\bar{s}'$, s' are in any case small because of the rapid decrease in the factors $(s'-s)^{-1}$, $(\bar{s}'-\bar{s})^{-1}$ in the dispersion integrals. Thus, a 100% change in $\mathrm{Im}A_2$ at an excitation energy corresponding to one oscillator spacing ($\sim$25 MeV) results in only a 5% change in $\mathrm{Re}A$. This suppression of the effects of highly excited states is a direct consequence of the low energy of the decay, and the resultant long wavelengths of the positron.[40]

[39] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

It will be convenient to evaluate the dispersion integrals in Eq. (126) first in the approximations $F_1=F_2=1$. The calculations may be simplified further by the introduction of the usual dimensionless variables $t=-e\cdot p_2/m_em_2$ and $\bar{t}=-e\cdot p_1/m_em_1$. The physical regions for t and $\bar{t}$ are identical if we neglect the recoil of the nucleus. Then $t=\bar{t}=e_0/m_e$, $1\leq t$, $\bar{t}\leq t_m$. For the decay of O^{14}, $t_m=(m_1-m_2)/m_e\sim 4.53$. With the introduction of appropriate integration variables

$$t'=(s'-m_2^2-m_e^2)/(2m_2m_e),$$
$$\bar{t}'=(\bar{s}'-m_1^2-m_e^2)/(2m_1m_2),$$

the ranges of integration can be converted into the interval $1\leq t'$, $\bar{t}'<\infty$. The integrals are easily performed, and we obtain for $\mathrm{Re}A$ the result

$$\begin{aligned}\mathrm{Re}A(t,\bar{t})\\ &=A(t_0,\bar{t}_0)\{1-\tfrac{1}{2}Z\alpha\pi t(t^2-1)^{-1/2}+\tfrac{1}{2}Z\alpha t_0(1-t_0^2)^{-1/2}\\ &\times\ln[(4m_e^2/\lambda^2)(1-t_0^2)]-Z[D'(t,\lambda^2)-D'(t_0,\lambda^2)]\\ &-(Z\alpha/2\pi)[tR(t)-t_0R(t_0)]\\ &+(Z+1)[D'(\bar{t},\lambda^2)-D'(\bar{t}_0,\lambda^2)]\\ &+(Z+1)(\alpha/2\pi)[\bar{t}R(\bar{t})-\bar{t}_0R(\bar{t}_0)]\},\end{aligned}\quad(128)$$

where $R(t)$ and $D'(t,\lambda^2)$ are defined in Eqs. (27) and (92) and Z is the charge of the final nucleus. We have assumed that $|t_0|<1$, and will, as usual, neglect terms in the functions D which are negligible in the physical region. The subtraction points t_0 and $\bar{t}_0$ are to be determined by the condition that the positron spectrum and decay rate for O^{14}, as calculated including the effects of inner bremsstrahlung, be independent of λ. This condition leads to the transcendental equation

$$\begin{aligned}1=&Zt_0(1-t_0)^{-1/2}[(\pi/2)+\sin^{-1}t_0]\\ &+(Z+1)\bar{t}_0(1-\bar{t}_0^2)^{-1/2}[\pi/2-\sin^{-1}\bar{t}_0].\end{aligned}\quad(129)$$

We will choose a subtraction point which corresponds to the value $(m_1-m_2)^2$ for the momentum transfer variable Q^2; $A(t_0,\bar{t}_0)$ will then involve the usual nuclear matrix

[40] It is interesting to compare this result with that obtained in perturbation theory by Chern, reference 10, following a procedure applied to the mass 12 system by M. Gell-Mann and S. M. Berman [Phys. Rev. Letters **3**, 99 (1959)]. The perturbation expansion for the electromagnetic corrections to nuclear beta decay involves a sum over excited nuclear states with energy denominators which depend on the energy of those states and the energy of the intermediate electron. If it is assumed that the main contributions to the sum are associated with large electron energies, the nuclear energies may be neglected, and the sum evaluated using closure. (This approximation leads directly to the single particle model for the decay which was discussed in Sec. IVA.) Consequently, if the region of large electron energies yields the dominant contributions to the logarithmically divergent integrals encountered in perturbation theory, many excited states of the nuclei contribute to the electromagnetic renormalization of the weak vertex. On the other hand, it is clear from our results that the momentum-dependent corrections to the renormalized decay amplitudes are associated almost entirely with energies comparable to the physical energy of the decay positron. The selection rules for the weak and electromagnetic interactions then restrict the sum over intermediate states to essentially only one significant term.

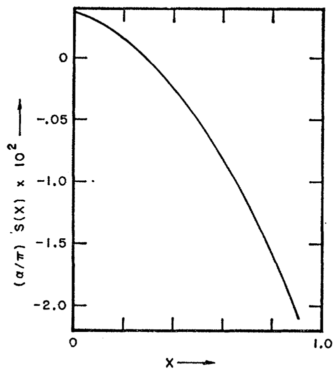

FIG. 4. The radiative correction to the positron spectrum in the decay of O^{14} as calculated from Eq. (133). The radiative contribution to the spectrum is $(dN/dx)_{\rm rad}=(\alpha/\pi)s(x)(dN/dx)_0$, where $(dN/dx)_0$ is the statistical spectrum not including the Fermi factor, and x is the variable $x=e_0/e_{0,\max}$.

element M as a factor. For this choice, $\bar{t}_0\sim t_0$, and one obtains as an approximate solution for Eq. (128),

$$t_0\sim\bar{t}_0\sim(2/\pi)(2Z+1)^{-1}\times[1+(4/\pi^2)(2Z+1)^{-2}+\cdots]\sim 0. \quad (130)$$

Recalling that $\bar{t}$ and t are equal in the physical region to within terms of order $E_m/m_{1,2}$, we obtain for $\mathrm{Re}A(t,\bar{t})$ the result

$$\mathrm{Re}A(\bar{t},t)\to(G_\beta/\sqrt{2})M\{1-\tfrac{1}{2}Z\alpha\pi t(t^2-1)^{-1/2}+D'(t,\lambda^2)+(\alpha/2\pi)[tR(t)-1]+(\alpha/2\pi)[2\ln 2-2]\}, \quad (131)$$

with

$$(G_\beta/\sqrt{2})M=A(t_0,\bar{t}_0), \quad (132)$$

and M the usual nuclear matrix element evaluated for $Q^2=(m_1-m_2)^2$, $M=\langle 1\rangle\sim\sqrt{2}$. The Z-dependent term in this expression arises from the expansion of the Fermi function $F^+(Z,t)$. As emphasized by Berman,[7] the remaining terms, which represent the radiative corrections, are of order α rather than $Z\alpha$.

The positron spectrum corresponding to the above form for $A(\bar{t},t)$ is given by

$$dN(t)=\tfrac{1}{2}m_e^5\pi^{-3}G_\beta^2|M|^2(t_m-t)^2(t^2-1)^{1/2}tF^+(Z,t)dt\times[1+(\alpha/\pi)m(t)+(\alpha/\pi)(2\ln 2-2)]. \quad (133)$$

The factor in square brackets is shown in Fig. 4 for the decay of O^{14}. The foregoing result for the spectrum differs from that obtained by treating the nucleus as a collection of free particles [Eq. (123)] in the small final terms in the square brackets, and in the appearance of G_β, the renormalized coupling constant for the nuclear beta decay, rather than G_V in the leading factor. One can, in principle, attempt to relate the two coupling constants by choosing the decay of the neutron as fundamental, and calculating the nuclear beta decay including such diagrams as those in Figs. 5(b) and 5(c).[41] The normal decay is in this case described by diagrams similar to that of Fig. 5(a). The transition to the structureless nucleus considered above converts the diagram of Fig. 5(b) into that of Fig. 3(c), and that of Fig. 5(c) into that of Fig. 5(d). The contribution of the last diagram to the transition amplitude is independent of t and $\bar{t}$, but leads to a change in the value of the effective coupling constant, as yet unknown. The results of a detailed calculation of this type would be of considerable interest.

The radiative correction factor for the O^{14} decay rate calculated from Eq. (133) differs by 0.24% from the rate calculated from Eq. (123),

$$\Gamma(O^{14})=\tfrac{1}{2}m_e^5\pi^{-3}G_\beta^2|M|^2f[1-0.0103]. \quad (134)$$

This difference represents at least in part a difference between G_V and G_β, but, pending a complete calculation of the nuclear decay rate in terms of G_V, we have no means of making a detailed comparison.

C. Corrections for Finite Nuclear Size

We have thus far restricted our discussion to a theory in which the initial and final nuclei in a decay have no *nuclear* structure in the sense that the nuclear matrix element $M(Q^2)$ was assumed to be independent of the momentum transfer variable $Q^2=-(p_1-p_2)^2$. The *electromagnetic* structure of the nuclei was also ignored when the form factors $F_{1,2}(q^2)$ were set equal to unity in Eq. (127). However, the structure effects are not negligible even for the $O^{14}(\beta^+)N^{14*}$ transition, and must be considered.

The changes in the radiative corrections associated with the nuclear electromagnetic form factors are easily determined. We will assume that the form factors satisfy spectral representations with anomalous thresholds,[22,42]

$$F_{1,2}(q^2)=1-q^2\int_{x_0}^{\infty}\frac{\sigma_{1,2}(x)}{x(x+q^2)}dx. \quad (135)$$

For simplicity, we also assume that $\sigma_1\sim\sigma_2$; the error

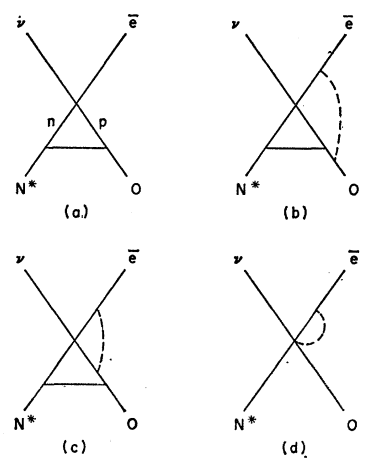

FIG. 5. Representative diagrams for a calculation of the radiative corrections to the decay of O^{14} in which the structure of the nucleus is not ignored.

[41] Such a calculation would represent a relativistic generalization of the work of Chern, reference 10.

[42] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

incurred thereby is negligible. Upon changing variables from s and $\bar{s}$ to t and $\bar{t}$ and noting that t and $\bar{t}$ are in the physical region, we obtain from Eqs. (126), (127), and (135) the change in $\mathrm{Re}A(t,\bar{t})$ associated with the form factors,

$$\Delta[\mathrm{Re}A(t,\bar{t})] = (Z\alpha/2\pi)(G_\beta/\sqrt{2})MtP\int_1^\infty (t'^2-t^2)^{-1}(t'^2-1)^{1/2}dt'^2 \times\int_0^1 u^{-1}(1-u)du\int_{x_0}^\infty \sigma(x)x^{-1} \times[t'^2-1+(x/4m_e^2u)]^{-1}dx +(\alpha/2\pi)(G_\beta/\sqrt{2})Mt\int_1^\infty (t'+t)^{-1}(t'^2-1)^{1/2}dt' \times\int_0^1 u^{-1}(1-u)du\int_{x_0}^\infty \sigma(x)x^{-1} \times[t'^2-1+(x/4m_e^2u)]^{-1}dx+O(E_m/m_{1,2}). \quad (136)$$

Integrating successively over t' and u, expanding the result in powers of the small parameter $4m_e^2/x$, and noting that the moments of the nuclear charge distribution are given by

$$\langle r^n\rangle_{\mathrm{ch}}=(n+1)!\int_{x_0}^\infty \sigma(x)x^{-(n+2)/2}dx, \quad (137)$$

we obtain

$$\Delta[\mathrm{Re}A(t,\bar{t})]\to(G_\beta/\sqrt{2})M\{\tfrac{2}{3}(Z+\tfrac{1}{2})\alpha m_e t\langle r\rangle_{\mathrm{ch}} +(\alpha/\pi)m_e^2t^2\langle r^2[\ln\gamma m_e r-(31/72)]\rangle_{\mathrm{ch}} +(\alpha/6\pi)m_e^2t(t^2-1)R(t)\langle r^2\rangle_{\mathrm{ch}} +O(Z\alpha m_e^3\langle r^3\rangle_{\mathrm{ch}}),\quad \ln\gamma=0.5772\cdots. \quad (138)$$

Only the leading term is of significant size for the decay of O^{14}; this changes the positron spectrum, Eq. (133), by a factor $(1+\Delta_{\mathrm{ch}})$, where

$$\Delta_{\mathrm{ch}}=\tfrac{4}{3}(Z+\tfrac{1}{2})\alpha m_e t\langle r\rangle_{\mathrm{ch}}. \quad (139)$$

The electromagnetic structure corrections have been calculated for the beta decays of O^{14}, Al^{26*}, and Cl^{34}, assuming the equivalent uniform charge distributions as given by Hofstadter[43]; the results are summarized in Table I.[44]

It should be emphasized that the foregoing corrections to $A(t,\bar{t})$ arise from the *electromagnetic* structure of the nucleus, and are present even if, as has been assumed, the *decaying matter* is concentrated at a single point. The Z-dependent term arises from the rescattering of the positron emitted by this point nucleus, and represents the change in the value of $\psi_{\bar{e}}(0)$, the value of the positron wave function at the origin, associated with the finite size of the charge distribution.[45] The remaining terms, of order α would be absent had we calculated the structure corrections using the Dirac equation with the static nuclear field of charge Ze. The positive and negative frequency components of the wave function in that case each involve the charge Ze, rather than charges differing by e as in the decay problem. It is interesting in this connection to note that the leading term in Eq. (138) depends on the average of the initial and final charges. The same should probably be true of a corresponding term in the nuclear structure corrections, as will be noted below.

We shall not discuss the calculation of the nuclear structure corrections to $A(t,\bar{t})$ in any detail, but will take over the results of Morita[20] obtained by the usual multipole expansion of the nuclear matrix element using for the positron the Dirac wave functions for a point Coulomb field. These corrections result in the replacement of $|M|^2=|M((m_1-m_2)^2)|^2$ in the positron spectrum by

$$|M|^2\to|M|^2(1+\Delta_{1n}+\Delta_{2n}), \quad (140)$$

where Δ_{1n} represents for the most part the "finite de Broglie wavelength effect," and Δ_{2n} arises from the interference of allowed and second forbidden components of the transition amplitude. To order $Z\alpha m_e^2t_m^2\langle r^2\rangle_n$, one obtains[46]

$$\Delta_{1n}=\tfrac{1}{2}(\gamma-1)+\tfrac{1}{3}Z\alpha m_e\langle r\rangle_n[4t+t^{-1}+t_m]-\tfrac{1}{3}m_e^2\langle r^2\rangle_n \times[(t^2-1)+(t_m-t)^2+\tfrac{2}{3}t^{-1}(t^2-1)(t_m-t)], \quad (141)$$

and

$$\Delta_{2n}=Z\alpha\langle i\boldsymbol{\alpha}\cdot\hat{r}\rangle_n-\tfrac{2}{3}m_e\langle ir\boldsymbol{\alpha}\cdot\hat{r}\rangle_n[t_m-t+t^{-1}(t^2-1)]. \quad (142)$$

Here $\gamma=[1-(Z\alpha)^2]^{1/2}$, $m_e(t_m-t)$ is the momentum of the neutrino, $\langle r^m\rangle_n$ is the mth moment of the spacial distribution of the decaying nucleon in the nucleus,

$$\langle r^m\rangle_n=\mathfrak{M}(r^m)/\mathfrak{M}(1),\quad \mathfrak{M}(1)=M, \quad (143)$$

and $\langle ir^m\boldsymbol{\alpha}\cdot\hat{r}\rangle_n$ is the mth moment of the second-forbidden matrix element $\mathfrak{M}(i\boldsymbol{\alpha}\cdot\hat{r})$,[47]

$$\langle ir^m\boldsymbol{\alpha}\cdot\hat{r}\rangle_n=\mathfrak{M}(ir^m\boldsymbol{\alpha}\cdot\hat{r})/\mathfrak{M}(1). \quad (144)$$

It is customary to remark at this point that changes in the positron wave function are to be expected within the range of the nuclear charge distribution, and to evaluate

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[43] R. Hofstadter, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1957), Vol. 7.

[44] This correction was not included in Table I of reference 18.

[45] The electromagnetic structure correction has been calculated previously for heavy nuclei by numerical integration of the Dirac equation assuming specific charge distributions. On the other hand, the simple analytic expressions for the correction given in Eqs. (138) and (139) has not, so far as the authors are aware, been obtained before. The separation of the electromagnetic structure corrections from such purely nuclear corrections as the "finite deBroglie wavelength effect" is no longer possible in order $Z\alpha t_m^2m_e^2R_{\mathrm{ch}}R_n$.

[46] We have retained the proper r dependence of the wave functions in the following expression in order to show clearly which moments of the nuclear matter distribution are actually relevant. Morita, on the other hand, follows the customary usage in reference 20, with the result that the actual r dependence is not always clear; for example, $\mathfrak{M}(r)$ is at one point replaced by $R^{-1}\mathfrak{M}(r^2)$.

[47] See, for example, E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

TABLE I. Corrections and ft values for nuclear beta decay.[a]

Nucleus	n	O^{14}	Al^{26*}	Cl^{34}
Half-life, sec[b]	702±18	71.36±0.09[c]	6.374±0.0016	1.53±0.02
Total end-point energy, keV[b]	1293±1	2323.6±1.4	3719.0±2.3	5011±30
$f(Z,E_m)$, uncorrected[d]	1.688±0.006	42.97±0.13	473.0±1.5	2036±61
Corrections, % of f				
Competituon from K capture	···	+0.090	0.078	0.068
Electron screening	···	0.093	0.113	0.127
Nuclear electromagnetic form factors, Δ_{ch}[e]	−0.002	0.110	0.355	0.680
Finite nuclear size, Δ_{1n}[f]	−0.002	0.079	0.253	0.466
"Second forbidden" nuclear matrix elements, Δ_{2n}[g]	···	−0.028	−0.074	−0.117
Total electronic and nuclear corrections	0.00	+0.34	+0.73	+1.22
Electromagnetic (radiative)[h]	−0.60	−0.79	−1.01	−1.38
Total corrections[i] to f, %	−0.60	−0.45	−0.28	−0.16
$f_c(Z,E_m)$, corrected f value	1.678±0.006	42.78±0.13	471.6±1.5	2033±61
$f_c t$	1178±30	3052±10	3006±12	3110±110
G_β,[j] 10^{-49} erg cm^3	1.356±0.068	1.419±0.002	1.430±0.003	1.406±0.025

[a] This table supersedes Table I of reference 18, and should be used in preference to it.
[b] The half-lives and end-point energies for the decays of the neutron and Cl^{34} are those summarized by O. C. Kistner and B. M. Rustad, reference 50, while those for the decay of O^{14} are taken from Bardin *et al.*, reference 3. For Al^{26*}, we have used the recent, unpublished values of Freeman *et al.*, reference 49.
[c] Weighted average of the results of references 3, corrected for the $(0.6\pm1)\%$ branch to the ground state of N^{14} [R. Sherr, J. B. Gerhart, H. Horie, and W. F. Hornyak, Phys. Rev. **100**, 945 (1955)].
[d] Calculated by numerical integration using *Tables for the Analysis of Beta Spectra*, National Bureau of Standards Applied Mathematics Ser. 13 (U. S. Government Printing Office, 1952). The f value for O^{14} is taken from Bardin *et al.*, reference 3, that for Al^{26*}, from Freeman *et al.*, reference 49.
[e] Evaluated using Eq. (139) with $\langle r\rangle_{ch}=\frac{3}{4}R$, and R the radius of an equivalent uniform charge distribution as given by Hofstadter, reference 43, $R=1.33A^{1/3}\times10^{-13}$ cm. This model may be inaccurate by 5–10%.
[f] Evaluated using Eq. (141) with Z replaced by $Z+\frac{1}{2}$, $\langle r\rangle_n=\frac{3}{4}R$, $\langle r^2\rangle_n=\frac{3}{5}R^2$, and R given by the electromagnetic radius $R=1.33A^{1/3}\times10^{-13}$ cm. This procedure may be inaccurate by 10–20%.
[g] Evaluated using Eqs. (142)–(146). A different evaluation of Λ used in reference 18 leads to corrections about 30% smaller in magnitude; the probable uncertainty in these terms is of this order. In addition, those terms in Δ_{1n} which depend on $\langle r^2\rangle_n$ were grouped with the "second-forbidden" corrections in reference 18.
[h] The radiative corrections for the nuclear transitions are given for the single-particle model of the decay discussed in Sec. IVA. These corrections are smaller in magnitude by 0.24% than those obtained from the "elementary particle" model of Sec. IVB. The possible effects of nuclear structure on the radiative corrections are as yet unknown, but we would regard the single-particle model of the decay as the more realistic. The electromagnetic structure corrections calculated in Sec. IVC using the elementary-particle model are equally applicable to the single-particle model. Use of the radiative corrections from the elementary-particle model would increase the quoted values of G_β by 0.12%, that is, by 0.0017×10^{-49} erg cm^3.
[i] The "total corrections" for O^{14}, Al^{26*}, and Cl^{34} do not include the "Coulomb" corrections to the nuclear matrix elements. The magnitude of these corrections is still uncertain (references 11–14, and footnote 51).
[j] Evaluated using $\rho=1.25\pm0.06$, reference 30.

the positron wave function at the nuclear surface. The indicated matrix elements can then be expressed in terms of $\langle r^2\rangle_n$ and $\langle i\boldsymbol{\alpha}\cdot\mathbf{r}\rangle_n$. However, the following remarks should be made. First, the dominant electromagnetic structure correction is given by Δ_{ch}, Eq. (139). This correction is in fact roughly equal in magnitude to the nuclear structure corrections [cf. Table I]. Second, the combined effects of the nuclear and electromagnetic structure can first enter Δ in order $Z\alpha m_e^2 t_m^2\langle r^2\rangle$, and are entirely negligible for our purposes. In a strictly consistent calculation, the quantities $\langle r^m\rangle_n$ and $\langle ir^m\boldsymbol{\alpha}\cdot\hat{r}\rangle_n$ in Eqs. (14) and (142) should, therefore, be evaluated by calculating the appropriate matrix elements in the form in which they are given. (It is interesting in this connecnection to note that a calculation based on diagrams such as those of Fig. 5 would yield results unambiguous in this sense.) We shall nevertheless make the usual approximations in the evaluation of Δ_{2n}; thus,

$$\langle ir^m\boldsymbol{\alpha}\cdot\hat{r}\rangle_n\rightarrow R^{m-1}\langle i\boldsymbol{\alpha}\cdot\mathbf{r}\rangle=\tfrac{1}{4}\Lambda Z\alpha R^{m-2}\langle r^2\rangle_n, \quad (145)$$

where R is a suitable defined nuclear radius and Λ is given by Ahrens and Feenberg[48] as

$$\Lambda=1-(W_1-W_2)A^{1/3}Z^{-1}. \quad (146)$$

Although the uncertainties in this procedure are rather large, the second-forbidden contributions to the decay rate are very small, and we require only a rough estimate of their magnitude. The changes in the nuclear decay rates associated with Δ_{1n} and Δ_{2n} have been calculated for the $0+\rightarrow0+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$, and are summarized in Table I. In this calculation, it was assumed that $\langle r^2\rangle_n\sim\langle r^2\rangle_{ch}=\frac{3}{5}R^2$; where R is the nuclear electromagnetic radius as defined by Hofstadter.[43] In addition, we have replaced Z in Δ_{1n} by the average value $[Z+\frac{1}{2}]$ in the expectation that a proper calculation which took account of the different Coulomb fields seen by the positive and negative frequency components of the wave function would lead to this result, as in the case of the electromagnetic structure corrections.

D. The ft Values of the $0+\rightarrow0+$ Transitions

The very accurate data available for the $O^{14}(\beta^+)N^{14*}$ transition,[3,4] and the relatively small uncertainties in the theoretical calculation of the decay rate, make this transition especially favorable for the determination of the vector weak coupling constant for nuclear beta decay. Data of comparable accuracy are now available for the $Al^{26*}(\beta^+)Mg^{26}$ transition and may soon be available also for the transition $Cl^{34}(\beta^+)S^{34}$.[49] Although some-

[48] T. Ahrens and E. Feenberg, Phys. Rev. **86**, 64 (1952), especially Eqs. (20) and (21).

[49] J. M. Freeman, J. H. Montague, D. West, and R. E. White, Phys. Letters **3**, 136 (1962), and (private communication to L. D. from Miss Freeman). A new measurement of the end-point energy and half of the $Cl^{34}(\beta^+)S^{34}$ transition has been undertaken by the same group.

what less favorable than the O^{14} decay from a theoretical point of view, these $0+ \rightarrow 0+$ decays provide a test of the results obtained from the former, and a comparison of the three may in addition yield important information on the Coulomb corrections to the nuclear matrix element. The relevant data[3,49,50] and the known theoretical corrections to the ft values for these decays and that of the neutron are listed in Table I. The corrected value f_c of f was in each case calculated using for the complete positron spectrum the form which corresponds to the single-particle model of the decay discussed in Sec. IVA, including, however, the various structure corrections. Thus,

$$dN(t)=\tfrac{1}{2}m_e^5\pi^{-3}G_\beta^2|M|^2(t_m-t)^2(t^2-1)^{1/2}tF^+(Z,t)dt \times[1+\Delta_{\text{ch}}+\Delta_{1n}+\Delta_{2n}+(\alpha/\pi)m(t)+(\alpha/8\pi)(\rho-1)n(t)].$$

The coupling constant G_β can be determined from the equation

$$G_\beta^2=2m_e^{-5}\pi^3|M|^{-2}(\ln 2)(f_ct)^{-1}. \quad (148)$$

If it is assumed that O^{14} and N^{14} are components of a pure $T=1$ system, hence, that the nuclear wave functions are identical, the nuclear matrix element M is equal to $\sqrt{2}$. The values of G_β derived on this assumption are given in Table I. Because of the Coulomb field of the nucleus, the assumption of strict charge independence is not valid. Nevertheless, shell-model calculations by MacDonald[11] and Weidenmüller[14] show that the corrections to $|M|^2$ for the O^{14} decay are very small ($\sim -0.05\%$) in the absence of significant configuration mixing. However, Weidenmüller has suggested that such mixing could be important, and could lead to a 1–2% decrease in $|M|^2$, depending on the excitation energy of the first collective breathing mode of the nuclei. Other estimates based on the nuclear compressibility[14] and on the known asymptotic behavior of the nuclear wave functions[14,51] yield somewhat smaller corrections. It has also been suggested by Blin-Stoyle and LeTourneux[12] that small deviations from charge independence in the nuclear forces could lead to 1–2% changes in $|M|^2$, but Altman and MacDonald[13] have shown that such a mechanism is probably incompatible with the charge dependence of the beta decays of the remaining members of the $C^{14}-N^{14*}-O^{14}$ triplet to the ground state of N^{14}. It seems unlikely in any case that $|M|^2$ will be found to differ from its charge-independent value, two, by more than about 1%. The situation is less clear with respect to the beta decays of Al^{26*} and Cl^{34}. The Coulomb corrections to the nuclear matrix elements have been calculated in part for the latter,[11] but no calculations have so far dealt with the former.

The presumably accurate values of G_β obtained from the $O^{14}(\beta^+)N^{14*}$ and the $Al^{26*}(\beta^+)Mg^{26}$ transitions differ by $(0.8\pm0.5)\%$. The bare coupling constants should be equal for a universal Fermi interaction with a conserved vector current.[1] Because the electromagnetic renormalizations would be the same for the two nuclei in the single-particle model of the decay discussed in Sec. IVA, and would probably not be much different in a more detailed model, the renormalized coupling constants G_β should also be essentially equal for O^{14}, Al^{26*}, and Cl^{34}, and should equal G_V as determined from the decay of the neutron. The apparent discrepancy between the values of G_β for O^{14} and Al^{26*} may reflect, to the extent to which it is real, the difference between the corrections to the nuclear matrix elements in the two cases. It is interesting to note that, because of the smallness of the *negative* correction to M expected for O^{14}, the discrepancy suggests that the correction for Al^{26*} is *positive*. This in turn implies that the dominant correction must arise from $T=2$ isotopic spin impurities in the relevant states of Al^{26*} and Mg^{26}, a result consistent with that obtained by MacDonald[11] for Cl^{34}. A more detailed study of this problem would be of considerable interest. Because of the large uncertainties in the ft values for Cl^{34} and the neutron, and the additional uncertainty in the value of $\rho=-G_A/G_V$ for the latter,[30,50] it is not possible to check in detail the expected equality of the coupling constants for these decays with those discussed above.

[50] O. C. Kistner and B. M. Rustad, Phys. Rev. **114**, 1329 (1959).

[51] The Coulomb corrections to the nuclear matrix element have been considered by one of the authors (L. D., unpublished) using a crude single-particle model of the nucleus. In this model, the decaying nucleon was taken in a P state, and the effect of this particle on the remaining "N^{13}" core was ignored. Single-particle neutron and proton wave functions were obtained by matching the asymptotic wave functions, which depend only on the known binding energies of the last nucleon in N^{14*} and O^{14}, with square-well wave functions for the interior region. If the Coulomb field of the nucleus is ignored in obtaining the exterior proton wave function, direct calculation of the overlap integration yields a 0.55% reduction in M for a matching radius $R=1.4[(13)^{1/3}+1]$F $\sim$4.7F, the "interaction radius." The result is essentially unchanged (-0.57%) if the matching radius is reduced to $R=1.4(13)^{1/3}$F$\sim$3.3F, a choice which gives the proper mean square electromagnetic radius for N^{14}. In a second model, which has also been considered by Weidenmüller, reference 14, the exterior proton wave function is matched in magnitude, but not in slope, to the interior wave function for the last neutron in N^{14*}, determined as above. The sudden change in the proton wave function corresponds to a sharing of the Coulomb energy among all the protons when the proton in question is inside the nucleus. This model would, therefore, appear to be more realistic than that considered above. Using for the matching radius $R=3.3$F, one obtains a 0.31% reduction in M. [This result disagrees with that of reference 14, apparently because of the use of an incorrect binding energy in the latter calculation.] For $R=4.7$F, the correction is reduced to -0.25%. The correction to M arises in both models from the considerably different asymptotic behavior of the neutron and proton wave functions associated with the different binding energies. If the effects of the nuclear Coulomb field are included in the proton wave function, the added potential barrier leads to a more rapid decrease of that wave function with increasing radius, and to better overlap with the more tightly bound neutron wave function. Recent calculations by Ian McGee, using WKB Coulomb wave functions, yield correction to M of -0.05% for $R=3.3$ F, smaller than that quoted above by a factor of six. A correction of this size is consistent with the estimates obtained in perturbation theory, references 11 and 14. Even if some allowance is made for a failure of the N^{13} core wave functions to overlap perfectly when the extra nucleon is inside the core, it appears unlikely that the Coulomb corrections to M could be larger by as much as an order of magnitude. We would like to thank Isaac Cole for performing the original calculations, and Ian McGee for his careful work on the Coulomb wave function problem.

V. DISCUSSION

We have been concerned in this paper with the calculation of electromagnetic corrections to weak interactions using the techniques of dispersion theory. The method, as has been seen, has the advantage that it yields relatively unambiguous results for the momentum-dependent corrections to the weak vertex functions, hence, to the decay spectra and correlation parameters. However, this desirable feature of our results entails the use of renormalized rather than bare coupling constants, with the consequence that we are unable in the end to make any statements about the universality of the Fermi interaction without first appealing to cutoff perturbation theory for information about the renormalization constants. The concept of renormalization of the weak coupling constant was examined in detail in the case of the muon; similar arguments apply to the neutron, but the problem is there complicated by the presence of strong interactions. The renormalized coupling constants were in each case defined in terms of the value of a weak vertex function of an appropriate subtraction point. The latter was determined by the requirement that physically significant quantities not be infrared divergent when calculated including the effects of processes in which inner bremsstrahlung is emitted. The fact that the acceptable subtraction points are unique (with the exception of $\bar{s}_0$ for the neutron, reference 36) makes the resulting definitions of the renormalized coupling constants much more attractive than would otherwise be the case. We were also able in the present method to study the effects on the transition amplitudes of the finite electromagnetic structure of the particles in question, without being forced to make any assumptions about the behavior of form factors off the mass shell.

The electromagnetic corrections to the decay, and the renormalization problem, appear to be well understood in the case of the muon. No other significant corrections to the decay rate are known. The status of the theory of the $0+\rightarrow 0+$ nuclear transitions is less clear. Aside from the still-outstanding uncertainties with respect to the magnitude of the Coulomb corrections to the nuclear matrix elements,[11–14,51] there remains the question of the possible influence of the structure of the nucleus on the electromagnetic corrections. We have considered two models, in one of which the nucleon directly involved in the beta transition was treated as a free particle (Sec. IVA), and in the other of which the nuclear structure was disregarded altogether, the nuclei being treated as elementary particles with spin $0+$ (Sec. IVB). The subtraction point in the second model is somewhat artificial, and it is probable that the single-particle model is to be preferred. Some support for this view may be found in the work of Gell-Mann and Berman,[40] and Chern.[10] The difference between the results obtained with the two models is fortunately quite small ($\sim$0.24% in the decay rates). The known theoretical corrections to the decay rates of the $0+\rightarrow 0+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$ are summarized in Table I. The as yet uncertain Coulomb corrections to the nuclear matrix elements[11–14,51] are not included.

The theoretical results of the present paper have been used in conjunction with the very accurate results of recent experiments on the beta decays of the muon,[5] O^{14},[3,4] and Al^{26*},[49] to derive values for the *renormalized* weak coupling constants G_μ and G_β. These values, and the less accurate value of G_V obtained from the decay if the neutron,[30] are summarized in Table II. As noted previously, there is a discrepancy of $(0.8\pm0.5)\%$ between the values of G_β obtained from the data on O^{14} and Al^{26*}. This may disappear when the Coulomb corrections to the nuclear matrix elements are known. Because the theoretical uncertainties are probably less in the case of O^{14}, we will use the value of G_β obtained from that decay in the following discussion. With this choice, the "discrepancy" between G_μ and G_β is seen to be $(1.0\pm0.2)\%$ (there is in fact no reason to expect the renormalized coupling constants to be equal). The renormalization factors obtained by comparing our results with those of perturbation theory,[6] and choosing for the ultraviolet cutoff in the case of the neutron and the nuclear beta decays the value $\Lambda=m_p$, are listed in the third column of Table II. The extent to which one can rely upon the cutoff theory is essentially unknown.[37] If the results are accepted, one obtains for the bare coupling constants the values listed in the fourth column of Table II,[52] and a discrepancy between $G_{\mu,\text{bare}}$ and $G_{\beta,\text{bare}}$ (O^{14}) of $(1.9\pm0.2(\pm0.5))\%$, where the final uncertainty is theoretical.[9] A part of the discrepancy may be removed when the Coulomb corrections to the nuclear matrix element for O^{14} are better known, but it seems unlikely that it will be removed altogether.[51] However, it should be recalled, first, that $G_{\beta,\text{bare}}$ as defined above is actually renormalized with respect to the strong interactions, and that the renormalization constant need not be unity[1,2] if electromagnetic corrections to the strong interactions are considered; and second, that the effect on the decay rate of diagrams such as that in Fig. 2(d) have been ignored altogether. Although such effects

[52] The expressions in Eqs. (118) and (119) relate the completely renormalized coupling constants $G_V(1+\rho)$ and $G_V(1-\rho)$ to the partially renormalized coupling constants $G'(1+\rho')$ and $G'(1-\rho')$ introduced in Sec. IIIA. If we write $G_V(1\pm\rho)=Z^{\pm}G'(1\pm\rho')$, we can solve for G_V and ρ as follows: $G_V=\frac{1}{2}(Z^++Z^-)G'+\frac{1}{2}(Z^+-Z^-)G'\rho'$, $G_V\rho=\frac{1}{2}(Z^++Z^-)G'\rho'+\frac{1}{2}(Z^+-Z^-)G'$. The difference (Z^+-Z^-) is of order α, as are the differences between G_V, $G_V\rho$ and G', $G'\rho'$. We may, therefore, replace the partially renormalized quantities by the renormalized quantities in the second terms on the right-hand sides of the equation, and obtain at once the desired relations, correct to order α, $G'=2G_V(Z^++Z^-)^{-1}[1-\frac{1}{2}\rho(Z^+-Z^-)]$, $\rho'=\rho+\frac{1}{2}(Z^+-Z^-)(\rho^2-1)$. These expressions relate the partially renormalized quantities to the measureable, renormalized quantities G_V and ρ. If electromagnetic corrections to the strong interactions do not significantly affect the strong renormalization, then $G'\sim G$, where G is the bare coupling constant of the universal Fermi interaction. Using a cutoff $\Lambda=m_p$, we obtain from Eqs. (118) and (119) the values $Z^+=1.0117$ and $Z^-=1.0102$ for the renormalization constants, thus, assuming that $G_V=1.42\times10^{-49}$ erg cm^3 and $\rho=1.25$, the ratios of renormalized to partially renormalized coupling constants $G_V/G'=1.0120$ and $\rho/\rho'=0.9996$.

TABLE II. Values for the vector coupling constant.

Particle	G, uncorrected[a] (10^{-49} erg cm^3)	G, corrected[b] (10^{-49} erg cm^3)	Renormalization factor, $G_{\text{ren}}/G_{\text{bare}}$ (%)	G_{bare}[e] (10^{-49} erg cm^3)
Muon	1.428 ± 0.001	1.436	$+0.32$[c]	1.431 ± 0.001
O^{14}	1.416 ± 0.002	1.419	$1.20(\pm0.50)$[d]	$1.402\pm0.002(\pm0.007)$
Al^{26*}	1.428 ± 0.002	1.430	$1.20(\pm0.50)$[d]	$1.413\pm0.002(\pm0.007)$
Neutron	1.352 ± 0.068	1.356	$1.20(\pm0.50)$[d]	$1.340\pm0.068(+0.007)$

[a] Uncorrected values for G for the muon and O^{14} from Bardin *et al.*, reference 3. Uncorrected value of G for Al^{26*} from Freeman *et al.*, reference 49.
[b] Total corrections for the neutron, O^{14} and Al^{26*} from Table I.
[c] Calculated using the result of perturbation theory, Eq. (43).
[d] Calculated using the result of perturbation theory for the neutron, Eqs. (118) and (119), with the cutoff chosen as $\Lambda = m_p$. The method is discussed in footnote 52. The indicated uncertainty in the renormalization factor is the estimated uncertainty in the value of the cut off given in reference 9.
[e] The values of G given for the neutron, O^{14}, and Al^{26*} are "bare" values only if the effect of electromagnetic interactions on the strong renormalization factor, equal to unity in the absence of such effects, can be neglected. See, for example, the discussion in Sec. IIIA.

might account for the remaining discrepancy between the "bare" coupling constants, they are difficult to estimate in any reliable fashion.

It is interesting to note that the discrepancy between $G_{\mu,\text{bare}}$ and $G_{\beta,\text{bare}}$ (O^{14}) could be eliminated if the weak interaction is mediated by a charged vector meson.[15,17] Lee and Yang[15] have considered this possibility, and find that the decay rate for the muon in the absence of electromagnetic corrections is given by

$$\Gamma_\mu(LY) = (G^2/192)\pi^{-3} m_\mu^5 [1 + \tfrac{3}{5}(m_\mu/M_W)^2], \quad (149)$$

where M_W is the mass of the intermediate boson. The radiative corrections to this expression have been studied by Lee[16] on the basis of the theory of the electromagnetic interactions of charged vector mesons developed by Lee and Yang.[27] The results are essentially those of perturbation theory,[8] except for the appearance of the extra factor in Eq. (149) and some additional terms which depend on the anomalous magnetic moment κ of the meson and diverge for $\kappa \to 0$. An analogous calculation leads to results for the neutron similar to those of perturbation theory, with the ultraviolet cutoff Λ replaced in the leading correction by M_W. Because of the small momentum transfer in this decay, the nonlocality of the interaction does not lead to any significant change in the uncorrected matrix element. Lee[16] finds that the discrepancy between the values of G_μ and G_β (which now have the significance of effective coupling constants for a local four-fermion interaction) would disappear for a meson mass on the order of 500–600 MeV, but the result cannot be regarded as decisive pending the discovery of the intermediate vector meson, and a more careful consideration of nuclear and electromagnetic corrections which have so far been omitted. The intermediate meson approach nevertheless appears to be the most promising, if one is to insist on the universality of the Fermi interaction.

It may be noted finally that, to the extent to which the nonlocality of the interaction may be ignored (cf. Sec. IVC), the momentum-dependent radiative corrections are given correctly by our calculation even if the intermediate vector meson exists. (The effect of the meson on the uncorrected matrix element may be treated in the usual fashion.) Our results are in this sense universal. The precise relation of the effective coupling constants G_μ, G_V, and G_β to more fundamental quantities of course depends on the details of the underlying interaction, but the determination of this relation is a separate problem, and need be considered only in connection with questions concerning the universality of the basic weak interactions.

ACKNOWLEDGMENTS

Two of the authors (L. D. and L. F. L.) would like to thank Robert Craig and the Physics Division of the Aspen Institute for their generous hospitality during the summer of 1962, during which this work was completed.