one obtains

$$\lambda_3=\frac{J^3S^2(S+1)^2Nz}{12}[-1+8S(S+1)q_3]$$
$$=\lambda_3^{(2)}+\lambda_3^{(3)}, \quad (79)$$

$$\lambda_4=\frac{J^4S^2(S+1)^2Nz}{15}\bigg\{[1-\tfrac{7}{2}S(S+1)+\tfrac{3}{2}S^2(S+1)^2]$$
$$-\frac{5}{3}[q_3+S(S+1)(z-1)]+\frac{5}{3}q_4S^2(S+1)^2\bigg\}$$
$$=\lambda_4^{(2)}+\lambda_4^{(3)}+\lambda_4^{(4)}. \quad (80)$$

The individual terms inside the curly brackets correspond to increasing numbers of particles. For the simple cubic lattice $z=6$, $q_3=0$, $q_4=12$, for the face-centered cubic lattice $z=12$, $q_3=0$, and $q_4=6$. Inspection of Eqs. (80) and (79) shows that there is no particular predominance of any one sort of diagram over another as was observed in the longer range dipole-dipole potential problem.

The Curie point has been inferred by Rushbrooke and Wood[17] to be proportional to $S(S+1)$. A sufficient condition for this is that the ring diagrams predominate in their contributions, for the ring diagram with n vertices has a factor $[S(S+1)]^n$. The results on λ_3 and λ_4, however, confirm one's intuitive feeling that for a very short range potential the cycle diagrams do not predominate. In contrast to the situation with the dipolar lattice, it is, therefore, not possible to obtain a natural explanation for the $S(S+1)$ dependence of the Curie temperature for an exchange-coupled lattice.

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Use of Green Functions in the Theory of Ferromagnetism. III. *s-d* Interactions

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We use the method of Part I of this series of papers to study the influence of *s-d* interactions, thus extending the work by Potapkov and Tyablikov to higher spin values and that of Vonsovskii and Izyumov to higher temperatures. Expressions are given for the energy shift and damping caused by the *s-d* interaction, using the first nontrivial approximation to the Green-functions equations of motion.

1. INTRODUCTION

IN the first two papers of this series[1] (we use throughout the same notations as in I and II and refer to these papers for the definition of the various symbols) we discussed an ideal ferromagnet with a Heisenberg Hamiltonian, that is, the interaction between the spins was assumed to be an isotropic exchange interaction. It is, however, well known[2–6] that, on the one hand, in crystals of metals and alloys of the iron group as well as direct-exchange interaction there is also an indirect interaction produced through *s-d* exchange while, on the other hand, this *s-d* exchange mechanism may well be the dominant one in crystals of rare-earth elements and for the case of solutions of paramagnetic ions in diamagnetic crystals where the direct exchange is small. Potapkov and Tyablikov[7] have used a Green-function method to discuss this problem for the case where $S=\frac{1}{2}$,

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[1] R. A. Tahir-Kheli and D. ter Haar, Phys. Rev. **127**, 88 and 95 (1962). These papers are referred to as I and II and their equations are quoted as (I3, 5), (II2.11), and so on. We should like to use this opportunity to rectify an incorrect statement in Appendix B of I and to apologize to Dr. Kawasaki and Dr. Mori for incorrectly criticizing their work. We have now found that their theory gives, indeed, the correct high-temperature expansion, at least up to terms of order $1/\tau^2$; our misinterpretation was caused by a misprint in their paper.

[2] S. V. Vonsovskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **16**, 981 (1946).

[3] S. V. Vonsovskii and E. A. Turov, J. Exptl. Theoret. Phys. (U.S.S.R.) **24**, 419 (1953).

[4] J. Owen, M. Browne, W. D. Knight, and C. Kittel, Phys. Rev. **102**, 1501 (1956).

[5] K. Yosida, Phys. Rev. **106**, 893 (1957).

[6] K. Yosida, Phys. Rev. **107**, 396 (1957).

[7] N. A. Potapkov and S. V. Tyablikov, Fiz. Tverd. Tela **2**, 2733 (1960) [translation: Soviet Phys.—Solid State **2**, 2433 (1961)].

using a Hamiltonian involving the so-called Pauli operators[8] (also see I). On the other hand, Vonsovskii and Izyumov[9] discussed the case of a general spin value at low temperatures, using spin-wave variables (compare II). In the present paper we shall extend Potapkov and Tyablikov's work to higher spin values (or, alternatively, Vonsovskii and Izyumov's work to higher temperatures).

In the next section we shall introduce the Hamiltonian for our system. In Sec. 3 we find an expression for the magnetization of the system, and for the energy shift and damping of the spin-wave-like boson excitations. In Sec. 4 we discuss the Green functions involving the conduction-electron creation and annihilation operators and the damping of the fermion excitations. Finally, in Sec. 5 we discuss our results.

2. THE HAMILTONIAN

The Hamiltonian we shall use in the following consists of three parts,

$$H=H_s+H_d+H_{sd}. \tag{2.1}$$

In (2.1) H_s is the unperturbed Hamiltonian for the conduction electrons which we take to be of the form

$$H_s=\sum_{\mathbf{k},\sigma}(e_{\mathbf{k}}-\mu)c_{\mathbf{k},\sigma}{}^\dagger c_{\mathbf{k},\sigma} -g_s\mu_B B\sum_{\mathbf{k}}[c_{\mathbf{k},-}{}^\dagger c_{\mathbf{k},-}-c_{\mathbf{k},+}{}^\dagger c_{\mathbf{k},+}], \tag{2.2}$$

where $e_{\mathbf{k}}$ is the unperturbed single-electron energy of an electron with wave vector $\mathbf{k}$ (we assume $e_{\mathbf{k}}$ to be independent of the electron spin), the $c_{\mathbf{k},\sigma}{}^\dagger(c_{\mathbf{k},\pm}{}^\dagger)$ and $c_{\mathbf{k},\sigma}(c_{\mathbf{k},\pm})$ are the creation and annihilation operators for electrons with wave vector $\mathbf{k}$ and spin σ ($\sigma=+\frac{1}{2}$ or $-\frac{1}{2}$, according to whether the orientation of the electron spin is parallel or antiparallel to the z axis; in indices we drop the $\frac{1}{2}$ and σ stands for $+$ or $-$), and μ is the chemical potential.

For the Hamiltonian, H_d, of the localized spins we use (I3.1), and for the interaction Hamiltonian H_{sd} we use Vonsovskii and Izyumov's expression[9,10]

$$H_{sd}=-\sum_{\mathbf{l}}\sum_c D'(\mathbf{l},c)(\mathbf{S}_{\mathbf{l}}\cdot\mathbf{S}_c), \tag{2.3}$$

where the summation is over all lattice sites of the localized spins and over all conduction electrons (c). Introducing in (2.3) the $c_{\mathbf{k},\sigma}{}^\dagger$ and $c_{\mathbf{k},\sigma}$ and combining Eqs. (2.1) to (2.3), we find for the total Hamiltonian[10]

$$\begin{aligned}H=&-\sum_{\mathbf{l},\mathbf{m}}I(\mathbf{l},\mathbf{m})(\mathbf{S}_{\mathbf{l}}\cdot\mathbf{S}_{\mathbf{m}})-(g_d\mu_B B/\hbar)\sum_{\mathbf{l}}S_{\mathbf{l}}{}^z\\&+\sum_{\mathbf{k},\sigma}(e_{\mathbf{k}}-\mu)c_{\mathbf{k},\sigma}{}^{\dagger\rho}c_{\mathbf{k},\sigma}\\&-g_s\mu_B B\sum_{\mathbf{k}}[c_{\mathbf{k},-}{}^\dagger c_{\mathbf{k},-}-c_{\mathbf{k},+}{}^\dagger c_{\mathbf{k},+}]\\&-(\hbar/N)\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{f}}\exp[-i\mathbf{f}\cdot(\mathbf{k}_1-\mathbf{k}_2)]D(\mathbf{k}_1,\mathbf{k}_2)\\&\times[c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{f}}{}^-+c_{\mathbf{k}_1,+}{}^\dagger c_{\mathbf{k}_2,-}S_{\mathbf{f}}{}^+\\&\qquad+c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,-}S_{\mathbf{f}}{}^z-c_{\mathbf{k}_1,+}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{f}}{}^z],\end{aligned} \tag{2.4}$$

where

$$D(\mathbf{k}_1,\mathbf{k}_2)=D^*(\mathbf{k}_2,\mathbf{k}_1) \tag{2.5}$$

is the Fourier transform of the s-d exchange integral.

When we introduce our Green functions we must bear in mind that we are now no longer dealing with boson-type excitations only, and we shall distinguish—as we did not do in I and II—between two possible types of Green functions[8,11] which differ in the value of Zubarev's parameter η. The equations of motion for the Green functions are instead of (I2.9) of the form

$$E\langle\langle A;B\rangle\rangle^{-\eta}=(1/2\pi)\langle[A,B]_{-\eta}\rangle+\langle\langle[A,H]_-;B\rangle\rangle^{-\eta}, \tag{2.6}$$

where $[A,B]_+$ denotes the anticommutator. In I and II we only consider the case $\eta=+1$, but now it is convenient to consider both $\eta=+1$ and $\eta=-1$ at the same time.

3. THE MAGNETIZATION; THE BOSON-LIKE EXCITATIONS

We saw in I that, in order to evaluate the magnetization, we had to study the Green functions $\langle\langle S_{\mathbf{g}}{}^+;(S_{\mathbf{l}}{}^-)^n(S_{\mathbf{l}}{}^+)^{n-1}\rangle\rangle^-$. From (2.4) and (2.6) we find for them the equation of motion

$$\begin{aligned}\langle\langle S_{\mathbf{g}}{}^+;A_n\rangle\rangle^-[E-g_d\mu_B B]&\\=\frac{\delta_{\mathbf{l},\mathbf{g}}}{2\pi}Q^{(n)}&-2\hbar\sum_{\mathbf{f}}I(\mathbf{g}-\mathbf{f})\langle\langle S_{\mathbf{g}}{}^zS_{\mathbf{f}}{}^+-S_{\mathbf{f}}{}^zS_{\mathbf{g}}{}^+;A_n\rangle\rangle^-\\&-\frac{2\hbar^2}{N}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}e^{-i\mathbf{g}\cdot(\mathbf{k}_1-\mathbf{k}_2)}D(\mathbf{k}_1,\mathbf{k}_2)\\&\times\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{g}}{}^z;A_n\rangle\rangle^-\\&+\frac{\hbar^2}{N}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}e^{-i\mathbf{g}\cdot(\mathbf{k}_1-\mathbf{k}_2)}D(\mathbf{k}_1,\mathbf{k}_2)\\&\times\langle\langle[c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,-}-c_{\mathbf{k}_1,+}{}^\dagger c_{\mathbf{k}_2,+}]S_{\mathbf{g}}{}^+;A_n\rangle\rangle^-,\end{aligned} \tag{3.1}$$

where the first term on the right-hand side of (3.1) is the same as the first term on the right-hand side of (I3.4), and where

$$A_n=(S_{\mathbf{l}}{}^-)^n(S_{\mathbf{l}}{}^+)^{n-1}. \tag{3.2}$$

[8] V. L. Bonch-Bruevich and S. V. Tyablikov, *Green Function Methods in Statistical Mechanics* (Moscow, 1961) [English translation: North-Holland Publishing Company, Amsterdam, 1962].

[9] S. V. Vonsovskii and Ya. A. Izyumov, Fiz. Metal. Metalloved. Akad. Nauk S.S.S.R. Ural. Filial. **10**, 321 (1960).

[10] See also T. Kasuya, Progr. Theoret. Phys. (Kyoto) **16**, 45 (1956); A. H. Mitchell, Phys. Rev. **105**, 1439 (1957).

[11] D. N. Zubarev, Usp. Fiz. Nauk **71**, 71 (1960) [translation: Soviet Phys.—Usp. **3**, 320 (1960)].

We now introduce the following decoupling. First of all, we use (I3.5), and secondly, for the mixed Green functions we write

$$\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{g}}{}^z;A_n\rangle\rangle^- \doteqdot \langle S^z\rangle\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+};A_n\rangle\rangle^-, \quad (3.3)$$

$$\langle\langle c_{\mathbf{k}_1,\sigma_1}{}^\dagger c_{\mathbf{k}_2,\sigma_2}S_{\mathbf{g}}{}^+;A_n\rangle\rangle^- \doteqdot \langle c_{\mathbf{k}_1,\sigma_1}{}^\dagger c_{\mathbf{k}_2,\sigma_2}\rangle\langle\langle S_{\mathbf{g}}{}^+;A_n\rangle\rangle^- \doteqdot \bar{\zeta}_{\mathbf{k}_1}(\sigma_1)\delta_{\mathbf{k}_1,\mathbf{k}_2}\delta_{\sigma_1,\sigma_2}\langle\langle S_{\mathbf{g}}{}^+;A_n\rangle\rangle^-, \quad (3.4)$$

where $\bar{\zeta}_{\mathbf{k}_1}(\sigma_1)$ is the average occupation number of conduction electrons with wave vector $\mathbf{k}_1$ and spin σ_1.

We now get from (3.1) to (3.4)

$$\langle\langle S_{\mathbf{g}}{}^+;A_n\rangle\rangle^- \times\left\{E-g_d\mu_{\mathrm{B}}B-\frac{\hbar^2}{N}\sum_{\mathbf{k}}D(\mathbf{k},\mathbf{k})[\bar{\zeta}_{\mathbf{k}}(-)-\zeta_{\mathbf{k}}(+)]\right\} =\frac{\delta_{1,\mathbf{g}}}{2\pi}Q^{(n)}-2\hbar\sum_{\mathbf{f}}I(\mathbf{g}-\mathbf{f})\langle\langle S_{\mathbf{g}}{}^zS_{\mathbf{f}}{}^+-S_{\mathbf{f}}{}^zS_{\mathbf{g}}{}^+;A_n\rangle\rangle^- -\frac{2\hbar^2}{N}\sum_{\mathbf{k}_1}\sum_{\mathbf{k}_2}e^{-i\mathbf{g}\cdot(\mathbf{k}_1-\mathbf{k}_2)}\langle S^z\rangle D(\mathbf{k}_1,\mathbf{k}_2) \times\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+};A_n\rangle\rangle^- \quad (3.5)$$

If we neglect the remaining mixed Green functions $\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+};A_n\rangle\rangle^-$, (3.5) differs from (I3.4) only in a shift in the energies of the boson-like (spin-wave) excitations, in that $E_{\mathbf{k}}{}^{(S)}$ of (I3.11) is replaced by $E_{\mathbf{k}}$ given by

$$E_{\mathbf{k}}=E_{\mathbf{k}}{}^{(S)}+\frac{\hbar^2}{N}\sum_{\mathbf{k}}D(\mathbf{k},\mathbf{k})[\bar{\zeta}_{\mathbf{k}}(-)-\bar{\zeta}_{\mathbf{k}}(+)]. \quad (3.6)$$

To find the mixed Green functions we must write down their equations of motion which are

$$\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+};A_n\rangle\rangle^-[E-e_{\mathbf{k}_2}+e_{\mathbf{k}_1}-2g_s\mu_{\mathrm{B}}B] =\frac{\hbar}{N}\sum_{\mathbf{f},\mathbf{k}}\{D(\mathbf{k},\mathbf{k}_1)e^{-i\mathbf{f}\cdot(\mathbf{k}-\mathbf{k}_1)} \times\langle\langle c_{\mathbf{k},+}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{f}}{}^+ + c_{\mathbf{k},-}{}^\dagger c_{\mathbf{k}_2,+}S_{\mathbf{f}}{}^z;A_n\rangle\rangle^- +D(\mathbf{k}_2,\mathbf{k})e^{-i\mathbf{f}\cdot(\mathbf{k}_2-\mathbf{k})} \times\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k},+}S_{\mathbf{f}}{}^z - c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k},-}S_{\mathbf{f}}{}^+;A_n\rangle\rangle^-. \quad (3.7)$$

Using the decoupling (3.3) and (3.4) this equation reduces to the form

$$\langle\langle c_{\mathbf{k}_1,-}{}^\dagger c_{\mathbf{k}_2,+};A_n\rangle\rangle^-[E-\epsilon_{\mathbf{k}_2,+}+\epsilon_{\mathbf{k}_1,-}] \doteqdot\frac{\hbar}{N}\sum_{\mathbf{f}}e^{-i\mathbf{f}\cdot(\mathbf{k}_2-\mathbf{k}_1)}D(\mathbf{k}_2,\mathbf{k}_1) \times[\bar{\zeta}_{\mathbf{k}_2}(+)-\bar{\zeta}_{\mathbf{k}_1}(-)]\langle\langle S_{\mathbf{f}}{}^+;A_n\rangle\rangle^-, \quad (3.8)$$

where

$$\epsilon_{\mathbf{k},\sigma}=(e_{\mathbf{k}}-\mu)+2[g_s\mu_{\mathrm{B}}B+\hbar\langle S^z\rangle D(\mathbf{k},\mathbf{k})]\sigma \quad (\sigma=\pm\tfrac{1}{2}). \quad (3.9)$$

We now have a set of coupled equations for our Green functions which can be solved in the usual way by using an inverse-lattice Fourier transformation. If $G_{\mathbf{k}}{}^{(n)}(E_\pm)$ is the Fourier transform of $\langle\langle S_{\mathbf{g}}{}^+;A_n\rangle\rangle^-$ for $E=E\pm i\epsilon$ in the limit as ϵ tends to zero, we find

$$G_{\mathbf{k}}{}^{(n)}(E_\pm)=\frac{Q^{(n)}/2\pi}{E-E_{\mathbf{k}}-R_{\mathbf{k}}(E)\pm\pi i\gamma_{\mathbf{k}}(E)}, \quad (3.10)$$

with

$$R_{\mathbf{k}}(E)=\mathcal{P}\frac{2\hbar^3}{N}\sum_{\mathbf{k}',\mathbf{k}''}\frac{\langle S^z\rangle D(\mathbf{k}'',\mathbf{k}')D(\mathbf{k}',\mathbf{k}'')[\bar{\zeta}_{\mathbf{k}''}(-)-\zeta_{\mathbf{k}''}(+)]\delta_{\mathbf{k}+\mathbf{k}'',\mathbf{k}'}}{E-\epsilon_{\mathbf{k}',+}+\epsilon_{\mathbf{k}'',-}} \quad (3.11)$$

$$\gamma_{\mathbf{k}}(E)=\frac{2\hbar^3}{N}\pi\sum_{\mathbf{k}',\mathbf{k}''}\langle S^z\rangle D(\mathbf{k}'',\mathbf{k}')D(\mathbf{k}',\mathbf{k}'')[\zeta_{\mathbf{k}''}(-)-\zeta_{\mathbf{k}''}(+)]\delta_{\mathbf{k}+\mathbf{k}'',\mathbf{k}'}\delta(E-\epsilon_{\mathbf{k}',+}+\epsilon_{\mathbf{k}'',-}) \quad (3.12)$$

($\mathcal{P}$ indicates the principal value).

From our discussion in II it follows that $R_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ can be considered to give the energy shift and damping of the boson-like excitations.

If we now compare (3.10) with (I3.10), we see that we can apply the same reasoning as in I, and the result is that we can use Eqs. (I3.16), (I3.21) to (I3.25), but with $\Phi(S)$ replaced by $\bar{\Phi}(S)$ given by

$$\bar{\Phi}(S)=(1/N)\sum_{\mathbf{k}}[\exp(\beta\bar{E}_{\mathbf{k}})-1]^{-1}, \quad (3.13)$$

where $\bar{E}_{\mathbf{k}}$ is the root of the equation

$$E-E_{\mathbf{k}}-R_{\mathbf{k}}(E)=0. \quad (3.14)$$

4. THE FERMION-LIKE EXCITATIONS

In the previous section we saw how we can use the results of I to obtain expressions for the magnetization. As $\bar{E}_{\mathbf{k}}$ depends on $\langle S^z\rangle$ we have, as in I, an implicit equation for $\langle S^z\rangle$ which we can solve in the same way as was done in I. However, $\bar{E}_{\mathbf{k}}$ also contains the $\bar{\zeta}_{\mathbf{k}}$, and we need to know these before we can discuss the final results. To find the $\bar{\zeta}_{\mathbf{k}}$ we shall study the Green functions $\langle\langle c_{\mathbf{k},\sigma};c_{\mathbf{k}',\sigma'}{}^\dagger\rangle\rangle^+$. The equation of motion for these functions is

$$\langle\langle c_{\mathbf{k},\sigma};c_{\mathbf{k}',\sigma'}{}^\dagger\rangle\rangle^+[E-e_{\mathbf{k}}+\mu-2\sigma g_s\mu_{\mathrm{B}}B] =(\delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}/2\pi)-\frac{\hbar}{N}\sum_{\mathbf{k}'',\mathbf{f}}e^{-i\mathbf{f}\cdot(\mathbf{k}-\mathbf{k}'')}D(\mathbf{k},\mathbf{k}'') \times[\langle\langle c_{\mathbf{k}'',-\sigma}S_{\mathbf{f}}{}^\sigma;c_{\mathbf{k}',\sigma'}{}^\dagger\rangle\rangle^+ -2\sigma\langle\langle S_{\mathbf{f}}{}^zc_{\mathbf{k}'',\sigma};c_{\mathbf{k}',\sigma'}{}^\dagger\rangle\rangle^+]. \quad (4.1)$$

We use the decoupling

$$\langle\langle S_f{}^z c_{k,\sigma}; c_{k',\sigma'}{}^\dagger\rangle\rangle^+ \doteq \langle S^z\rangle\langle\langle c_{k,\sigma}; c_{k'\sigma'}{}^\dagger\rangle\rangle^+, \quad (4.2)$$

and get from (4.1)

$$\langle\langle c_{k,\sigma}; c_{k',\sigma'}{}^\dagger\rangle\rangle^+[E-\epsilon_{k,\sigma}] = \frac{1}{2\pi}\delta_{k,k'}\delta_{\sigma,\sigma'} - \frac{\hbar}{N}\sum_{k'',f} e^{-if\cdot(k-k'')}D(\mathbf{k},\mathbf{k}'') \times\langle\langle c_{k'',-\sigma}S_f{}^\sigma; c_{k',\sigma'}{}^\dagger\rangle\rangle^+. \quad (4.3)$$

We discuss the influence of the last sum in the Appendix; it produces both an energy shift and damping of the fermion-like excitations. For the moment we shall neglect it. We then get (compare the discussion in reference 11) for the $\bar{\zeta}_k$ the Fermi distribution

$$\bar{\zeta}_k(\sigma) = [\exp(\beta\epsilon_{k,\sigma})+1]^{-1}. \quad (4.4)$$

If there were no s-d-coupling, i.e., $D=0$, $\epsilon_{k,+}=\epsilon_{k,-}$ and $\bar{\zeta}_k(+)=\bar{\zeta}_k(-)$. However, if $D\neq 0$ the system of conduction electrons is magnetized, and the total magnetization $M(\beta)$ is now given by the relation

$$M(\beta) = (Ng_d\mu_B/\hbar)\langle S^z\rangle + Ng_s\mu_B\Delta\bar{\zeta}, \quad (4.5)$$

where

$$\Delta\bar{\zeta} = (1/N)\sum_k[\bar{\zeta}_k(-)-\bar{\zeta}_k(+)]. \quad (4.6)$$

5. DISCUSSION

We saw that the magnetization will contain the energies $\bar{E}_k$ for which we have the equation [cf. Eqs. (3.14) and (3.6)]

$$\bar{E}_k \doteq E_k{}^{(S)} + \frac{\hbar^2}{N}\sum_{k'} D(\mathbf{k}',\mathbf{k}')[\bar{\zeta}_{k'}(-)-\bar{\zeta}_{k'}(+)] + R_k(\bar{E}_k). \quad (5.1)$$

The difference $\bar{\zeta}_{k'}(-)-\bar{\zeta}_{k'}(+)$ will be nonzero only if $\mathbf{k}'$ lies sufficiently close to the Fermi surface. If $\mathbf{k}_1'$ and $\mathbf{k}_2'$ are defined by the equations

$$\epsilon_{k_1',-}=\epsilon_{k_2',+}=0, \quad (5.2)$$

we have, first of all,

$$|\mathbf{k}_1'|-|\mathbf{k}_2'| \ll k_F, \quad (5.3)$$

where k_F is the wave number on the Fermi surface. This follows from the fact that

$$|\mathbf{k}_1'|-|\mathbf{k}_2'| = \frac{2g_s\mu_B B + 2\hbar\langle S^z\rangle D}{\mu}|k_F|, \quad (5.4)$$

where the chemical potential μ follows from the usual equation

$$\sum_k[\bar{\zeta}_k(+)+\bar{\zeta}_k(-)] = N_e. \quad (5.5)$$

(N_e: total number of conduction electrons.)

Secondly, we may assume that $\bar{\zeta}_{k'}(-)-\bar{\zeta}_{k'}(+)$ vanishes unless

$$|\mathbf{k}_1'| < |\mathbf{k}'| < |\mathbf{k}_2'|. \quad (5.6)$$

As this restricts $\mathbf{k}'$ to a rather narrow range, we shall replace in (5.1) $D(\mathbf{k}',\mathbf{k}')$ by its average value $\tilde{D}$, say, and we get

$$(\hbar^2/N)\sum_{k'} D(\mathbf{k}',\mathbf{k}')[\bar{\zeta}_{k'}(-)-\bar{\zeta}_{k'}(+)] \doteq \hbar^2\tilde{D}\Delta\bar{\zeta}. \quad (5.7)$$

As long as the normal condition $k_BT \ll \mu$ is satisfied, we have[12]

$$\Delta\bar{\zeta} = \tfrac{3}{2}(N_e/\mu)[\tilde{D}\hbar\langle S^z\rangle + g_s\mu_B B]. \quad (5.8)$$

From (3.11), (3.9), and (5.1), (5.7) we now get for $\bar{E}_k$ for the case $\mathbf{k}=0$

$$\bar{E}_0 = E_0{}^{(S)} + \hbar^2\tilde{D}\Delta\bar{\zeta} + \frac{2\hbar^3\langle S^z\rangle\tilde{D}^2\Delta\zeta}{\bar{E}_0 - 2g_s\mu_B B - z\langle S^z\rangle\hbar\tilde{D}}. \quad (5.9)$$

If we assume $g_d=2g_s=g_0$ we get from (5.9) two solutions:

$$\bar{E}_0 = g_0\mu_B B, \quad (5.10a)$$

$$\bar{E}_0 = g_0\mu_B B + 2\langle S^z\rangle\hbar\tilde{D} + \hbar^2\tilde{D}\Delta\bar{\zeta}. \quad (5.10b)$$

The situation is more complicated if $\mathbf{k}\neq 0$ and we refer to the paper by Vonsovskii and Izyumov[9] who solve a similar equation.

It is interesting to note that we get *two* boson-excitation branches: one, corresponding to (5.10a) without, and the other, corresponding to (5.10b) with, an energy gap.[13] Potapkov and Tyablikov did not find the first branch, and it is clear from (5.9) how this happens: At first sight the term arising from $R_k(\bar{E})$ looks like being *second order* in D, but it turns out to be only of first order for $\bar{E}_0=2g_s\mu_B B$ which is just the case for the gapless branch. From this it follows that the magnetization will at sufficiently low temperatures again show a spin-wave behavior, that is,

$$M(\beta) = M(\infty)[1-aT^{3/2}+\cdots], \quad (5.11)$$

and not

$$M(\beta) = M(\infty)[1-ae^{-\beta\Delta}T^{3/2}+\cdots], \quad (5.12)$$

as found by Potapkov and Tyablikov.

From (4.5) and (5.8) we get for the magnetization the expression

$$M(\beta) = \frac{3}{4}\frac{N_e}{\mu}g_0\mu_B B + N\frac{\langle S^z\rangle}{\hbar}\mu^*, \quad (5.13)$$

[12] See, for instance, A. J. Dekker, *Solid State Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1957), p. 219.

[13] The presence of the second branch is connected with the possibility of the conduction electron gas supporting spin-density (fluctuation) waves. These spin-density waves are through the s-d coupling coupled with the usual ferromagnetic spin waves; this renormalizes both the energies of the spin waves and those of the spin-density waves. The renormalization of the spin waves is seen in the second branch. It is clear from the physical nature of the spin-density waves that the case with $\mathbf{k}\neq 0$ differs from that with $\mathbf{k}=0$: In the former case, the spontaneous magnetization of the system as a whole will not be changed, as regions with a higher spin-polarization density will be balanced by those with a smaller density. However, in the $\mathbf{k}=0$ case the spin-polarization density will uniformly be lowered or increased.

This problem is at the moment studied by Dr. H. B. Callen and one of us (R.A.T.-K.) and we should like to express our thanks to Dr. H. B. Callen for pointing out that this point needs further discussion.

with μ^*, the effective magnetic moment of the d-electrons, given by

$$\mu^* = \mu_B[1+\tfrac{3}{2}(N_e/N\mu)\tilde{D}\hbar^2]. \tag{5.14}$$

With $\mu \sim 1$ to 10 eV, $N_e\tilde{D}\hbar^2/N \sim 10^{-14}-10^{-18}$ erg, we find that μ^* is a few percent larger than μ_B, in order of magnitude agreement with experiment.

A discussion of the behavior of $\gamma_{\mathbf{k}}(E)$, especially near the Curie temperature, has been given by one of us.[14]

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APPENDIX

In this Appendix we study the solution of Eq. (4.3) taking the terms involving the mixed Green functions into account. We use the notation

$$\langle\langle c_{\mathbf{k},\pm} S_{\mathbf{f}}^{\mp};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} = T_{\mathbf{k};\mathbf{k}',\sigma'}^{\pm}(\mathbf{f}). \tag{A1}$$

If we look at the equations of motion for these functions, we find that even after making the following decouplings,

$$\langle\langle S_{\mathbf{f}}^{z} S_{\mathbf{p}}^{\pm} c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} \doteqdot \langle S^{z}\rangle\langle\langle S_{\mathbf{p}}^{\pm} c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}, \quad \mathbf{f}\neq\mathbf{p}, \tag{A2}$$

$$\langle\langle S_{\mathbf{f}}^{-} S_{\mathbf{p}}^{+} c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} \doteqdot \langle\langle c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}\langle S_{\mathbf{f}}^{-} S_{\mathbf{p}}^{+}\rangle, \tag{A3}$$

$$\langle\langle c_{\mathbf{k}_1,\sigma}^{\dagger} c_{\mathbf{k}_2,\sigma} c_{\mathbf{k}_3,-\sigma} S_{\mathbf{p}}^{z};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} \doteqdot \langle c_{\mathbf{k}_1,\sigma}^{\dagger} c_{\mathbf{k}_2,\sigma}\rangle\langle S^{z}\rangle\langle\langle c_{\mathbf{k}_3,-\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}, \tag{A4}$$

$$\langle\langle c_{\mathbf{k}_1,\sigma}^{\dagger} c_{\mathbf{k}_2,\sigma} c_{\mathbf{k}_3,\sigma} S_{\mathbf{f}}^{\pm};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} \doteqdot (1-\delta_{\mathbf{k}_2,\mathbf{k}_3})[\bar{\zeta}_{\mathbf{k}_1}(\sigma)\delta_{\mathbf{k}_1,\mathbf{k}_2}\langle\langle c_{\mathbf{k}_3,\sigma} S_{\mathbf{f}}^{\pm};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+} - \bar{\zeta}_{\mathbf{k}_1}(\sigma)\delta_{\mathbf{k}_1,\mathbf{k}_3}\langle\langle c_{\mathbf{k}_2,\sigma} S_{\mathbf{f}}^{\pm};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}], \tag{A5}$$

neglecting terms of relative order N^{-1} (compare the discussion in II), using the approximation

$$\langle S_{\mathbf{f}}^{\pm}\rangle = 0, \tag{A6}$$

and writing

$$\langle S_{\mathbf{f}}^{-} S_{\mathbf{p}}^{+}\rangle = N^{-1}\textstyle\sum_{\mathbf{k}''} F(\mathbf{k}'') e^{i\mathbf{k}''\cdot(\mathbf{p}-\mathbf{f})}, \tag{A7}$$

$$T_{\mathbf{k};\mathbf{k}',\sigma'}^{\pm}(\mathbf{p}) = N^{-1}\textstyle\sum_{\mathbf{k}''} T_{\mathbf{k};\mathbf{k}',\sigma'}^{\pm}(\mathbf{k}'') e^{i(\mathbf{p}\cdot\mathbf{k}'')}, \tag{A8}$$

we cannot solve these equations. We find a relation between $T^{\pm}$ and the inverse lattice sum Σ given by the equation

$$\Sigma = N^{-1}\sum_{\mathbf{k}_1(\neq\mathbf{k})} D(\mathbf{k},\mathbf{k}_1) T_{\mathbf{k}_1;\mathbf{k}',\sigma'}^{\pm}(\mathbf{k}-\mathbf{k}_1+\mathbf{k}''). \tag{A9}$$

The terms involving this particular sum were dropped by Potapkov and Tyablikov and as a result their energy shifts and damping coefficients are different from ours.

Although we cannot solve for the $T^{\pm}$, it is possible to solve for the double sums $\Lambda_{\pm}$ given by the relation

$$\Lambda_{\pm} = N^{-1}\textstyle\sum_{\mathbf{k}''}\sum_{\mathbf{f}} e^{i(\mathbf{k}''-\mathbf{k})\cdot\mathbf{f}} D(\mathbf{k},\mathbf{k}'') T_{\mathbf{k}'';\mathbf{k}',\sigma'}^{\pm}(\mathbf{f}). \tag{A10}$$

These are actually the sums occurring in Eq. (4.3) after a few transformations. We can then finally solve for the $\langle\langle c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}$ and find

$$\langle\langle c_{\mathbf{k},\sigma};\, c_{\mathbf{k}',\sigma'}^{\dagger}\rangle\rangle^{+}[E_{\pm} - W_{\mathbf{k}}^{\pm}(\sigma)] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}/2\pi, \tag{A11}$$

where

$$W_{\mathbf{k}}^{\pm}(\sigma) = \epsilon_{\mathbf{k},\sigma} - 2\frac{\hbar\sigma}{N}\sum_{\mathbf{k}'}\sum_{\mathbf{k}''}\frac{|D(\mathbf{k},\mathbf{k}'')|^2\delta_{\mathbf{k}',\mathbf{k}-\mathbf{k}''}\{2\hbar^2\langle S^z\rangle[\bar{\zeta}_{\mathbf{k}''}(-\sigma)-\frac{1}{2}-\sigma]-2\sigma\hbar F(-\mathbf{k}')\}}{E_{\pm}-\epsilon_{\mathbf{k}'',\sigma}-2\sigma E_{\mathbf{k}'}^{0}+(\hbar^2/N)\sum_{\mathbf{k}_1(\neq\mathbf{k}'')}[D(\mathbf{k},\mathbf{k}_1)D(\mathbf{k}_1,\mathbf{k}'')/D(\mathbf{k},\mathbf{k}'')]\bar{\zeta}_{\mathbf{k}_1}(\sigma)}. \tag{A12}$$

From (A10) to (A12) we find that the excitation-energies are "renormalized" and are found from (A12) by putting $W^{\pm} = E_{\pm} = \tilde{\epsilon}_{\mathbf{k},\sigma}$, and taking the principal part of the double sum. The damping is obtained by taking the modulus of the double sum in (A12) and replacing the denominator by a delta function and $E_{\pm}$ by the value $\tilde{\epsilon}_{\mathbf{k},\sigma}$ found for the excitation energies.

As long as the damping is small compared to the energy shift, we may regard the averages of the conduction-electron occupation number as being given by a Fermi distribution, smeared out over a region of the order of the damping, and we can put [compare Eq. (4.4)]

$$\bar{\zeta}_{\mathbf{k}}(\sigma) \doteqdot [1+\exp\beta\tilde{\epsilon}_{\mathbf{k},\sigma}]^{-1}. \tag{A13}$$

[14] R. A. Tahir-Kheli, Phys. Letters (to be published).