ON THE VIBRATIONS OF ELASTIC SHELLS PARTLY FILLED WITH LIQUID.

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I. Introduction.

THE problem considered in this paper is chiefly of acoustical interest in relation to the theory of "musical glasses." This class of instrument consists of a series of thin-walled elastic shells whose gravest modes of vibration are tuned to form a musical seal by partially filling them with a liquid and are excited either by striking or by tangential friction on the ruins. The principal features of interest requiring elucidation are (1) the dependence of the pitch of the vibration upon the quantity of liquid contained in the vessel and (2) the mode of vibration of the liquid itself. These features are discussed in this paper for the three cases in which the elastic shell is respectively (1) a hemispherical shell, (2) a cylindrical vessel with a flat bottom and (3) a conical cup, these forms approximating more or less closely to those used in practice. The analytical expressions show that the motion of the liquid is very marked near the margin of the vessel and is almost imperceptible near the center and also at some depth inside the liquid. This feature becomes more and more marked in the case of the higher modes of vibration of the vessel. Numerical results have also been obtained and tabulated showing the theoretical relation between the quantity of liquid contained in the vessel and the vibration frequency. These show that the rapidity with which the frequency falls on addition of liquid is greatest when the vessel is nearly full, this being specially noticeable in the case of the higher modes of vibration.

The general principle of the analytical method used is similar to that adopted by Lord Rayleigh[1] in treating the two-dimensional case of a long cylinder completely filled with liquid which was studied by Auerbach.[2] This case has also been recently discussed by Nikoloi.[3] The lowering of

[1] Lord Rayleigh, Phil. Mag., XV., pp. 385–389 (1883). Scientific Papers, Vol. 2, pp. 208–211.

[2] Auerbach, Wied. Ann., 3, p. 157, 1878, and also Wied. Ann., 17, p. 964, 1882. Reference may also be made to the papers by Montigny, Bull. del' Acad. de Belg., [2], 50, 159, 1880, and by Koläcek, Wied. Ann., 7, 23, 1879, and also Sitz. math. naturw. d. Wien, 87, Abath. 2, 1883.

[3] Nilokoi, Journ. Russk. Fisik Chimicesk, 41, 5, pp. 214–227, 1909.

the pitch produced by the liquid is of course due to the added inertia exactly as in the related case of the vibrations of a bar or a string immersed in a liquid which have been studied by Northway,[1] Mackenzie and Kalähue.[2]

Musical glasses are sometimes excited by rotating them about a fixed vertical axis, the tangential friction being produced by a rubber kept in a fixed position. No attempt is made in this paper to consider this somewhat complicated case,[3] which I hope to be able to deal with on a future occasion.

2. Hemispherical Cups.

The force which a thin sheet of matter subjected to stress opposes to extension is very great in comparison with that which it opposes to bending. From this Lord Rayleigh concluded that the middle surface of a vibrating shell remains unstretched and proposed a theory[4] of flexural vibrations of curved plates and shells in accordance with this condition. As the direct application of the Kirchhoff-Gehring method led to equations of motion and boundary conditions which were difficult to reconcile with Lord Rayleigh's theory, his theory gave rise to much discussion. Later investigations have, however, shown that any extension that may occur must be limited to a region of infinitely small area near the edge of the shell and that the greater part of the shell vibrates according to Lord Rayleigh's type.

Let the radius of the hemisphere be equal to a. Let a point whose natural coördinates are a, θ, ϕ be displaced to the position $a + u$, $\theta + v$, $\phi + w$, where u, v, w are to be treated as small.

From the condition of inextension

$$\begin{aligned}(\delta s)^2 &= a^2(\delta\theta)^2 + a^2 \sin^2\theta(\delta\phi)^2 \\ &= (a + u)^2(\delta\theta + \delta v)^2 + (\delta u)^2 + (a + u)^2 \sin^2(\theta + v)\,(\delta\phi + \delta w)^2, \quad (1)\end{aligned}$$

Lord Rayleigh obtains the three differential equations

$$\frac{\partial v}{\partial \theta} + \frac{u}{a} = 0,$$

[1] Northway and Mackenzie, Phys. Rev., 13, pp. 145–164, 1901.

[2] Kalähne, Ann. d. Physik, 46, 1, pp. 1–38, 1914.

[3] Reference may be made in this connection to papers by Prof. Love on "The Free and Forced Vibrations of an Elastic Spherical Shell Containing a Given Mass of Liquid," Proc. Lond. Math. Soc., Vol. XIX., where he has studied the case of a rotating spherical shell completely full of liquid, and by Prof. Bryan on "The Beats in the Vibrations of a Revolving Cylinder or Bell," Proc. Comb. Phil. Soc., Vol. VII., 1892.

[4] Lord Rayleigh, Proc. Lond. Math. Soc., Vol. XIII., p. 4, 1881. See also Proc. Roy. Soc., Vol. 45, pp. 45 and 443, 1881, Theory of Sound, Vol. I., Chap. XV, and Love's Elasticity, Chap. XXXIII.

$$\frac{\partial v}{\partial \phi} + \sin^2\theta \frac{\partial w}{\partial \theta} = 0,$$

$$\frac{u}{a} + \cot\theta \cdot v + \frac{\partial w}{\partial \phi} = 0,$$

which can be integrated in the forms

$$\frac{u}{a} = \begin{matrix} \sin \\ -\cos \end{matrix} m\phi[A_m(m + \cos\theta)\tan^m \tfrac{1}{2}\theta - B_m(m - \cos\theta)\cot^m \tfrac{1}{2}\theta],$$

$$\frac{v}{\sin\theta} = \begin{matrix} -\sin \\ \cos \end{matrix} m\phi[A_m \tan^m \tfrac{1}{2}\theta - B_m \cot^m \tfrac{1}{2}\theta],$$

$$w = \begin{matrix} \cos \\ \sin \end{matrix} m\phi[A_m \tan^m \tfrac{1}{2}\theta + B_m \cot^m \tfrac{1}{2}\theta],$$

A_m and B_m being arbitrary constants. These equations determine the character of the displacement of a point in the middle surface.

Since the pole $\theta = 0$ is included the constant B_m must be considered to vanish and the type of vibrations in a principal mode is expressed by the equations

$$\begin{aligned} u &= A_m a(m + \cos\theta)\tan^m \tfrac{1}{2}\theta \sin m\phi, \\ v &= -A_m \sin\theta \tan^m \tfrac{1}{2}\theta \sin m\phi, \\ w &= A_m \tan^m \tfrac{1}{2}\theta \cos m\phi, \end{aligned} \tag{4}$$

in which A_m is proportional to a simple harmonic function of the time.

The potential energy of bending of the vibrating shell is given by

$$\begin{aligned} V &= \frac{8}{3}\pi\mu\frac{\tau^3}{a^2}m^2(m^2 - 1)^2 A_m^2 \int_0^{\pi/2} \tan^{2m}\frac{\theta}{2}\frac{d\theta}{\sin^3\theta} \\ &= \frac{2}{3}\pi\mu\frac{\tau^3}{a^2}(m^3 - m)(2m^2 - 1)A_m^2, \end{aligned} \tag{5}$$

where τ = thickness of the shell and μ = rigidity.

The kinetic energy T is given by the expression

$$\begin{aligned} T &= \frac{1}{2}\pi\sigma a^4\tau\left(\frac{dA_m}{dt}\right)^2 \int_0^{\pi/2} \sin\theta\{2\sin^2\theta + (\cos\theta + m)^2\}\tan^{2m}\tfrac{1}{2}\theta d\theta \\ &= \frac{1}{2}\pi\sigma a^4\tau\left(\frac{dA_m}{dt}\right)^2 \int_1^2 \frac{(2 - x)^m}{2^m}[(m - 1)^2 + 2(m + 1)x - x^2]dx \\ &= \frac{1}{2}\pi\sigma a^4\tau f(m)\left(\frac{dA_m}{dt}\right)^2, \end{aligned} \tag{6}$$

where σ represents the density of the shell, and

$$f(m) = \int_1^2 \frac{(2 - x)^m}{x^m}[(m - 1)^2 + 2(m + 1)x - x^2]dx, \tag{7}$$

which can be evaluated for any integral value of m.

Since the types of vibrations of the shell are entirely determined by the geometry of the middle surface of the shell, it is obvious that the types can under no circumstances be altered by the presence of the liquid in the shell. The liquid gives rise to a surface traction and affects only the arbitrary constant A_m, that is to say, the amplitude and the frequency of vibration of the shell.

The motion of the liquid will depend upon a velocity potential which satisfies the equation

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{\operatorname{cosec}^2\theta}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}=0. \tag{8}$$

A solution of this differential equation which will correspond to the type of vibration of the shell can be obtained by assuming Φ to be of the form

$$\Phi=\left(C_m r+\frac{D_m}{r^2}\right)\sin m\phi\cdot\Delta_\theta,$$

where Δ_θ is a function of θ only. Substituting in the differential equation we find that Δ_θ satisfies the equation

$$\frac{d^2\Delta_\theta}{d\theta^2}+\cot\theta\frac{d\Delta_\theta}{d\theta}+(2-m^2\operatorname{cosec}^2\theta)\Delta_\theta=0.$$

The general solution of this differential equation is

$$\Delta_\theta=E_m\tan^m\tfrac{1}{2}\theta(m+\cos\theta)+F_m\cot^m\tfrac{1}{2}\theta(m-\cos\theta).$$

Neglecting solutions of the type $\cot^m\frac{1}{2}\theta(m-\cos\theta)$, we see that Φ is of the form

$$\Phi=\left(C_m r+\frac{D_m}{r^2}\right)\sin m\phi\tan^m\tfrac{1}{2}\theta(m+\cos\theta), \tag{9}$$

where C_m and D_m are two arbitrary constants. Let us first take

$$\Phi=C_m\frac{r}{a}\tan^m\tfrac{1}{2}\theta(m+\cos\theta)\sin m\phi\cos pt. \tag{10}$$

The relation between C_m and A_m of (4) is readily found by equating the value of $\partial\Phi/\partial r$, when $r=a$, to $\partial u/\partial t$, both of which represent the normal velocity at the circumference. We get

$$C_m\cos pt=a^2\frac{dA_m}{dt}. \tag{11}$$

The expression (10) determines the principal mode of vibration of the liquid. The simple character of the fluid motion as determined by this expression will however be a little disturbed on account of the existence of a free surface and we shall have to add a small correction to this expression. The condition to be satisfied at the free surface is

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0, \text{ when } z = h,$$

where h denotes the depth of the liquid surface below the center of the hemisphere. We shall neglect the force of gravity, inasmuch as the period of free waves of length comparable with the diameter of the shell is much greater than that of the actual motion. The condition to be satisfied at the free surface then becomes simply

$$\Phi = 0, \quad \text{when } z = h.$$

Hence we must have

$$\Phi = C_m\frac{r}{a}\tan^m \tfrac{1}{2}\theta(m + \cos\theta)\sin m\phi\cos pt + f(r, \theta, \phi)\cos pt, \tag{12}$$

where $f(r, \theta, \phi)$ is a solution of $\Delta^2\Phi = 0$ and is such that its differential coefficient with respect to r vanishes on the spherical boundary and it has the value

$$-C_m\frac{h\sec\theta}{a}\tan^m \tfrac{1}{2}\theta(m + \cos\theta)\sin m\phi \tag{13}$$

on the free surface.

In the particular case, when the shell is completely full of liquid, the differential coefficient of $f(r,\theta,\phi)$ with respect to r vanishes on the spherical surface and $f(r, \theta, \phi)$ has the value

$$-mC_m\frac{r}{a}\sin m\phi \tag{14}$$

on the surface defined by $\theta = \pi/2$.

For the determination of the function $f(r, \theta, \phi)$, spherical harmonics of the complex degree $-\frac{1}{2} + p\sqrt{-1}$ are extremely suitable. The properties of these harmonics and their applications to some physical problems have been investigated by Hobson.[1] Solutions of Laplace's equation of the form

$$\frac{1}{\sqrt{r}}\sin(p\log Ar)\frac{\sin}{\cos}m\phi K_p^m(\cos\theta),$$

where $K_p^m(\cos\theta)$ is a harmonic of degree $-\frac{1}{2} + p\sqrt{-1}$ and order m, and is defined by the hypergeometric series

$$K_p^m(\cos\theta) = F(m + \tfrac{1}{2} + pi, m + \tfrac{1}{2} - pi, m + 1, \sin^2 \tfrac{1}{2}\theta),$$

are suitable for our present purpose. These solutions are finite and continuous for all points in the space inside the hemispherical shell (except

[1] Hobson, "On a Class of Spherical Harmonics of Complex Degree with Application to Physical Problems," Trans. Camb. Phil. Soc., Vol. 14, pp. 212–236, 1889.

infinitely near the origin which may be supposed excluded by surrounding it by an infinitely small sphere).

Let us assume

$$f(r, \theta, \phi) = \Sigma_p B_p \sqrt{\frac{a}{r}} \sin\left(p \log\frac{r}{h}\right) \frac{K_p{}^m(\cos\theta)}{K_p{}^m(\cos\alpha)} \sin m\phi, \qquad (15)$$

where $h/a = \cos\alpha$. Then

$$\frac{\partial}{\partial r} f(r, \theta, \phi) = 0,$$

when $r = a$, if the p's are the roots of the equation

$$\tan(p \log a/h) - 2p = 0,$$

and the summation in the above series extends over all the roots of this equation.

The values of the constant B_p have to be obtained from the equation

$$-Cm\frac{\cos\alpha}{\cos\theta}\tan^m \tfrac{1}{2}\theta(m + \cos\theta) = \Sigma_p B_p \sqrt{\frac{a}{h}} \frac{\sin[p \log(\sec\theta)]}{\sqrt{\sec\theta}} \frac{K_p{}^m(\cos\theta)}{K_p{}^m(\cos\alpha)},$$

which must be satisfied for all values of θ between the limits $0 < \theta < \alpha$. Approximate values of the constants B_p's can be easily obtained from this equation. In the particular case when the shell is completely full of liquid, the values of the constants B_p's can be obtained in a very simple form. Since the origin is a singular point, we exclude the point by surrounding it with a small sphere of radius ϵ, and assume that $f(r, \theta, \phi)$ vanishes on the surface of this sphere. Since in this case $\alpha = \pi/2$, we can assume

$$f(r, \theta, \phi) = \Sigma_p B_p \sqrt{\frac{a}{r}} \sin\left(p \log\frac{r}{\epsilon}\right) \frac{K_p{}^m(\cos\theta)}{K_p{}^m(0)} \sin m\phi,$$

where

$$K_p{}^m(0) = \frac{\sqrt{\pi}\,\Pi(m)}{\Pi(\frac{1}{2}m - \frac{1}{4} + \frac{1}{2}pi)\Pi(\frac{1}{2}m - \frac{1}{4} - \frac{1}{2}pi)}$$

$$= \frac{\sqrt{\pi}\,\Pi(m)}{\left\{\frac{(2m-1)^2 + p^2}{2^2}\right\}\left\{\frac{(2m-3)^2 + p^2}{2^2}\right\}\cdots}$$

and the summation extends for all values of p which are the roots of the equation

$$\frac{d}{da}\left[\frac{\sin[p \log(a/\epsilon)]}{\sqrt{a}}\right] = 0,$$

that is to say, the equation

$$\tan\left(p \log\frac{a}{\epsilon}\right) - 2p = 0. \qquad (16)$$

The constants B_p's have to be determined by the condition that $f(r, \theta, \phi)$ must have the value $-mC_m(r/a)\sin m\phi$ on the free surface which is given by $\theta = \pi/2$. Hence B_p's are given by

$$-mC_m\frac{r}{a} = \Sigma_p B_p\sqrt{\frac{a}{r}}\sin\left(p\log\frac{r}{\epsilon}\right).$$

Putting $r = \epsilon\cdot e^{\lambda}$, it is easy to see that

$$B_p = -\frac{2(p^2+\frac{1}{4})\cdot mC_m}{p^2\log(a/\epsilon)+\frac{1}{2}[\frac{1}{2}\log(a/\epsilon)-1]}\left(\frac{\epsilon}{a}\right)^{3/2}\int_0^{\log a/\epsilon} e^{3/2\lambda}\sin p\lambda d\lambda$$

$$= -\frac{8(4p^2+1)\cdot mC_m}{4p^2\log(a/\epsilon)+[\log(a/\epsilon)-2]}\left[\frac{\sin[p\log(a/\epsilon)]+p(\epsilon/a)^{3/2}}{9+4p^2}\right]. \quad (17)$$

To obtain an idea of the magnitude of the constant B_p, we shall obtain its value when a/ϵ is a very large quantity. It is easy to see by the method of successive approximation that the roots of the equation (16) are given by

$$p\log\frac{a}{\epsilon} = X - \frac{1}{2X}\log\frac{a}{\epsilon} - \left(\log\frac{a}{\epsilon}\right)^2\left[\frac{1}{4}-\frac{1}{24}\log\frac{a}{\epsilon}\right]\frac{1}{X^3}$$
$$-\left(\log\frac{a}{\epsilon}\right)^3\left[\frac{1}{4}-\frac{1}{12}\log\frac{a}{\epsilon}+\frac{1}{160}\left(\log\frac{a}{\epsilon}\right)^2\right]\frac{1}{X^5}$$
$$-\text{ etc.},$$

where $X = (s+\frac{1}{2})\pi$, s being any integer. Now, if we take $a/\epsilon = 10^5$, the roots are successively the following:

$$p_1 = \cdot 321,\ p_2 = \cdot 628,\ p_3 = \cdot 960,\ p_4 = 1.205, \text{ etc.}$$

Hence we easily find that the constants B_{p1}, B_{p2}, B_{p3}, etc., have approximately the values $B_{p_1} = -\cdot 09\ C_m$, $B_{p_2} = -\cdot 08\ C_m$, $B_{p_3} = -\cdot 06\ C_m$, etc., from which we infer that the surface correction $f(r, \theta, \phi)$ is a small one. The principal mode ofvibration of the liquid is therefore expressed by

$$\Phi = C_m\frac{r}{a}\tan^m\tfrac{1}{2}\theta(m+\cos\theta)\sin m\phi\cos pt. \quad (18)$$

If q represent the velocity of the liquid as given by this expression, we have

$$q^2 = C_m{}^2\frac{1}{a^2}\tan^{2m-2}\tfrac{1}{2}\theta[(m+\cos\theta)^2(\sin^2 m\phi\tan^2\tfrac{1}{2}\theta+\tfrac{1}{4}m^2\cos^2 m\phi\sec^2\tfrac{1}{2}\theta)$$
$$+\{\tfrac{1}{2}m(m+\cos\theta)\sec^2\tfrac{1}{2}\theta-\tan\tfrac{1}{2}\theta\sin\theta\}^2\sin^2 m\phi]\cos^2 pt. \quad (19)$$

Since q is independent of r, the velocity of the liquid at any point in a given radius vector is constant. We see that the velocity varies as $\tan^{m-1}\frac{1}{2}\theta$. Hence if we move along any given meridian, the velocity increases from a zero value at the pole at first very slowly then rather abruptly to a large value at the surface, the abruptness of rise being greater the larger the quantity m, that is to say, the higher the mode of vibration of the liquid. Since the velocity of the liquid is constant along any given radius vector, we see that if we consider the motion of the liquid on the surface of a cone of semi-vertical angle θ, and trace the motion of the liquid as a whole as θ increases, the velocity remains small as θ increases and assumes a large value only at or near the surface. It is obvious therefore that in every case when the cup is not quite filled to the brim, the velocity of the liquid has a very large value near the margin of the vessel and is almost imperceptible near the center and at some depth in the liquid. In the particular case, when the shell is almost filled to the brim, the velocity of the liquid as given by this expression at a point which is near the center and also near the free surface is not small. But in this case the free surface correction $f(r, \theta, \phi)$ to the expression for the velocity potential becomes of some importance and has a sign opposite to it. Consequently the velocity of the liquid near the center always remains very small. These indications of theory are all confirmed by experiment.

To calculate the kinetic energy of the liquid, we have to integrate $\Phi \times \partial\Phi/\partial n$ over the whole boundary of the fluid. At the free surface $\Phi = 0$. We have therefore only to consider the spherical surface.

Therefore

$$\begin{aligned} T &= \tfrac{1}{2}\rho \int\int \Phi \frac{\partial \Phi}{\partial n} dS \\ &= \tfrac{1}{2}a\rho \cos^2 pt \int_0^{2\pi}\int_0^{\alpha} [C_m \sin m\phi \tan^m \tfrac{1}{2}\theta(m + \cos\theta) + f(r, \theta, \phi)] \\ &\quad \times C_m \sin m\phi \tan^m \tfrac{1}{2}\theta(m + \cos\theta) \sin\theta d\theta d\phi \\ &= \frac{\pi}{2} a\rho \cos^2 pt\, C_m^2 \int_0^{\alpha} \tan^{2m} \tfrac{1}{2}\theta(m + \cos\theta)^2 \sin\theta d\theta \\ &\quad + \frac{\pi}{2} a\rho \cos^2 pt\, C_m \sum_p \beta_p \frac{\sin[p \log(a/h)]}{K_p^m(\cos\alpha)} \\ &\qquad\qquad \times \int_0^{\alpha} \tan^m \tfrac{1}{2}\theta(m + \cos\theta) K_p^m(\cos\theta) \sin\theta d\theta, \end{aligned} \tag{20}$$

ρ being the density of the liquid.

Since the liquid is supposed to be incompressible, the potential energy is zero.

The sum of the kinetic and potential energies of the solid and liquid together must be independent of the time. Thus we get

$$\left[a^5\rho \int_0^{\alpha} \tan^{2m} \tfrac{1}{2}\theta(m + \cos\theta)^2 \sin\theta d\theta + a^5\rho K + a^4\tau\sigma f(m)\right] \frac{d^2 A_m}{dt^2} + \frac{4}{3}\mu \frac{\tau^3}{a^2}(m^3 - m)(2m^2 - 1)A_m = 0,$$

where

$$K = \sum_p \frac{B_p}{C_m} \frac{\sin[p \log (a/h)]}{K_p{}^m (\cos\alpha)} \int_0^\alpha \tan^m \tfrac{1}{2}\theta(m + \cos\theta) K_p{}^m(\cos\theta)\sin\theta d\theta$$

$$= \sum_p \frac{B_p}{C_m}\frac{4\sin^2\alpha}{9+4p^2}\cdot\frac{\sin[p\log(a/h)]}{K_p{}^m(\cos\alpha)}\Big[K_p{}^m(\cos\alpha)\frac{d}{d\cos\alpha}\{\tan^m\tfrac{1}{2}\alpha(m+\cos\alpha)\}$$

$$-\tan^m\tfrac{1}{2}\alpha(m+\cos\alpha)\frac{d}{d\cos\alpha}K_p{}^m(\cos\alpha)\Big].$$

If we put

$$F(\alpha, m) = \int_0^\alpha \tan^{2m}\tfrac{1}{2}\theta(m+\cos\theta)^2\sin\theta d\theta$$

$$= \int_{1+\cos\alpha}^{2}\left(\frac{2-x}{x}\right)^m (m-1+x)^2 dx,$$

and if A_m varies as $\cos(pt+\epsilon)$, we get

$$[a\rho F(\alpha, m) + \tau\sigma f(m) + \alpha\rho K]p^2 = \frac{4}{3}\frac{\mu}{\alpha^3}\left(\frac{\tau}{\alpha}\right)^3 (m^3 - m)(2m^2 - 1). \quad (21)$$

This equation gives the frequency of vibration of the shell with different quantities of liquid.

The fall of pitch for the three gravest tones given by $m = 2$, $m = 3$ and $m = 4$ for a brass hemispherical shell 10 cm. in radius, 2 mm. in thickness and of density 8.6 with different quantities of liquid are shown in Table I. In Fig. 1, the frequencies have been plotted against the quantity of water in the vessel for these three modes of vibrations.

Table I.

α.	Quantity of Water in the Shell.	$m = 2$.			$m = 3$.			$m = 4$.		
		$F(\alpha, m)$.	$f(m)$.	$p \times$ Const.	$F(\alpha, m)$.	$f(m)$.	$p \times$ Const.	$F(\alpha, m)$.	$f(m)$.	$p \times$ Const.
90°	$\pi a^3 \times .667$	1.114	1.53	1.80	1.580	1.88	4.63	2.030	2.296	8.76
80°	$\pi a^3 \times .494$	.570	"	2.24	.641	"	6.50	.479	"	14.51
70°	$\pi a^3 \times 338$	.291	"	2.75	.125	"	9.53	.097	"	19.42
60°	$\pi a^3 \times 208$	.123	"	3.29	.032	"	10.57	.009	"	21.45
50°	$\pi a^3 \times .133$	.058	"	3.61	.008	"	10.95	.002	"	21.64
40°	$\pi a^3 \times .034$	.026	"	3.81	.003	"	11.14	.000	"	21.68
30°	$\pi a^3 \times .014$	.015	"	3.88	.001	"	11.22	.000	"	21.69
20°	$\pi a^3 \times .003$	.013	"	3.90	.000	"	11.23	.000	"	21.69
10°	$\pi a^3 \times .001$	.011	"	3.91	.000	"	11.23	.000	"	21.69
0°	0	0	"	3.93	0	"	11.24	0	"	21.69

The frequencies of a brass hemispherical shell of about the same radius and thickness loaded with different quantities of water have also been determined experimentally by a photographic method. The results showed a general agreement with the calculated values. As a shell of uniform thickness and of uniform elastic properties throughout could not be procured, and the one that was used was very much deficient in these respects, the slight discrepancy that was noticed between the calculated and the observed values of the frequency, was probably due to these defects.

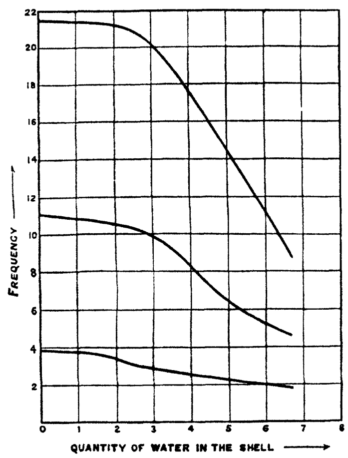

Fig. 1.

3. Cylindrical Cups.

The problem of the flexural vibrations of a cylindrical shell is considered in Lord Rayleigh's Theory of Sound, Vol. I., §235C. If the displacements at any point a, θ, z of the cylinder be δr, $a\delta\theta$, δz, then

$$\begin{aligned} \delta r &= -\ n(A_n a + B_n z) \sin n\theta, \\ a\delta\theta &= \quad (A_n a + B_n z) \cos n\theta, \\ \delta z &= -\ n^{-1} B_n a \sin n\theta. \end{aligned} \tag{22}$$

Supposing now that the cup has been formed by an inextensible disk being attached to the cylinder at $z = 0$, the displacements δr, $a\delta\theta$ must vanish for that value of z. Hence $A_n = 0$, and

$$\delta r = -\ nB_n z \sin n\theta,\ a\delta\theta = B_n z \cos n\theta,\ \delta z = -\ n^{-1}B_n a \sin n\theta, \tag{23}$$

the constant B_n is proportional to a simple harmonic function of the time, say, $\cos pt$.

Since the displacements δr and $a\delta\theta$ are proportional to z and the displacement δz is independent of z, it is obvious, that when z is large the displacements δr and $a\delta\theta$ are also very large compared to δz, that is to say, near the free end of the shell, the displacement δz is negligible compared to δr or $a\delta\theta$. But at the bottom of the shell, the displacements δr and $a\delta\theta$ vanishes and δz remains constant. We conclude, therefore, from the law of continuity, that the disk at the bottom of the shell must have a small normal vibration. If w denote the normal displacement of the disk, it is well known that w satisfies the differential equation

$$\frac{\partial^2 w}{\partial t^2} + c^4 \nabla^4 w = 0, \tag{24}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2} \text{ and } C^4 = \frac{E\tau^{12}}{3\sigma(1 - \epsilon^2)},$$

E being Young's modulus, σ the density, τ^1 the thickness and ϵ the Poisson's ratio.

If $w \alpha \cos(pt + e)$, then the equation becomes

$$\nabla^4 w - v^4 w = 0,$$

where $v^4 = p^2/c^4$. A solution of this differential equation is known to be

$$w = C_n J_n(vr) \sin n\theta.$$

Hence we shall take

$$w = C_n J_n(vr) \sin n\theta \cos pt. \tag{25}$$

The value of the constant C_n can be obtained from the condition that w and δz must be continuous at the boundary.

This gives

$$C_n \cos pt = -\frac{B_n}{n} \frac{a}{J_n(va)}. \tag{26}$$

We can assume that $J_n(va)$ is very large and consequently that the normal vibration of the disk is very small. The potential energy of deformation for a length l of the cylinder is

$$V = \frac{4\pi\mu\tau^3}{3a}(n^2 - 1)^2\left[\frac{\lambda + \mu}{\lambda + 2\mu}\frac{n^2 l^2}{3a^2} + 1\right]B_n^2. \tag{27}$$

The potential energy of vibration of the disk is given by

$$\frac{E\tau^{13}}{3(1 - \epsilon^2)}\int_0^a \int_0^{2\pi}\left[(\nabla^2 w)^2 - 2(1 - \epsilon)\left\{\frac{\partial^2 w}{\partial n^2}\frac{\partial^2 w}{\partial y^2}\right\} - \left(\frac{\partial^2 w}{\partial n \partial y}\right)^2\right] r d\theta dr.$$

The value of this integral can be easily obtained. But as we regard the vibration of the disk compared to that of the cylindrical surface to be very small, the value of this expression is also very small.

If the volume density be σ, we get the expression for the kinetic energy in the form

$$T = \tfrac{1}{2}\pi\sigma\tau l a[\tfrac{1}{3}l^2(1 + n^2) + n^{-2}a^2]\left(\frac{dB_n}{dt}\right)^2 + \tfrac{1}{2}\pi\sigma\tau' \frac{a^2}{n^2[J_n(va)]^2}\int_0^a [J_n(vr)]^2 r dr \left(\frac{dB_n}{dt}\right)^2. \tag{28}$$

If the cylinder contains frictionless incompressible fluid, the motion of the liquid will depend upon a velocity potential Φ which satisfies the equation $\nabla^2\Phi = 0$, or in cylindrical coördinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0.$$

The solution of this differential equation can be written in either of the forms

$$\Phi = \alpha_n zr^n \sin n\theta \cos pt, \tag{29}$$

$$\Phi = \beta_n e^{-kz} J_n(kr) \sin n\theta \cos pt. \tag{30}$$

The boundary conditions to be satisfied by Φ are

$$\text{(i)} \quad \frac{\partial \Phi}{\partial r} = \frac{d\delta r}{dt}, \text{ when } r = a;$$

$$\text{(ii)} \quad \frac{\partial \Phi}{\partial z} = \frac{dw}{dt}, \text{ when } z = 0; \tag{31}$$

$$\text{(iii)} \quad \Phi = 0, \text{at the free surface, } u, \text{ when } z = h.$$

We assume that

$$\Phi = \alpha_n zr^n \sin n\theta \cos pt + \sum_k J_n(kr)[D_k \cosh kz + E_k \sinh kz] \sin n\theta \cos pt, \tag{32}$$

where the summation extends for all values of k which are the roots of the equation

$$\frac{d}{da} J_n(ka) = 0. \tag{33}$$

We at once get by condition (*i*)

$$\alpha_n \cos pt = -\frac{1}{a^{n-1}} \frac{dB_n}{dt}.$$

The condition (*ii*) gives

$$-\left[\frac{a}{n}\frac{J_n(vr)}{J_n(va)} - \frac{r^n}{a^{n-1}}\right]\frac{1}{k}\frac{dB_n}{dt} = \sum_k E_k J_n(kr) \cos pt.$$

This equation must be satisfied for all values of r between the limits $(0 < r < a)$ and will give the value of the constant E_k. Now since

$$\int_0^a r^{n+1} J_n(kr)dr = \frac{na^n}{k^2} J_n(ka),$$

$$\int_0^a J_n(kr)J_n(vr)rdr = \frac{av}{k^2 - v^2} J_n(ka)J_n'(va),$$

and

$$\int_0^a [J_n(kr)]^2 rdr = \tfrac{1}{2}a^2\left(1 - \frac{n^2}{k^2a^2}\right)[J_n(ka)]^2,$$

we get

$$-\frac{1}{k}\frac{dB_n}{dt}\left[\frac{a}{nJ_n(va)}\int_0^a J_n(kr)J_n(vr)rdr - \frac{1}{a^{n-1}}\int_0^a r^{n+1}J_n(kr)dr\right.$$
$$= E_k \cos pt \int_0^a [J_n(kr)]^2 rdr,$$

and therefore

$$E_k \cos pt = -\frac{2a}{ka}\frac{dB_n}{dt}\frac{k^2a^2}{(k^2a^2 - n^2)J_n(ka)}\left[\frac{v}{(k^2 - v^2)n}\frac{J_n'(va)}{J_n(va)} - \frac{na}{k^2a^2}\right]. \tag{35}$$

The condition (iii) gives

$$\alpha_n h r^n + \sum_k (D_k \cosh kh + E_k \sinh kh)\, J_n(kr) = 0$$

for all values of r between the limits $(0 < r < a)$.
Therefore we get

$$D_k \cosh kh + E_k \sinh kh = -\,\alpha_n \frac{2na^n h}{(k^2a^2 - n^2)J_n(ka)}. \tag{36}$$

The equations (35) and (36) give the values of the constants D_k and E_k.

To calculate the kinetic energy we have to integrate $\Phi \times \partial\Phi/\partial n$ over the boundary of the shell. At the free surface $\Phi = 0$. We have therefore only to consider the cylindrical surface and the bottom. The expression can be written in the form

$$T = \tfrac{1}{2}\pi\rho \cos^2 pt \Bigg[n\alpha_n^2 a^{2n}\frac{h^3}{3} + n\alpha_n a^n \Sigma J_n(ka) \Bigg\{ D_k\Bigg(\frac{h \sinh kh}{k} - \frac{1}{k^2}\cosh kh + \frac{1}{k^2}\Bigg) + E_k\Bigg(\frac{h\cosh kh}{k} - \frac{1}{k^2}\sinh kh\Bigg)\Bigg\} - \sum_k \Bigg\{ \alpha_n D_k \frac{na^n}{k^2} J_n(ka) + \tfrac{1}{2}ka^2 E_k D_k \Bigg(1 - \frac{n^2}{k^2a^2}\Bigg)[J_n(ka)]^2 \Bigg\}\Bigg]. \tag{37}$$

The constants E_k and D_k are very small compared to α_n. If we neglect E_k and D_k, the expression for the kinetic energy reduces to the simple form

$$T = \frac{\pi}{2}\rho n \alpha_n^2 a^{2n} \frac{h^3}{3}\cos^2 pt. \tag{38}$$

In this case the expression for Φ reduces to the form

$$\Phi = \alpha_n z r^n \sin n\theta \cos pt. \tag{39}$$

This expression represents the principal mode of vibration of the liquid and all the other coexistent modes are very small compared to this one. Since the expression for the velocity varies as $(r/a)^{n-1}$, the velocity is very marked near the margin of the vessel and is almost imperceptible near the center. Using the principle that the sum of the kinetic and potential energies of the solid and liquid together must be independent of the time, we easily obtain an expression for the frequency of vibrations in the most general case from the expressions for the kinetic and potential energies already given. If we neglect E_k and D_k, the frequency equation takes a very simple form. The expression in this case is

$$[\sigma\tau l a\{\tfrac{1}{3}l^2(1 + n^2) + n^{-2}a^2\} + \tfrac{1}{3}\rho n h^3 a^2]p^2 = \frac{8\mu\tau^3 l}{3a}(n^2 - 1)^2\left[\frac{\lambda + \mu}{\lambda + 2\mu}\frac{n^2l^2}{3a^2} + 1\right]. \tag{40}$$

Thus we see that the law of variation of the frequency with the height of water in the vessel can be expressed in the form

$$p^2 = \frac{1}{A + B\,(h/l)^3},$$

where A and B are two constants for the vessel.

For a glass cylinder whose dimensions are given by $l/a = 4$, $\tau/a = .02$ and which has the density $\sigma = 2.6$ and the elastic constants $\mu = 1.8$ and $\lambda = 1.53$, we easily find that the frequency p_n is given by

$$\left[.052\left\{(1 + n^2) + \frac{.1875}{n^2}\right\} + n\left(\frac{h}{l}\right)^3\right] p_n^2 = \frac{8\mu\tau^3}{3a^2l^2}(n^2 - 1)^2\,[10.4\,n^2 + 3].$$

For the three gravest tones given by $n = 2$, $n = 3$ and $n = 4$, the values of the frequencies p_2, p_3 and p_4 with different quantities of water in the cylinder are shown in Table II.

TABLE II.

h/l.	$p_2 \times$ Const.	$p_3 \times$ Const.	$p_4 \times$ Const.
0	12.47	34.44	65.63
.1	12.33	34.40	65.48
.2	12.02	33.98	64.47
.3	11.29	32.05	61.95
.4	10.15	29.45	57.79
.5	8.85	26.26	52.45
.6	7.61	22.99	46.68
.7	6.56	19.97	41.09
.8	5.60	17.34	36.05
.9	4.97	15.13	31.67
1.0	4.21	13.25	27.93

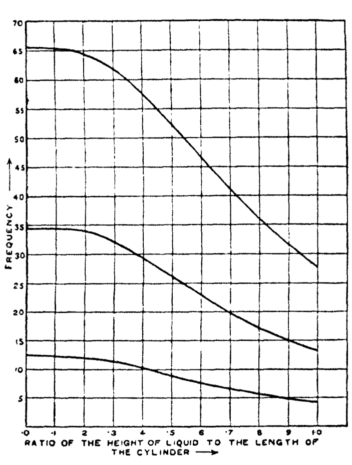

Fig. 2.

The values of the frequencies given in Table II. have been plotted in Fig. 2.

4. CONICAL CUPS.

It is shown in Lord Rayleigh's Theory of Sound in the article already referred to that if a cone for which $\rho = \tan\gamma\cdot z$, γ being the semi-vertical angle, executes flexural vibrations, the displacements $\delta\rho$, $\delta\phi$, δz at any point whose cylindrical coördinates are (ρ, ϕ, z) are given by

$$\delta\rho = n \tan\gamma\,(A_n z + B_n) \sin n\phi,$$

$$\delta\phi = (A_n + B_n z^{-1}) \cos n\phi,$$

$$\delta z = \tan^2\gamma[n^{-1}B_n - n\,(A_n z + B_n)] \sin n\phi.$$

If the cone be complete up to the vertex at $z = 0$, then $B_n = 0$, so that

$$\delta\rho = n \tan \gamma \cdot A_n z \sin n\phi,$$
$$\delta\phi = A_n \cos n\phi,$$
$$\delta z = - n A_n \tan^2 \gamma \cdot z \sin n\phi.$$

If the displacements in polar coördinates (r, θ, ϕ) be denoted by δr, $\delta\theta$, $\delta\phi$, we easily obtain

$$\delta\phi = A_n \cos n\phi$$
$$\delta r = \delta\rho \sin \gamma + \delta z \cos \gamma = 0,$$
$$r\delta\theta = \delta\rho \cos \gamma - \delta z \sin \gamma = n A_n \tan \gamma \cdot r \sin n\phi.$$

It is easy to see that the potential energy of deformation for a length l of the cone

$$W = \frac{4\pi}{3} \mu\tau^3 \frac{\lambda + \mu}{\lambda + 2\mu} A_n^2 \sin \gamma \left[\left(- n^3 \frac{\tan \gamma}{\sin^2 \gamma} + n \tan \gamma + n \cot \gamma \right)^2 + \cos^2 \gamma \right] \log \frac{2l}{\tau},$$

where τ = thickness.[1]

The expression for the kinetic energy of vibration of the shell can be easily obtained in the form

$$T = \frac{\pi}{8} \sigma\tau l^4 \sin^3 \gamma \, [n^2 \sec^2 \gamma + 1] \left(\frac{dA_n}{dt} \right)^2. \tag{45}$$

If the cup contains frictionless incompressible fluid, the velocity potential of the fluid must satisfy Laplace's equation. Let us assume that the velocity potential is given by

$$\Phi = C_n r^2 \phi_n (\cos \theta) \sin n\phi \cdot \cos pt \tag{46}$$

where $\phi n (\cos \theta)$ is a function of θ only. It is easy to see by substitution in the differential equation

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial \Phi}{\partial \theta} + \frac{\operatorname{cosec}^2 \theta}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0,$$

that $\phi n (\cos \theta)$ satisfies the equation

$$\frac{\partial^2 \phi_n}{\partial \theta^2} + \cot \theta \frac{\partial \phi_n}{\partial \theta} + (6 - n^2 \operatorname{cosec}^2 \theta) \phi_n = 0.$$

[1] This expression can be readily deduced from a very general expression for the potential energy due to strain in curvilinear coördinates obtained by Prof. Love. (Vide his paper on "The Small Free Vibrations and Deformation of a Thin Elastic Shell," Phil. Trans., Vol. 179, 1888, A.) The expression has been criticized by Prof. Basset (Phil. Trans., Vol. 181, 1890, A) on the ground that Prof. Love has omitted several terms which involve the extension of the middle surface. As the inextensional vibrations only have been considered in this paper, this criticism does not affect us in any way.

A solution of this differential equation can be easily obtained in the form

$$\phi n\,(\cos\theta) = \tan^n \tfrac{1}{2}\theta\left[\;(1-n)(2-n) - 6(2-n)\cos^2\frac{\theta}{2} + 12\cos^4\frac{\theta}{2}\right].$$

The relation between C_n and A_n can be easily obtained by equating the value of $\partial\Phi/r\partial\theta$, when $\theta = \gamma$, to $d(r\delta\theta)/dt$, both of which represent the normal velocity at the boundary. We thus get

$$C_n \cos pt\frac{\partial}{\partial\gamma}\left[\tan^n \tfrac{1}{2}\gamma\left\{(1-n)(2-n) - 6(2-n)\cos^2\frac{\gamma}{2} + 12\cos^4\frac{\gamma}{2}\right\}\right]$$
$$= n\tan\gamma\frac{dA_n}{dt}. \quad (48)$$

The principal mode of vibration of the liquid will therefore be expressed by (46) except for a small correction to be introduced on account of the existence of a free surface. At the free surface the condition to be satisfied is given by

$$\Phi = 0, \text{ when } z = h,$$

where h denotes the height of the liquid.

To satisfy this condition, we assume

$$\Phi = C_n r^2\phi_n(\cos\theta)\sin n\phi\cos pt + \sum_m D_m r^m P_m^n(\cos\theta)\sin n\phi\cos pt, \quad (49)$$

where the summation extends for all values of m which are the roots of the equation

$$\frac{\partial}{\partial\gamma}P_m^n(\cos\gamma) = 0. \quad (50)$$

The constants D_m's have to be determined by means of the equation

$$C_n(h\sec\theta)^2\phi_n(\cos\theta) + \sum_m D_m(h\sec\theta)^m P_m^n(\cos\theta) = 0, \quad (51)$$

which must be satisfied for all values of θ between the limits $0 < \theta < \gamma$. Approximate values of the constants D_m's can be eas ly obtained from this equation. To get an idea of the magnitude of the constant D_m, we shall obtain its value in the particular case when the semi-vertical angle γ of the cone is small and the height h of the liquid is large compared to the radius of the cross-section of the cone by the free surface. In this case the free surface can be taken to be practically coincident with the surface of the sphere $r = h$. The equation for determining D_m is then

$$C_n h^2\phi_n(\cos\theta) + \sum_m D_m h^m P_m^n(\cos\theta) = 0.$$

Now since

$$\int_{\cos\gamma}^{1} P_m^n(\cos\theta)P_{m'}^n(\cos\theta)\sin\theta d\theta = 0,$$

m, m^1 being two different roots of the equation (50),
and

$$\int_{\cos\gamma}^{1} [P_m{}^n (\cos\theta)]^2 \sin\theta d\theta = \frac{1 - \cos^2\gamma}{2m + 1} P_m{}^n (\cos\gamma) \frac{\partial^2}{\partial m \partial \cos\gamma} P_m{}^n (\cos\gamma),$$

we easily get

$$D_m = - \frac{2m + 1}{1 - \cos^2\gamma} \frac{C_n h^{2-m} \int_{\cos\gamma}^{1} \phi_n (\cos\theta) P_m{}^n (\cos\theta) \sin\theta d\theta}{P_m{}^n (\cos\gamma) \dfrac{\partial^2}{\partial m \partial \cos\gamma} P_m{}^n (\cos\gamma)} .$$

Neglecting the small correction introduced by the free surface, we see that the kinetic energy of the fluid motion is

$$\frac{\pi}{2} \rho C_n{}^2 \cos^2 pt \phi_n (\cos\gamma) \frac{\partial\phi_n (\cos\gamma)}{\partial\gamma} \sin\gamma \sec^5\gamma \frac{h^5}{5} .$$

Since the sum of the kinetic and potential energies of the solid and liquid together must be independent of the time, we easily obtain, on assuming that $A_n \alpha \cos pt$, the frequency equation in the form

$$\left[\tfrac{1}{4}\sigma\tau l^4 \sin^3\gamma (n^2 \sec^2\gamma + 1) + \rho n^2 \tan^2\gamma \sin\gamma \frac{\phi_n (\cos\gamma)}{\dfrac{\partial}{\partial\gamma} \phi_n (\cos\gamma)} \frac{H^5}{5} \right] p^2$$

$$= \frac{8}{3} \mu\tau^3 \frac{\lambda + \mu}{\lambda + 2\mu} A_n{}^2 \sin\gamma \left[\left(- n^3 \frac{\tan\gamma}{\sin^2\gamma} + n \tan\gamma + n \cot\gamma \right)^2 + \cos^2\gamma \right] \log\frac{2l}{\tau} .$$

where $H = h \sec\gamma$, H being the slant height of the liquid. In this case, we see that the law of variation of frequency with the height of liquid can be expressed in the form

$$p^2 = \frac{1}{A + B(h/l)^5},$$

A and B being two constants for the particular shell.

The frequencies p_2, p_3 and p_4 with different quantities of water for the three gravest modes of vibrations given by $n = 2$, $n = 3$ and $n = 4$ have been calculated from this expression for a cone of semi-vertical angle 30°, the ratio of the thickness of the sides of the cone to the slant height being equal to .02 and are shown in Table III.

The curves showing the fall of frequency for these three modes of vibrations of the cone when loaded with different quantities are plotted in Fig. 3.

TABLE III.

H/l.	$p_2 \times$ Const.	$p_3 \times$ Const.	$p_4 \times$ Const.
0	5.030	13.58	26.75
.1	5.030	13.58	26.75
.2	5.028	13.57	26.73
.3	5.008	13.54	26.69
.4	4.937	13.41	26.47
.5	4.761	13.08	25.94
.6	4.428	12.43	24.87
.7	3.942	11.64	23.13
.8	3.399	10.07	20.79
.9	2.808	8.63	18.07
1.0	2.310	7.26	15.41

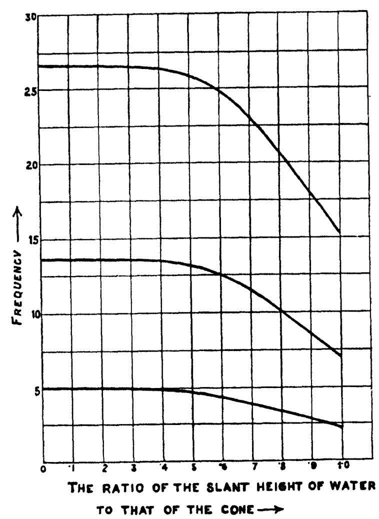

Fig. 3.

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