THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 129, NO. 3 — 1 FEBRUARY 1963

Resonances of a Small Plasma Sphere in a Magnetic Field

MANUEL CARDONA AND BRUCE ROSENBLUM

RCA Laboratories, Princeton, New Jersey

(Received 6 September 1962)

Resonances of a sphere of plasma in a magnetic field are discussed for a radius small compared to the wavelength of the incident radiation. There are two classes of resonances: radius independent and radius dependent. The former are the electrical analog of the resonant modes originally observed in ferrites by White and Solt. The latter are related to the electromagnetic resonances of a dielectric sphere. Although some of the approximations used in calculating the electromagnetic resonances are not obviously valid, they give remarkably good agreement with experiments on small spheres of indium antimonide. In particular, it is shown that a strong resonance is associated with a mode which is essentially a rotating magnetic dipole.

INTRODUCTION

THE effective dielectric constant of a plasma in a magnetic field is a tensor. The mathematical difficulty of solving boundary value problems involving finite plasmas in a magnetic field is therefore great, and very few have been solved. The resonances of a spherical plasma in a magnetic field is still an unsolved problem. We have observed such magnetoplasma resonances in small spheres of indium antimonide where the plasma consists of the free electrons and the immobile, positively charged, donor impurities.

The resonances were observed by placing a small sphere of indium antimonide in a waveguide or a resonant cavity and observing maxima in the absorption of K-band ($f\approx 25$ kMc/sec) radiation as a function of applied magnetic field. In order to reduce the scattering frequency the sample was cooled to about 60°K. At this temperature the scattering frequency was generally smaller than the plasma frequency and the cyclotron frequency, but almost a few times greater than the frequency of the exciting radiation. The resonances were studied as a function of sphere diameter, electron concentration, and the symmetry of the exciting microwave field. The electrons in indium antimonide have an isotropic effective mass equal to 0.014 times the free-electron mass. This low effective mass makes this material particularly useful for plasma studies since one can have a high plasma frequency with fewer carriers (and therefore fewer scattering impurities) and a high cyclotron frequency with a relatively small magnetic field.

The standard treatment of the electromagnetic resonances of a dielectric sphere[1] shows that these resonant modes may be divided into electric (TM) modes where the magnetic field is always perpendicular to the radius, and magnetic (TE) modes where the electric field is always perpendicular to the radius. When the "dielectric" sphere is a plasma in an applied magnetic field the dielectric constant is a tensor and the separation into electric and magnetic modes is not generally possible. However, if the radius of the sphere is much smaller than the wavelength of the radiation in the material of the sphere, the electric field in the electric modes is much larger than the magnetic field. We can, therefore, neglect the magnetic field and all propagation effects and solve the problem in the electrostatic approximation using Laplace's equation. If the dielectric constant is a tensor, we solve a generalized Laplace equation. In the case of a plasma where elements of the dielectric constant tensor are negative, we obtain radius independent resonances. For the magnetic modes the magnetic field is much larger than the electric field, and the problem can be treated in the magnetostatic approximation. For a plasma the permeability is positive, and one does not obtain magnetostatic resonances. In ferrites one has a positive dielectric constant and a tensor permeability with negative components and one observes magneto-

[1] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, New York, 1941).

static resonances.[2] These have been extensively treated[3–7] and we adapt these solutions for the case of the plasma.

One of the electrostatic modes of a plasma sphere is the well-known plasma resonance of Langmuir and Tonks[8] which has also been studied in InSb.[9] This mode can be identified with the lowest TM mode of a plasma sphere. We refer to it as the uniform mode; the displacement of the carriers is the same throughout the sphere. This mode is analogous to the uniform precession mode usually observed in ferromagnetic resonance.

If $\omega_p^2 \gg \omega_c\omega$ and $\omega_c > \omega$, where ω_p is the plasma frequency, ω_c the cyclotron frequency, and ω the frequency of the applied field, the effective dielectric constant of a plasma can be large and positive. In such a case the radius of a sphere may not be small compared to the wavelength inside the plasma in spite of the fact that the radius is small compared to the free-space wavelength. The electrostatic approximation is, of course, invalid and electromagnetic resonances are to be expected. These resonances are closely related to the "helicon" waves and resonances, which have been discussed recently.[10–13] We have not solved this boundary-value problem, but it appears that an assumption of an effective scalar dielectric constant fits the experiments surprisingly well. The lowest resonant mode observed in this case behaves very much like a magnetic dipole precessing about the applied magnetic field.

THEORY

A. Electrostatic Modes

The effective dielectric constant of a plasma without losses in an applied magnetic field can readily be computed from the equations of motion to be

$$\boldsymbol{\epsilon} = \left\| \begin{matrix} \epsilon_L - \dfrac{\omega_p^2}{\omega^2-\omega_c^2} & \dfrac{i\omega_c\omega_p^2}{(\omega^2-\omega_c^2)\omega} & 0 \\ -\dfrac{i\omega_c\omega_p^2}{(\omega^2-\omega_c^2)\omega} & \epsilon_L - \dfrac{\omega_p^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & \epsilon_L - \dfrac{\omega_p^2}{\omega^2} \end{matrix} \right\|, \tag{1}$$

where ϵ_L is the dielectric constant of the material in the absence of free carriers; ω is the frequency of the applied electric field; $\omega_c = eB_0/mc$, $\omega_p^2 = 4\pi Ne^2/m$; B_0 is the constant magnetic field in the z direction; N, e, and m are the electron concentration, charge, and mass, respectively. In the case of electrons in a solid, m would be the effective mass of the carriers.

If we have a circularly polarized incident field, one can write an effective dielectric constant tensor for each direction of circular polarization:

$$\boldsymbol{\epsilon}^{\pm} = \left\| \begin{matrix} \epsilon_L - \dfrac{\omega_p^2}{\omega(\omega\pm\omega_c)} & 0 & 0 \\ 0 & \epsilon_L - \dfrac{\omega_p^2}{\omega(\omega\pm\omega_c)} & 0 \\ 0 & 0 & \epsilon_L - \dfrac{\omega_p^2}{\omega^2} \end{matrix} \right\|. \tag{2}$$

Equations (1) and (2) are referred to the same system of coordinates.

If the size of the sphere is much smaller than the wavelength of the incident radiation, the time derivative in Maxwell's equations can be neglected, and one can merely solve the equation

$$\nabla \cdot \boldsymbol{\epsilon} \cdot \nabla\Phi = 0. \tag{3}$$

We first consider solutions of this equation when $\boldsymbol{\epsilon}$ is a scalar, which is the case for zero magnetic field. It turns out that there are also solutions for nonzero magnetic field which can be determined in this manner. In such a case the potential inside the sphere Φ_i and the potential outside Φ_0 can be written[1] as

$$\Phi_i = \sum_{n,m} a_{nm} r^n P_n^m(\cos\theta) e^{im\phi},$$
$$\Phi_0 = \sum_{n,m} b_{nm} r^{-(n+1)} P_n^m(\cos\theta) e^{im\phi},$$

where r, θ, and ϕ are the polar coordinates and P_n^m are the associated Legendre polynominals. When r is equal to the radius of the sphere R, the usual boundary conditions must be satisfied:

$$\Phi_i(R) = \Phi_0(R),$$
$$\epsilon(\partial\Phi_i/\partial r)_R = \epsilon_0(\partial\Phi_0/\partial r)_R, \tag{5}$$

where ϵ_0 is the dielectric constant of the material outside the sphere. We can find homogeneous solutions, i.e., resonances, whenever

$$\epsilon = -\epsilon_0(n+1)/n, \tag{6}$$

where n is an integer. For a positive ϵ_0 we can have resonances only when ϵ is negative, as in the case of a plasma. When there is no magnetic field these modes are $(2n+1)$-fold degenerate and $\epsilon = \epsilon_L - \omega_p^2/\omega^2$.

If a magnetic field is present, ϵ is a tensor and we cannot use Eqs. (4) as a solution of Laplace's equation. However, we can note that for modes where $m=n$ the

[2] R. L. White and I. H. Solt, Jr., Phys. Rev. **104**, 56 (1956).
[3] J. E. Mercereau and R. P. Feynman, Phys. Rev. **104**, 63 (1956).
[4] R. L. Walker, Phys. Rev. **105**, 390 (1957).
[5] P. C. Fletcher and R. Bell, J. Appl. Phys. **30**, 687 (1959).
[6] R. A. Hurd, Can. J. Phys. **36**, 1058 (1958).
[7] R. Plumier, Physica **28**, 423 (1961).
[8] L. Tonks and I. Langmuir, Phys. Rev. **33**, 195 (1929).
[9] G. Dresselhaus, A. F. Kip, and C. Kittel, Phys. Rev. **100**, 618 (1955).
[10] P. Aigrain, *Proceedings of the International Conference on Semiconductor Physics, Prague, 1960* (Czechoslovakian Academy of Sciences, Prague, 1961).
[11] A. Libchaber and R. Veilex, Phys. Rev. **127**, 774 (1962).
[12] F. E. Rose, M. T. Taylor, and R. Bowers, Phys. Rev. **127**, 1122 (1962).
[13] P. Cotti, P. Wyder, and A. Quattropani, Phys. Letters **1**, 50 (1962).

electric field is circularly polarized in a plane perpendicular to the z direction. The dielectric constant in the z direction is, therefore, irrelevant. For these modes we can thus write

$$\epsilon_L-\omega_p^2/\omega(\omega\pm\omega_c)=-\epsilon_0(n+1)/n, \qquad (7)$$

or, solving for ω_c,

$$\omega_c=\mp\left(\omega-\frac{n}{n\epsilon_L+(n+1)\epsilon_0}\omega_p^2\omega^{-1}\right). \qquad (8)$$

For the case of $n=1$ and $\epsilon_0=1$ we have

$$\omega_c=\mp\left(\omega-\frac{(4\pi/3m)Ne^2}{1+(4\pi/3)\chi}\omega^{-1}\right), \qquad (9)$$

where $\chi=(\epsilon_L-1)/4\pi$ is the electric susceptibility of the lattice. This is just the uniform mode of Langmuir and Tonks, which has also been studied in solids.[9,14] One can readily see that if $\epsilon_L\gg\epsilon_0$ all of the $n=m$ modes resonate at approximately the same magnetic field for a fixed ω. This is the usual case for a semiconductor in air.

When $m\neq n$ the situation is much more complicated. Fortunately, an analogous problem has already been solved for the case of a ferrite sphere and one can determine the solution for the case of the plasma from the results of the ferrite calculations. This is discussed in the Appendix.

The resonances of Eq. (7) do not depend on the radius of the sphere. However, if we increase the size of the sphere, the assumptions in deriving Eq. (7) do not hold, electromagnetic radiation effects appear, and ω_c shows a size dependence. The first-order correction in R^2 to Eq. (7) can be easily estimated by expanding the characteristic equation for the electric modes of a dielectric sphere[1]:

$$k_0^2[k_iRj_n(k_iR)]'/k_i^2j_n(k_iR) = [k_0Rh_n^{(1)}(k_0R)]'/h_n^{(1)}(k_0R), \qquad (10)$$

in powers of R. In Eq. (10) k_i and k_0 are the propagation constants of the material inside and outside the sphere, respectively, j_n is the spherical Bessel function, and $h_n^{(1)}$ the spherical Hankel function of the first kind and order n. Expanding Eq. (10) in powers of R and neglecting all terms higher than R^2 we obtain for $n=1$

$$\epsilon/\epsilon_0=-2[1+(6/5)k_0^2R^2]. \qquad (11)$$

A similar expression can be obtained for $n\neq 1$. Equation (11) and its equivalent for $n\neq 1$ show that the electrostatic plasma resonances are the lowest electric modes of a sphere with $k_0^2R^2\ll 1$. These modes only exist for negative dielectric constant. Using for ϵ the ϵ_{xx} component of Eq. (2), we obtain the magnetoplasma resonance for the uniform mode at

$$\mp\omega_c=\omega-\frac{\omega_p^2}{\omega(\epsilon_L+2\epsilon_0)}\left(1-\frac{12}{5}\frac{\epsilon_0}{\epsilon_L+2\epsilon_0}k_0^2R^2\right). \qquad (12)$$

Actually, when R becomes finite the electric field in the z direction is not zero, and it is not permissible to use a scalar dielectric constant. However, the result obtained by analogy to the more careful treatment for the ferrite[15,6] is identical to Eq. (12). We note that if $\epsilon_L\gg\epsilon_0$ (the usual case for a semiconductor in air) the size correction is quite small.

B. Electromagnetic Modes

When $\omega_p^2\gg\omega_c\omega$ and $\omega_c>\omega$, ϵ_{xx} and ϵ_{yy} in Eq. (2) can be large positive numbers, and the propagation constant for circularly polarized plane waves propagating in the direction of the magnetic field k_h may be positive and very large. (It is this mode of propagation which has been referred to as "helicon" waves[10] and observed in InSb,[11] sodium,[12] and other metals.[13]) Clearly, the treatment used in the previous section is no longer valid. In spite of the fact that an element of the dielectric constant tensor is a large negative number, one might expect electromagnetic resonance modes to exist when $k_hR>1$. In this case the division into electric and magnetic modes is no longer permissible, and the problem appears to be extremely complicated.

An "approximation" which we can justify only by its simplicity is to consider the plasma in the magnetic field to be a dielectric with a dielectric constant equal to ϵ_{xx} of Eq. (2). We, thus, choose the propagation constant for the plasma equal to k_h. We then calculate the resonant magnetic fields from the characteristic equations for the electric and magnetic modes of a dielectric sphere.[1] This is Eq. (10) with $k_i=k_h$ for the electric case. For the magnetic modes we have

$$[k_hRj_n(k_hR)]'/j_n(k_hR) = [k_0Rh_n^{(1)}(k_0R)]'/h_n^{(1)}(k_0R). \qquad (13)$$

Since $k_0R\ll 1$ we expand the right-hand side of Eqs. (10) and (13) and the conditions for resonance become

$$j_n(k_hR)=0, \qquad (14)$$

for the electric modes and

$$j_n'(k_hR)=0, \qquad (15)$$

for the magnetic modes. If $\omega_c\gg\omega$ we obtain

$$k_hR=(\omega/c)[\epsilon_L+\omega_p^2/\omega_c\omega]^{1/2}R=\beta_{nl} \text{ or } \alpha_{nl}, \qquad (16)$$

where β_{nl} and α_{nl} are the roots of Eqs. (14) and (15), respectively.[16] The first few of these roots are $\alpha_{1i}=2.08$,

[14] R. E. Michel and B. Rosenblum, Phys. Rev. **128**, 1646 (1962).

[15] J. E. Mercereau, J. Appl. Phys. **30**, 184S (1959).

[16] *Tables of Spherical Bessel Functions*, National Bureau of Standards Series (Columbia University Press, New York, 1941), Vol. I.

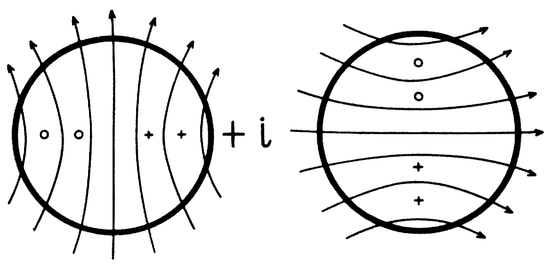

FIG. 1. Cross section of the proposed field distribution for the lowest size-dependent mode of a small plasma sphere in a magnetic field. The solid lines indicate the magnetic field and the dots and crosses the electric field. The applied field is perpendicular to the paper.

5.94, 9.21; $\alpha_{2i}=3.34$, 7.29; $\beta_{1i}=4.49$, 7.72, 10.9; $\beta_{2i}=5.76$, 9.10.

The lowest mode is the first magnetic mode (TE_{11}) corresponding to α_{11}. Corresponding to our choice of a propagation constant for a circularly polarized wave we expect this mode to be circularly polarized, i.e., to consist of the excitation of two TE_{11} modes at right angles to each other and 90° out of phase. This field configuration is shown in Fig. 1. The magnetic field is shown by the arrows and the electric field, which is in circles about the magnetic field and perpendicular to the paper in this cross section, is shown by the dots and crosses. The constant magnetic field is also perpendicular to the paper. The proposed mode is a magnetic dipole precessing about the constant magnetic field.

EXPERIMENTAL

The spheres of indium antimonide were prepared by cutting cubes of appropriate dimensions and placing them in a cylinder lined with a fine emery paper. A jet of air was used to rotate the cubes in the cylinder. Quite accurate spheres could readily be made in this manner. The spheres were mounted in a rectangular TE_{01n} microwave cavity or in a waveguide by suspending them on a 0.001-in.-thick Mylar sheet with a thin layer of GE No. 7031 lacquer. Most of the measurements were made at a temperature of about 60°K obtained by pumping on liquid nitrogen. At this temperature the carrier concentration is quite insensitive to temperature or magnetic field, and the scattering times are about as long as can be obtained in these samples. We did not operate at helium temperature since at this temperature the carrier concentration changes with magnetic field and the scattering time appeared to be somewhat shorter. Most of the measurements were taken at frequencies of about 25 kMc/sec. The resonances were usually observed by recording the change in the cavity reflection coefficient and therefore in sample absorption as a function of magnetic field.

We were not able to observe any of the electric modes other than the uniform mode. As was pointed out above, those with $m=n$ would be very close to the uniform mode and, therefore, unresolvable under our experimental conditions. When the sample was moved to an electric field node this resonance disappeared. Thus, if other modes with $n=m$ were present, but unresolved, their excitation at the node is very small (as would be expected). It is shown in the Appendix that with one exception we would have to go to frequencies about one-fifth of the plasma frequency to observe any of the modes with $m\neq n$, for $n=2$. The one exception is the $m=1$ resonance near zero magnetic field. We did observe in very small samples a broad maximum near zero magnetic field when the sample was at a node of electric field, but we do not feel that we can definitely assign it to this mode.

In the usual derivation of the magnetic field for uniform magnetoplasma resonance[8,9,14] it is assumed that the electromagnetic field can penetrate the sample. The criterion stated is that the sample must be small compared to the ordinary skin depth. According to Eq. (12) if $\epsilon_L\gg\epsilon_0$ the sample can be considerably larger than a skin depth and the resonant magnetic field is unchanged. To demonstrate this we took a sphere of indium antimonide ($\epsilon_L=19.6$)[17] with a carrier concentration of 6.4×10^{13} and observed the field for the uniform magnetoplasma resonance as the radius of the sphere was reduced in several steps from 0.050 to 0.015 in. The calculated skin depth for this material was 0.016 in. The resonant magnetic field changed less than 5%. There was a slight shift towards lower field, but our uncertainty as to the uniformity of the carrier concentration precludes our drawing any more quantitative conclusions.

The calculations for the resonances we observe were made with the assumption that the sphere is in free space; actually, of course, it is in a cavity or a waveguide. To demonstrate that any polarization charges induced on the walls were unimportant the uniform mode was measured for an $R=0.05$ in. sphere in a waveguide whose smaller dimension was 0.170 in. The sphere was thus less than its radius from the wall. Nevertheless, the resonant magnetic field was the same within 3% as the field measured for the same sphere in a larger waveguide. An estimate of the shift shows that it should be small for $\epsilon_L\gg\epsilon_0$.

In addition to the size-independent uniform mode, resonances were observed which depend strongly on sample size. Different resonances were excited depending on the position of the sample in the microwave cavity, e.g., whether it was at a node or an antinode of the electric field. When the sample was gradually moved from a node to an antinode, the relative intensities of the resonances changed, but the magnetic fields for resonance did not. Different resonances were also excited depending on the orientation of B_0 with respect to the microwave fields.

In Fig. 2 we show two typical experimental curves of

[17] T. S. Moss, *Optical Properties of Semiconductors* (Butterworths Scientific Publications, Ltd., London, 1959), p. 235.

power absorption versus magnetic field. The sample had a 0.015-in. radius and an electron concentration of 2.3×10^{14}/cc. [The stronger resonance in the solid curve is the one we identify with the first magnetic mode (Fig. 1) and the root α_{11} of Eq. (15).] As the sphere size is reduced these resonances move to lower field, decrease in strength, and disappear. The absolute width of the resonances decreases as the sphere size is reduced.

We expect the modes excited at a node of electric field to have electric fields of odd parity and magnetic fields of even parity. The reverse should be true of the modes excited at an electric antinode. Thus for odd n we expect magnetic modes to be excited at a node and electric modes at an antinode, and the opposite for even n. Using this rule and Eq. (16), we plot the expected resonant magnetic fields for the four field configurations in which resonances were observed for the first few values of n and l. It is a property of the spherical Bessel functions that each of the β_{nl} are approximately equal to an α_{nl} with n differing by unity. The one exception is β_{11}. Therefore, we have not plotted any electric modes except the first. In this plot we have chosen $\omega_p=7.3\times10^{12}$ sec^{-1}, $\omega=1.6\times10^{11}$ sec^{-1}, and $\epsilon_L=19.6$. The circles are the experimentally observed resonances for a sample of indium antimonide under these conditions. The plasma frequency of the sample was determined by a measurement of the uniform magnetoplasma mode. The corresponding carrier concentration was in substantial agreement with a concentration determined by a Hall measurement on an adjacent piece of material. There are no adjustable parameters used in comparing the data with Eq. (16).

The major discrepancies between the experiments and Eq. (16) is an extra resonance in Fig. 3(c) to which no electric or magnetic mode corresponds and the apparent absence of a resonance associated with β_{11}. As the radius is reduced the extra resonance appears to become a broad absorption maximum at low field which we have said might be due to the $n=2$, $m=1$ electrostatic mode. Since in this sample $|k_iR|>1$, the large size dependence would not be surprising.

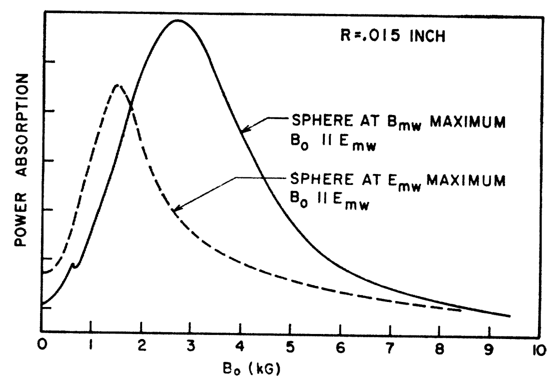

FIG. 2. Experimental microwave absorption as a function of magnetic field by a small sphere of indium antimonide.

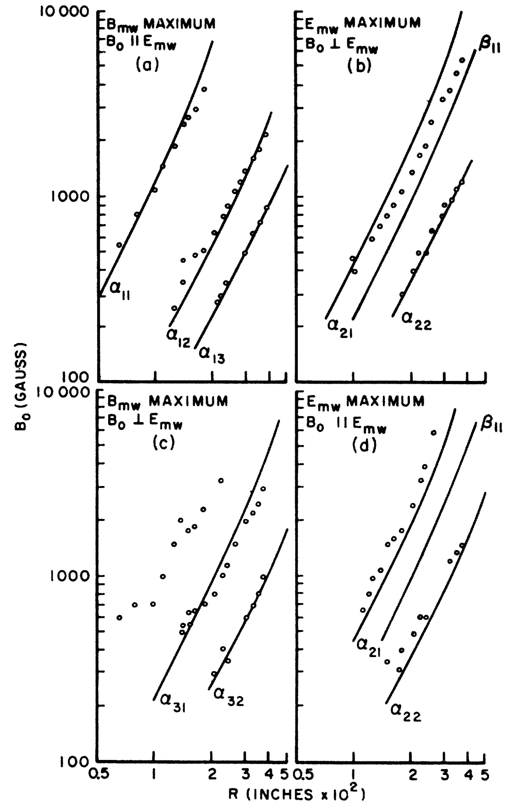

FIG. 3. Magnetic field for resonant absorption of 25-kMc/sec radiation vs radius for an indium antimonide sphere with $N=2.3\times10^{14}$/cc. The lines are calculated from Eq. (16). The points are experimental.

The form of the absorption curves for the various field configurations of Fig. 3 was generally quite dissimilar. At an antinode with B_0 perpendicular to the microwave electric field, and at a node with B_0 parallel to the microwave electric field, the lines were generally well resolved, and the intensity of the resonances changed in a fairly simple manner as the sample radius was reduced. This was not the case for the other two configurations.

The dependence of the resonant magnetic field on microwave frequency has also been observed to be approximately in accord with Eq. (16). Data were also taken on samples with carrier concentrations several times lower than that of the sample of Fig. 3. In these cases only the resonances corresponding to the lowest roots could be observed, but these occurred at approximately the fields predicted by Eq. (16).

We now consider the lowest and most prominent of the size-dependent modes, the one we assigned to the root α_{11}, in somewhat more detail. In the previous section we speculated that this mode has a field configuration very similar to that of a magnetic dipole precessing about the applied magnetic field. We would thus expect the microwave magnetic field to excite this resonance. This appears indeed to be the case. The resonance is excited whenever the applied field is perpendicular to the microwave magnetic field, whether or not it is perpendicular to the microwave electric field, as is the case for a magnetic dipole in an applied field (e.g., electron spin resonance). We would, furthermore, expect a magnetic dipole resonance to be excited by only one circular polarization. To check this, a sample was mounted in a waveguide halfway between the broad walls and one quarter of the long dimension away from a narrow wall. At such a point in a waveguide in its TE_{01} mode with a wave traveling in only one direction, the magnetic field is approximately circularly polarized, but the electric field is not. It was found that the resonance identified with α_{11} was considerably stronger for one direction of the applied magnetic field than for the other. This demonstrates that this resonance is excited by a circularly polarized microwave magnetic field.

The α_{11} resonance is associated with a stronger absorption than any of the other size-dependent resonances we have studied. On a sample with $R=0.008$ in. and $N=6\times10^{13}$/cc the α_{11} resonance with the sample at an electric field node had one eighth the peak intensity of the uniform electrostatic resonance at an antinode of the electric field. In several samples of different radii and electron concentration the ratio was not very different from this.

DISCUSSION

In the above treatment we assumed that the resonances of the actual plasma would occur at the same magnetic field as for a plasma with a very small collision frequency; Eq. (1) gives the dielectric constant of a lossless plasma. We would, generally, expect the position of resonances to be largely independent of the loss, which would determine their width. If loss is included in the dielectric constant, an imaginary part is added to the diagonal terms and a real part is added to the off-diagonal terms. The ratio of imaginary to real parts for the magnetic field dependent part of diagonal terms is approximately $\omega\tau$ while the ratio of real to imaginary parts for the off-diagonal terms is approximately $\frac{1}{2}\omega\tau(\omega_c/\omega)^2$, where τ is the scattering time. The material used in the present experiments had $\omega\tau\approx0.5$,[18] and since $\omega_c\gg\omega$ the magnetic dependence of the diagonal terms was loss dominated, while for the off-diagonal terms the effect of loss was small. The loss tangent of the material would be $\approx(\omega_c\tau)^{-1}$.

It was observed that the resonances at high magnetic field were quite broad and got considerably narrower when they moved to low field as the sample size was reduced. This seems to contradict what we said above, since the loss tangent decreases with increasing magnetic field. Actually, in going from Eq. (13) to Eq. (15) we have assumed $k_0R\ll1$, which is no longer true for large sphere radius. While Eq. (15) has real roots, as we increase the sphere radius this equation is not valid, and the resonance frequency, obtained as a solution of Eq. (13), becomes complex, the sphere radiates at resonance, and the resonance line is broadened.

For a gaseous plasma the relaxation time is usually much longer than for a solid and $\epsilon_L\approx1$. In such a case one should be able to resolve the electrostatic modes.

We would like to thank Harold Hanson for taking most of the experimental data, and Dr. Maurice Glicksman for stimulating discussions. We would also like to thank Dr. A. C. Beer and Dr. M. Glicksman for supplying the indium antimonide used in this work.

APPENDIX

The permeability tensor of a ferrite with a steady magnetic field H applied in the z direction is

$$\boldsymbol{\mu}=\begin{Vmatrix} 1-\dfrac{\Omega_H}{\Omega^2-\Omega_H^2} & i\dfrac{\Omega}{\Omega^2-\Omega_H^2} & 0 \\ -i\dfrac{\Omega}{\Omega^2-\Omega_H^2} & 1-\dfrac{\Omega_H}{\Omega^2-\Omega_H^2} & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{17}$$

where $\Omega=\omega/4\pi\gamma M_0$, $\Omega_H=(H/4\pi M_0)-\frac{1}{3}$, where γ is the gyromagnetic ratio, and M_0 is the steady magnetization in the z direction. The symmetry of Maxwell's equations with respect to $\mathbf{E}$ and $\mathbf{H}$ enables us to draw a close analogy between plasma resonance and ferromagnetic resonance. The electrostatic plasma resonance has its analog in the magnetostatic ferromagnetic resonance. The magnetostatic potential Ψ inside the sphere will satisfy $\nabla\cdot\boldsymbol{\mu}\cdot\nabla\Psi=0$ and outside $\nabla^2\Psi=0$. Ψ and the normal component of the magnetic induction must be continuous at the surface of the sphere. In order to establish the formal analogy between magnetostatic ferromagnetic resonance and electrostatic plasma resonance, it is convenient to write Eq. (1) as

$$\boldsymbol{\epsilon}=\epsilon_{zz}\boldsymbol{\epsilon}'=\epsilon_{zz}\begin{Vmatrix} 1-\dfrac{\omega_p^2\omega_c^2}{\omega^2(\omega^2-\omega_c^2)\epsilon_{zz}} & i\dfrac{\omega_p^2\omega_c}{\omega(\omega^2-\omega_c^2)\epsilon_{zz}} & 0 \\ -i\dfrac{\omega_p^2\omega_c}{\omega(\omega^2-\omega_c^2)\epsilon_{zz}} & 1-\dfrac{\omega_p^2\omega_c^2}{\omega^2(\omega^2-\omega_c^2)\epsilon_{zz}} & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{18}$$

The zz component of $\boldsymbol{\epsilon}'$ is 1 as is the corresponding component of $\boldsymbol{\mu}$. The electric potential Φ must satisfy $\nabla\cdot\boldsymbol{\epsilon}'\cdot\nabla\Phi=0$ and the boundary conditions, Φ continuous at the surface of the sphere and the normal component of $\boldsymbol{\epsilon}'\cdot\nabla\Phi$ inside the sphere equal to the normal component of $(\epsilon_0/\epsilon_{zz})\nabla\Phi$ outside. The electrostatic reso-

[18] This $\omega\tau$ is calculated from the measured dc Hall mobility on several adjacent samples. The linewidth of the uniform plasma mode was consistently found to be larger than predicted from the Hall mobility. The additional broadening may be due to the excitation of higher electrostatic modes. This excitation could be caused by microscopic fluctuations in the carrier concentration which have been reported for InSb.

nances are obtained by equating $\boldsymbol{\varepsilon}'$ to $\boldsymbol{\mu}$ and the permeability μ_0 of the medium around the sphere to ϵ_0/ϵ_{zz} in the equations for the magnetostatic resonances. By using this analogy and the characteristic equation for magnetostatic resonance,[5] we obtain the characteristic equation for electrostatic plasma resonance:

$$(n+1)\epsilon_0/\epsilon_{zz}+\xi[P_n{}^m(\xi)]'/P_n{}^m(\xi)=\pm m\nu,$$

where

$$\xi^2=1-\frac{\epsilon_{zz}\omega^2(\omega^2-\omega_c{}^2)}{\omega_p{}^2\omega_c{}^2},$$

$$\nu=\frac{\omega_p{}^2}{(\omega^2-\omega_c{}^2)}\frac{\omega_c}{\omega\epsilon_{zz}}.$$

This characteristic equation reduces to Eq. (6) for $n=m$. For $n=2$ and $\epsilon_0=1$ the characteristic equations are

$$\begin{aligned} y(y\pm\alpha)&=1, \quad (|m|=2)\\ y^3\mp\alpha y^2-y\pm\tfrac{1}{2}\alpha&=0, \quad (|m|=1)\\ 2\epsilon_L y^4-y^2[2(1+\alpha^2)\epsilon_L+3+2\epsilon_L]&\\ +\alpha^2(1+2\epsilon_L)+(3+2\epsilon_L)&=0, \quad (m=0) \end{aligned} \tag{19}$$

where $y=(\omega/\omega_p)(\frac{3}{2}+\epsilon_L)^{1/2}$ and $\alpha=(\omega_c/\omega_p)(\frac{3}{2}+\epsilon_L)^{1/2}$.

Figure 4 shows the solutions of Eqs. (19) as a function of α. These solutions for $|m|=2$ and $|m|=1$ are independent of ϵ_L. For $m=0$, ϵ_L appears explicitly in the characteristic equation. These solutions have been plotted for $\epsilon_L\to\infty$. In the experiments reported here $\omega\ll\omega_p$ and only $m=1$ and $m=2$ resonances could appear. The $m=1$ resonance, however, would appear at a very low magnetic field. Because of collision broadening it looks like a smooth decrease in the conductivity rather than like a resonance. The $|m|=2$ resonance is very close to the uniform magnetoplasma mode.

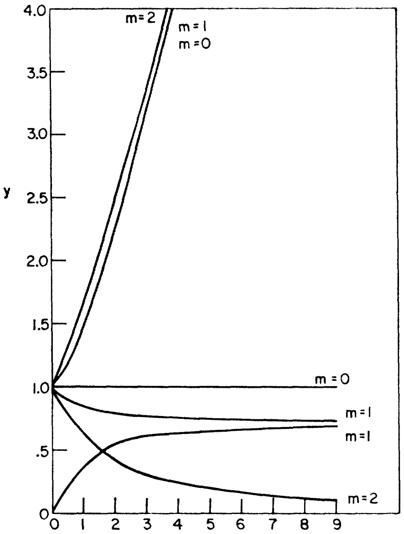

FIG. 4. Reduced frequency y vs reduced magnetic field α [defined by Eq. (19)] for the electrostatic modes of a plasma sphere for $n=2$.

PHYSICAL REVIEW VOLUME 129, NUMBER 3 1 FEBRUARY 1963

Oscillations of a Plasma in a Magnetic Field

M. J. STEPHEN
Physics Department, Yale University, New Haven, Connecticut
(Received 14 September 1962)

The plasma oscillations of a high-temperature and a degenerate plasma, both with and without an applied magnetic field, are discussed using a Green function technique. For the high-temperature plasma in lowest order the results reduce to those obtained via the linearized Boltzmann-Vlasov equation in the classical limit. The quantum-mechanical effects are important for strong fields where the quantization of the orbits of the electrons in the field must be taken into account. The dispersion relation of a degenerate plasma in a magnetic field is obtained within the random phase approximation. This dispersion relation is discussed for various special cases. The fluctuation spectrum of the plasma is obtained by making use of the fluctuation dissipation theorem.

1. INTRODUCTION

IN this paper the small oscillations of an electron gas in thermodynamic equilibrium are discussed. The two cases of a plasma in zero external magnetic field and a plasma situated in a strong uniform magnetic field are considered. The presence of the ions is neglected and we use the simple model where the ions are smeared out into a compensating positive background. The usual treatments of the oscillations of a high-temperature plasma are based on the collisionless Boltzmann-Vlasov (B.V.) equation or on the hydrodynamic equations of motion. Reviews of recent work in this field have been written by Thompson[1] and Oster.[2] Here we use a different ap-

[1] W. B. Thompson, *Reports on Progress in Physics* (The Physical Society, London, 1961), Vol. 24, p. 363.
[2] L. Oster, Rev. Mod. Phys. 32, 141 (1960).