Dispersion-Theoretic Impulse Approximation for Potential Scattering*

LEONARD ROSENBERG
Physics Department, University of Pennsylvania Philadelphia, Pennsylvania
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In the impulse approximation for the scattering of a particle by a bound system, the amplitude is a sum of integrals over two-body scattering amplitudes, off the energy shell, folded into bound-state wave functions. In the usual formulation, the nonphysical two-body amplitudes are replaced by physical amplitudes with no firm justification for this procedure. The dispersion-theoretic formulation presented here, for elastic scattering, removes this difficulty; for low values of t, the momentum transfer squared, the discontinuity across the cut in the t plane can be expressed in terms of the absorptive part of the physical two-body amplitude and the asymptotic form of the bound-state wave function. Working with a nonrelativistic model, it is shown that the Cutkosky method for finding absorptive parts of Feynman amplitudes applies here as well. The analyticity of the amplitude is a conjecture, based on a proof that the second and third Born approximations satisfy a Mandelstam representation. The method of this proof is an adaptation of techniques recently developed by Eden and others in the relativistic case.

1. INTRODUCTION

DISPERSION theory has recently assumed a central role in the description of processes involving strongly interacting particles. Since the ultimate validity of this approach is as yet unestablished, it would seem to be of interest to test these new techniques on more tractable model problems. In this spirit a study of the scattering of a particle by a static central potential, assumed to be a linear superposition of Yukawa potentials, has led to the result that the scattering amplitude has the analyticity properties in energy and momentum transfer which imply a Mandelstam representation.[1] Single variable dispersion relations have been applied to the Lee model,[2] to the scattering of electrons by hydrogen atoms,[3] and to the analysis of stripping reactions.[4] Blankenbecler, Goldberger, and Halpern[5] have applied dispersion relations, in a field-theoretic framework, to the study of low-energy elastic neutron-deuteron scattering.

An essential difficulty encountered in any attempt to extend the work of Blankenbecler *et al.* to higher energies is the increased importance of inelastic intermediate states. However, it has been demonstrated in the relativistic case[6] that for t, the square of the momentum transfer, sufficiently small the relevant elastic scattering diagrams involve only two-body intermediate states in the t channel even though the intermediate states are quite complex when approached from the s (energy) channel. In such a case the spectral function may be determined in terms of two-body scattering amplitudes, thereby providing, in the case of $n-d$ scattering, a dispersion theoretic analog of the impulse approximation.

We have attempted to pursue these ideas in a simple context; the model chosen here is the nonrelativistic scattering of a particle by a bound two-particle system, all three particles being spinless and neutral. We begin, as in the usual impulse approximation approach, by ignoring those contributions to the scattering amplitude which correspond to multiple scattering and "potential" corrections (see Sec. 2). The remainder, which is expected to give the dominant contribution for the high-energy scattering by a weakly bound system, is then assumed to be an analytic function of t with a cut on the negative real axis. Some support is given for this assumption in Sec. 3, where it is shown that the second Born approximation to this amplitude satisfies a Mandelstam representation. The method of proof is based on the Feynman parametrization of the integrals and an analysis of singularities used by Eden[7] and others in relativistic problems. The proof (assuming one exists) for higher terms in the Born series seems to involve no technical difficulties other than algebraic complexity. In the simpler two-body scattering problem we have in this way been able to reproduce the result that each term in the Born series satisfies the Mandelstam representation.[8] Of course, even a complete analysis of the Born series can, at best, only make plausible the analyticity of the amplitude itself. The analysis of singularities referred to above has the additional virtue that it provides a basis, first developed by Cutkosky[9] for the study of Feynman amplitudes, for obtaining the discontinuity across the cut. This is an essential point. The only other known technique for obtaining the spectral function in potential theory is based on the unitarity statement[1]; this, however, involves the inelastic intermediate states which we have been trying to avoid.

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[4] R. D. Amado, Phys. Rev. **127**, 261 (1962).
[5] R. Blankenbecler, M. L. Goldberger, and F. R. Halpern, Nucl. Phys. **12**, 647 (1959).
[6] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961); R. Cutkosky, Phys. Rev. Letters **4**, 624 (1960).
[7] R. J. Eden, Phys. Rev. **119**, 1763 (1960); **120**, 1514 (1960); **121**, 1567 (1961).
[8] We shall not present the details of this proof here.
[9] R. Cutkosky, J. Math. Phys **1**, 429 (1960).

One might expect that a dispersion-theoretic approach would provide the advantage that one need only employ two-body amplitudes which are on the energy shell. Indeed, we have found this to be the case within the zero-range approximation, in which only the tail of the bound-state wave function is retained. This approximation may be justified, for low momentum transfers,[10] in a manner characteristic of dispersion theory, namely, that effects of the finite range of the potential contribute to singularities which are "distant" and hence relatively unimportant. The use of physical two-body amplitudes removes a source of ambiguity which is present in the usual formulation of the impulse approximation.

It may be noted that the techniques presented here constitutes a nonrelativistic analog of the treatment of the impulse approximation given by Cutkosky.[11] Effects of the anomalous threshold are considerably simpler to deal with here since they are taken into account by the structure of the bound-state wave function.[12]

2. THE IMPULSE APPROXIMATION

We consider the elastic scattering of a particle of mass m (particle 1) from a system consisting of two particles (particles 2 and 3), each of mass m, bound in and s state with energy $-B$. The binding potential is assumed to be such that only one bound state exists. For simplicity we assume the particles to be spinless and neutral, and we ignore the Pauli principle. The interactions between particles 1 and 2 and between particles 2 and 3 are given by the potentials V and U, respectively. Both are taken to be of the Yukawa form, $\sim e^{-\mu r}/r$; the generalization to a linear superposition of Yukawa potentials for V and U is trivial. For simplicity, we assume that particles 1 and 3 do not interact. The Schrödinger equation takes the form

$$[-(\hbar^2/2m)(\nabla_1^2+\nabla_2^2+\nabla_3^2)+V+U-E_k+B]\Psi \equiv (K+V+U-E)\Psi=0, \quad (2.1)$$

where E_k is the initial kinetic energy of the incident particle. The integral equation, which incorporates Eq. (2.1) and the appropriate boundary conditions may be written symbolically as

$$\Psi=\Phi_i+GV\Phi_i, \quad (2.2)$$

where

$$G(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_1',\mathbf{r}_2',\mathbf{r}_3';E) = -\left\langle \mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3\left|\frac{1}{K+V+U-E-i\eta}\right|\mathbf{r}_1',\mathbf{r}_2',\mathbf{r}_3'\right\rangle \quad (2.3)$$

and

$$\Phi_i=e^{i\mathbf{k}_1\cdot\mathbf{r}_1}e^{i\mathbf{k}_{23}\cdot\frac{1}{2}(\mathbf{r}_2+\mathbf{r}_3)}\varphi(\mathbf{r}_2-\mathbf{r}_3). \quad (2.4)$$

$\hbar\mathbf{k}_1$ represents the initial momentum of particle 1 and $\hbar\mathbf{k}_{23}$ is the momentum associated with the center-of-mass motion of particles 2 and 3; $\varphi(\mathbf{r}_2-\mathbf{r}_3)$ is the bound-state wave function describing their relative motion. The scattering amplitude $T(\mathbf{k}_1,\mathbf{k}_{23};\mathbf{k}_1',\mathbf{k}_{23}')$ takes the form

$$T=(\Phi_f,V\Psi) \quad (2.5)$$

with Φ_f, the final-state wave function, given by

$$\Phi_f=e^{i\mathbf{k}_1'\cdot\mathbf{r}_1}e^{i\mathbf{k}_{23}'\cdot\frac{1}{2}(\mathbf{r}_2+\mathbf{r}_3)}\varphi(\mathbf{r}_2-\mathbf{r}_3). \quad (2.6)$$

Equation (2.5) becomes, in view of Eq. (2.2),

$$T=(\Phi_f,\ V[1+GV]\Phi_i). \quad (2.7)$$

We now make the "impulse" approximation,[13] according to which the effect of the binding potential U is negligible except for its determination of the bound-state function in Φ_i and Φ_f. The amplitude $\hat{T}$ in this approximation is

$$\hat{T}=(\Phi_f,\ V[1+\hat{G}V]\Phi_i) \equiv (2\pi)^3\delta(\mathbf{k}_1'+\mathbf{k}_{23}'-\mathbf{k}_1-\mathbf{k}_{23})R, \quad (2.8)$$

with $\hat{G}$ determined from Eq. (2.3) by dropping the potential U.

Let us now examine R in the center-of-mass frame. With the replacement

$$\varphi(\mathbf{r}_2-\mathbf{r}_3)=(2\pi)^{-3/2}\int d\mathbf{q}\ e^{i\mathbf{q}\cdot(\mathbf{r}_2-\mathbf{r}_3)}\tilde{\varphi}(\mathbf{q}), \quad (2.9)$$

and with the introduction of the variables

$$\begin{aligned}\mathbf{r}&=\mathbf{r}_1-\mathbf{r}_2,\\ \boldsymbol{\rho}&=\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\\ \boldsymbol{\Delta}&=\mathbf{k}_1-\mathbf{k}_1',\end{aligned} \quad (2.10)$$

we find, for $R\equiv R_B+R'$, the expressions

$$R_B=\int d\mathbf{q}\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}+\tfrac{1}{2}\boldsymbol{\Delta})\int d\mathbf{r}\exp(i\boldsymbol{\Delta}\cdot\mathbf{r})V(\mathbf{r}) \quad (2.11)$$

and

$$\begin{aligned}R'=-(2\pi)^{-6}\int & d\mathbf{r}d\boldsymbol{\rho}d\mathbf{r}'d\boldsymbol{\rho}'d\mathbf{q}d\mathbf{q}'\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}')\\ &\times\exp[-i(\boldsymbol{\alpha}'\cdot\mathbf{r}+\boldsymbol{\beta}'\cdot\boldsymbol{\rho})]V(r)\sum_\gamma\int d\mathbf{p}\psi_\gamma(\mathbf{r})\psi_\gamma^*(\mathbf{r}')\\ &\times[(3\hbar^2/4m)(p^2-k_\gamma^2-i\eta)]^{-1}\exp[i\mathbf{p}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')]\\ &\times V(r)\exp[i(\boldsymbol{\alpha}\cdot\mathbf{r}'+\boldsymbol{\beta}\cdot\rho')],\end{aligned} \quad (2.12)$$

where

$$\begin{aligned}\boldsymbol{\alpha}&=\tfrac{3}{4}\mathbf{k}_1-\tfrac{1}{2}\mathbf{q}, & \beta&=\tfrac{1}{2}\mathbf{k}_1+\mathbf{q},\\ \boldsymbol{\alpha}'&=\tfrac{3}{4}\mathbf{k}_1'-\tfrac{1}{2}\mathbf{q}', & \beta'&=\tfrac{1}{2}\mathbf{k}_1'+\mathbf{q}'.\end{aligned} \quad (2.13)$$

[10] The restriction to low momentum transfers is reasonable due to the dominance of the forward diffraction peak.

[11] R. Cutkosky, in *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester*, edited by E. C. G. Sudarshan, J. H. Tinlot, and A. C. Melissions (Interscience Publishers, Inc., New York, 1960), p. 236. See also, R. Blankenbecler, Phys. Rev. **122**, 983 (1961).

[12] R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960).

[13] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and M. L. Goldberger, *ibid.* **87**, 778 (1952).

$\psi_\gamma(\mathbf{r})$ satisfies the equation

$$[-(\hbar^2/m)\nabla_r^2+V(r)-E_\gamma]\psi_\gamma(\mathbf{r})=0, \qquad (2.14)$$

with

$$E=E_\gamma+(3\hbar^2/4m)k_\gamma^2. \qquad (2.15)$$

The summation over γ in Eq. (2.12) represents a sum over bound states and an integral over continuum states corresponding to solutions of Eq. (2.14). After performing the integrals over $\boldsymbol{\varrho}$ and $\boldsymbol{\varrho}'$, Eq. (2.12) becomes

$$R'=-\int d\mathbf{q}d\mathbf{r}d\mathbf{r}'\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}+\tfrac{1}{2}\boldsymbol{\Delta})\exp(-i\boldsymbol{\alpha}'\cdot\mathbf{r})V(r)\times\sum_\gamma[(3\hbar^2/4m)(\beta^2-k_\gamma^2-i\eta]^{-1}\times\psi_\gamma(\mathbf{r})\psi_\gamma^*(\mathbf{r}')V(r')\exp(i\boldsymbol{\alpha}\cdot\mathbf{r}'). \qquad (2.16)$$

We see that R has the form

$$R=\int d\mathbf{q}\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}+\tfrac{1}{2}\boldsymbol{\Delta})I[\boldsymbol{\alpha},\boldsymbol{\alpha}';E-(3\hbar^2/4m)\beta^2], \qquad (2.17)$$

where $I(k,k';z)$, the matrix element of the interaction operator, can be identified with the physical two-body center-of-mass scattering amplitude only when the relations

$$(\hbar^2/m)k^2=(\hbar^2/m)(k')^2=z \qquad (2.18)$$

are satisfied. In Sec. 4 we shall see how contact with physical two-body amplitudes may be established with the aid of dispersion relations.

Before going into the complex plane we wish to make an observation bearing on the convergence of the Born expansion for the scattering amplitude T given by Eq. (2.5). It is apparent that this series contains, as a subseries, the terms obtained by inserting the Born expansion for the interaction operator I in Eq. (2.17). Suppose the potential $V(|\mathbf{r}_1-\mathbf{r}_2|)$ is strong enough to support a two-body bound state, of energy $-|z_{12}|$. The Born expansion of $I(\mathbf{k},\mathbf{k}';z)$ then diverges for $z=-|z_{12}|$ and, for a wide class of potentials, diverges as well over a range of z values in that neighborhood.[14] Since β^2 runs through all positive values in Eq. (2.17), one is led to conclude that in the absence of fortuitous cancellations the Born expansion of the scattering amplitude in all likelihood diverges, *for arbitrarily large incident energies*. The situation here is identical to that discussed at length in reference 14 for the case of rearrangement collisions. Thus, the conclusions reached by Aaron, Amado, and Lee apply with equal validity to the case of direct collisions; the essential requirement is that the two-body potentials should be sufficiently strong.[15] These convergence difficulties point up the desirability of the development of nonperturbative approximation procedures in multiparticle scattering problems.

[14] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. 121, 319 (1961).

[15] The Born expansion of the two-body scattering amplitude may diverge over a range of energies even if no bound state exists. See, e.g., W. Kohn, Rev. Mod. Phys. 26, 292 (1954).

3. ANALYTICITY OF THE SECOND BORN AMPLITUDE

In the two-body scattering problem the analytic properties of the full amplitude in the momentum transfer variable are revealed (except for the behavior at infinity) by a study of the first and second Born amplitudes; this is true whether or not the Born series converges. We shall assume that a similar situation holds in the present case as well. It will now be shown that the second Born approximation to R satisfies a Mandelstam representation. This amplitude, which is obtained from Eq. (2.16) by replacing the functions ψ_γ by plane waves, takes the form (aside from constant factors)

$$\int d\mathbf{q}d\mathbf{l}\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}+\tfrac{1}{2}\boldsymbol{\Delta})\tilde{V}(\mathbf{l}-\boldsymbol{\alpha}')\times[\beta^2+\tfrac{4}{3}l^2-k_1^2+\tfrac{4}{3}\epsilon-i\eta]^{-1}\tilde{V}(\boldsymbol{\alpha}-\mathbf{l}), \qquad (3.1)$$

where $\tilde{V}$ is the Fourier transform of the potential and $\epsilon=(m/\hbar^2)B$. It is known[12] that $\tilde{\varphi}(\mathbf{q})$ has the analytic properties which allow the representation

$$\tilde{\varphi}(\mathbf{q})=\frac{C}{q^2+\epsilon}+\int_{(\epsilon^{1/2}+\mu)^2}^\infty\frac{\sigma(\nu)}{q^2+\nu}d\nu, \qquad (3.2)$$

where C is the asymptotic normalization of $\varphi(\mathbf{r})$. In the following, for convenience, only the pole term in Eq. (3.2) will be retained; it will be apparent how the proof goes for the entire function $\tilde{\varphi}(\mathbf{q})$.

The integral is now of the form which enables us to use the technique of Feynman parametrization, so that, after some trivial changes of variables, we are led to study the expression

$$R_2=\int d\mathbf{p}_1d\mathbf{p}_2\int_0^1d\alpha_1\cdots\int_0^1d\alpha_5\frac{\delta(1-\sum_1^5\alpha_i)}{Q^5}, \qquad (3.3)$$

with

$$Q=\alpha_1[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1')^2+4\epsilon]+\alpha_2[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k}_1)^2+4\epsilon]+\alpha_3[(\mathbf{k}_1'-\mathbf{p}_1)^2+\mu^2]+\alpha_4[p_1^2+\tfrac{1}{3}p_2^2-k_1^2+\tfrac{4}{3}\epsilon-i\eta]+\alpha_5[(\mathbf{k}_1-\mathbf{p}_1)^2+\mu^2]. \qquad (3.4)$$

The integrals over $\mathbf{p}_1$ and $\mathbf{p}_2$ can be performed in Eq. (3.3), leading to

$$R_2=\int_0^1d\alpha_1\cdots\int_0^1d\alpha_5\frac{[N(\alpha)]^{1/2}\delta(1-\sum\alpha_i)}{[D(\alpha,s,t)]^2}, \qquad (3.5)$$

with D of the form

$$D=sf(\alpha)+tg(\alpha)+h(\alpha). \qquad (3.6)$$

Here s is defined by

$$(3\hbar^2/4m)s=E,$$

and the functions f, g, and h, having been determined explicitly, are found to satisfy the inequalities

$$f(\alpha)<0, \quad g(\alpha)>0, \quad h(\alpha)>0 \qquad (3.7)$$

for $\alpha_i>0$. These inequalities imply that the function $R_2(s,t)$, as defined by Eq. (3.5) with $\eta=0$, is a real analytic function in the region $s<0$, $t>0$ since $D(\alpha,s,t)$ is positive in this region for $\alpha_i>0$. It may be shown, by examining the functions g and h, that D is in fact positive in the extended region $s<0$, $t>-16\epsilon$.

It is clear that with $s<0$ and $\mathrm{Im}t\neq0$, $D(\alpha,s,t)$ is nonvanishing so that $R_2(s,t)$ can be analytically continued into the complex t plane cut along the negative axis from $t=-\infty$ to $t=-16\epsilon$. Similarly, with $t>-16\epsilon$, $R_2(s,t)$ is analytic in s with a cut along the real axis from $s=0$ to $s=\infty$. The physical amplitude is recovered by letting s approach the positive real axis from above. This defines the physical sheet. It is easily established, by taking appropriate analytic continuations of $R_2(s,t)$, that singularities on the boundary of the physical sheet (s and t real) occur for undistorted α contours (i.e., $\alpha_i\geq0$). The location of these singularities is determined by the conditions that

$$\begin{aligned} \text{either} \quad & \alpha_i=0, \\ \text{or} \quad & \partial D/\partial\alpha_i=0, \end{aligned} \qquad (3.8)$$

which guarantee either a pinching or end-point singularity in each of the α integrations in Eq. (3.5). It is found that the curves of singularities on the boundary of the physical sheet are continuous, have positive slope, and are asymptotic to the threshold lines $s=0$, and $t=-16\epsilon$ or $t=-4\mu^2$.

With these properties established it may be shown that the curves of singularities have no extension into the complex region of the physical sheet. A simple way of showing this makes use of the method of analytic completion[7]; as this has been fully described we omit the details here. The Mandelstam representation then follows by a double application of Cauchy's theorem. The proof can also be carried through for the third Born term.[16] In the following section we examine the consequences of the assumption that the analytic properties in the variable t, with $s>0$, which have been established for the first few terms in the Born series, are retained in the full amplitude.[17]

4. DISPERSION-THEORETIC APPROACH

According to our assumption of analyticity (and, further, assuming no subtractions are necessary) we write

$$R(s,t)=R_B+\frac{1}{\pi}\int_{16\epsilon}^{\infty}\frac{A(s,t')}{t'+t}dt'. \qquad (4.1)$$

It is our purpose to evaluate $A(s,t)$ in terms of the absorptive part of the two-body scattering amplitude. The essential points in the method can, in fact, be well illustrated by an analysis of the Born term R_B to which we now turn our attention.

Keeping just the pole term in the expression for $\tilde{\varphi}(\mathbf{q})$ [see Eq. (3.2)] we obtain

$$R_B(t)=C^2\left[\int d\mathbf{r}\exp(i\mathbf{\Delta}\cdot\mathbf{r})V(r)\right]I(t), \qquad (4.2)$$

where

$$I(t)=\int d\mathbf{q}\frac{1}{q_1^2+\epsilon}\frac{1}{q_2^2+\epsilon}, \qquad (4.3)$$

with $\mathbf{q}_1=\mathbf{q}$ and $\mathbf{q}_2=\mathbf{q}+\frac{1}{2}\mathbf{\Delta}$. Direct evaluation of $I(t)$ (e.g., by Feynman parametrization) yields

$$I(t)=\frac{2\pi^2}{it^{1/2}}\ln\left[\frac{(-4\epsilon/t)^{1/2}+\frac{1}{2}}{(-4\epsilon/t)^{1/2}-\frac{1}{2}}\right]. \qquad (4.4)$$

We see that $I(t)$ is analytic in the complex t plane, with a cut running from $t=-16\epsilon$ to $t=-\infty$. $I(t)$ may, therefore, be represented as

$$I(t)=\frac{1}{\pi}\int_{-\infty}^{-16\epsilon}\frac{\mathrm{Im}I(t')}{t'-t}dt'. \qquad (4.5)$$

With the aid of Eq. (4.4) we get

$$I(t)=2\pi^2\int_{16\epsilon}^{\infty}(t')^{-1/2}\frac{1}{t'+t}dt'. \qquad (4.6)$$

The sign in Eq. (4.6) is determined by requiring that $I(t)$ be positive for $t>-16\epsilon$, the necessity of which can be seen from the parametrized form for $I(t)$.

To illustrate the technique for obtaining A in Eq. (4.1), we now re-evaluate $I(t)$ by making use of our knowledge of its singularities. By transforming variables of integration from q_1, θ, φ to q_1^2, q_2^2, and φ, with

$$q_2^2=q_1^2+\tfrac{1}{4}\Delta^2+q_1\Delta\cos\theta \qquad (4.7)$$

($\mathbf{\Delta}$ is taken to be along the z axis), Eq. (4.3) may be

[16] To study the general term in the Born series using the above method it would be necessary to obtain information about the functions $f(\alpha)$, $g(\alpha)$, and $h(\alpha)$, in particular the inequalities of Eq. (3.7). While such information can be readily obtained for each term in the Born expansion of the two-body amplitude (thereby providing another proof that these amplitudes satisfy a Mandelstam representation) we have as yet been unable to do so for the present case.

[17] When we apply the Landau-Bjorken conditions to the nth term in the Born series, $R_n(s,t)$, we find threshold singularities at $=-16\epsilon$ and $t=-n^2\mu^2$. Accordingly, a proof of the analyticity of $R(s,t)$ along the lines of that given for the two body problem (see reference 1) will be complicated by the fact that in this case the remainder, $R-R_n$, is *not* expected to be analytic inside a region which becomes arbitrarily large as $n\to\infty$.

written as

$$I(t)=\int_0^\infty dq_1^2\int_a^b dq_2^2\int_0^{2\pi} d\varphi\,\tfrac{1}{2}t^{-1/2}\times\frac{1}{q_1^2+\epsilon}\,\frac{1}{q_2^2+\epsilon},\qquad(4.8)$$

where the limits a and b are functions of q_1^2 and $\mathbf{\Delta}$ according to Eq. (4.7). The discontinuity across the cut starting at this singularity may be obtained from Eq. (4.8) by making the replacement

$$(q_1^2+\epsilon)^{-1}(q_2^2+\epsilon)^{-1}\rightarrow(2\pi i)\delta(q_1^2+\epsilon)(2\pi i)\delta(q_2^2+\epsilon).$$

To prove this statement, we follow the argument given by Cutkosky in his analysis of singularities in Feynman amplitudes.[9] We write Eq. (4.8) as

$$I(t)=\int_0^\infty F(q_1^2,t)(q_1^2+\epsilon)^{-1}dq_1^2.\qquad(4.9)$$

The singularity is due to the pinching of singularities in each of the factors in the integrand at $t=-16\epsilon$, which occurs at $q_1^2=-\epsilon$. The contour of integration in the q_1^2 plane may be taken to be the sum of two parts; the first part surrounds the point $q_1^2=-\epsilon$ and the second part is a contour which is not pinched as $t\rightarrow -16\epsilon$. Thus $I=I_1+I_2$, where

$$I_1=(2\pi i)F(-\epsilon,t),\qquad(4.10)$$

and I_2 doesn't contribute to the discontinuity across the cut. The limits a and b are determined by finding the maximum and minimum values of q_2^2 subject to the restriction that $q_1^2=-\epsilon$. The method of Lagrange's undetermined multipliers leads to the conditions

$$\mathbf{q}_2+\lambda\mathbf{q}_1=0,\qquad q_1^2=-\epsilon.\qquad(4.11)$$

Equations (4.11), along with the condition $q_2^2=-\epsilon$, imply, according to the Landau-Bjorken rules, that the singularity at $t=-16\epsilon$ due to the vanishing of $(q_2^2+\epsilon)$ occurs at one of the end points of the q_2^2 contour. This can also be seen directly from Eq. (4.7) by setting $\cos\theta=-1$. Since I_1 has the value

$$I_1(t)=(2\pi i)(2\pi)\tfrac{1}{2}t^{-1/2}\ln\frac{b+\epsilon}{a+\epsilon}\bigg|_{(q_1^2=-\epsilon)},\qquad(4.12)$$

the discontinuity across the cut starting at $t=-16\epsilon$ is just equal to the product of the factor $(2\pi^2 i)t^{-1/2}$ and the discontinuity of the logarithm across its cut starting at the origin. Since this latter factor is just $2\pi i$, the above-mentioned prescription for finding the discontinuity of $I(t)$ has been verified. The value of the discontinity thus obtained agrees with that obtained by direct evaluation of the integral, although we appeal to this latter calculation for the correct sign.

Turning to the integral of interest, we rewrite Eq. (2.16) as

$$R'=C^2\int d\mathbf{q}\,\frac{1}{q_1^2+\epsilon}\,\frac{1}{q_2^2+\epsilon}H(\mathbf{q};t)+R'',\qquad(4.13)$$

where we have isolated the contribution to R' due to the pole term in $\tilde{\varphi}(\mathbf{q})$. If we continue to assume that the second Born approximation to R' reveals the analytic structure of R' itself, we conclude that in the region $16\epsilon<t<4\mu^2$, $A(s,t)$ [see Eq. (4.1)] is identical with the absorptive part of the first term on the right-hand side of Eq. (4.13). [In fact, when the entire bound-state wave function is included the nearest additional singularity which is introduced lies at $t=-4(2\epsilon^{1/2}+\mu)^2$]. Further, the singularity at $t=-16\epsilon$ is due to the vanishing of the two denominators; H is analytic in the neighborhood of the singularity. [See Eq. (4.18).] In the expansion of H about the value of $\mathbf{q}$ determined by $q_1^2=q_2^2=-\epsilon$, only the leading term contributes to the absorptive part. We then find that

$$A(s,t)=\pi^2 t^{-1/2}\left[\int_0^{2\pi}d\varphi\,H(\mathbf{q}_0;\,-t)\right],\qquad 16\epsilon<t<4\mu^2,\qquad(4.14)$$

where

$$\mathbf{q}_0=(-\epsilon)^{1/2}(\cos\theta_0,\,\sin\theta_0\cos\varphi,\,\sin\theta_0\sin\varphi),\qquad \cos\theta_0=-(t/16\epsilon)^{1/2}.\qquad(4.15)$$

It may now be seen that $A(s,t)$ has the nice property that it can be expressed in terms of the analytic continuation of the physical two-body scattering amplitude. We observe that for $\mathbf{q}=\mathbf{q}_0$ the relations

$$(\alpha')^2=\alpha^2\equiv\alpha_0^2,\qquad (3\hbar^2/4m)(\beta^2-k_\gamma^2)=E_\gamma-(\hbar^2/m)\alpha_0^2,\qquad(4.16)$$

are valid. With $\mathbf{k}_1$ chosen as

$$\mathbf{k}_1=(\tfrac{1}{2}t^{1/2},\,[k_1^2-\tfrac{1}{4}t]^{1/2},\,0),$$

we find[18]

$$\alpha_0^2=(\tfrac{3}{4}k_1)^2-\tfrac{1}{4}\epsilon-\tfrac{3}{32}t+\tfrac{3}{4}\times\left[(k_1^2+\tfrac{1}{4}t)\left(\frac{t}{16}-\epsilon\right)\right]^{1/2}\cos\varphi.\qquad(4.17)$$

As a consequence of Eqs. (4.16), $H(\mathbf{q}_0;t)$ becomes

$$H(\mathbf{q}_0;t)=-\int d\mathbf{r}d\mathbf{r}'\exp(-\boldsymbol{\alpha}'\cdot\mathbf{r})V(r)\times\sum_\gamma[E_\gamma-(\hbar^2/m)\alpha_0^2-i\eta]^{-1}\times\psi_\gamma(\mathbf{r})\psi_\gamma^*(\mathbf{r}')V(r')\exp(i\boldsymbol{\alpha}\cdot\mathbf{r}').\qquad(4.18)$$

[18] Due to the subsequent integration over φ, no ambiguity arises with regard to the sign of the square root in Eq. (4.17).

Making use of the known analytic properties[1] of this function we may write Eq. (4.14) as

$$A(s,t)=(4\pi m/\hbar^2)C^2t^{-1/2}\int_0^{2\pi}d\varphi\int_0^\infty ds'\int_{4\mu^2}^\infty dt' \times\frac{\rho(s',t')}{(s'-\alpha_0^2-i\eta)(t'-t)},\quad 16\epsilon<t<4\mu^2,\quad (4.19)$$

where ρ is the double spectral function of the physical two-body scattering amplitude. It should be emphasized that the two-body amplitude is to be evaluated on its physical sheet; this is the instruction contained in the presence of the $i\eta$ term in the energy denominator of Eq. (4.18).[19] It is interesting to observe that for $t>16\epsilon$,α_0^2, as given by Eq. (4.17), can be positive only for positive values of $s=k_1^2-\frac{4}{3}\epsilon$. Thus, the right-hand side of Eq. (4.19) is analytic in the s plane cut along the positive axis. This is a necessary, though of course not sufficient, condition for $R(s,t)$ to satisfy a Mandelstam representation.

Note that the function

$$R_0=-2\left[\int_0^{2\pi}d\varphi\, f\left(\frac{\hbar^2}{m}\alpha_0^2,t\right)\right]\left[\int d\mathbf{q}\,\tilde{\varphi}(\mathbf{q})\tilde{\varphi}(\mathbf{q}+\tfrac{1}{2}\boldsymbol{\Delta})\right],$$

where f is the two-body scattering amplitude, has the same discontinuity across the cut in the region $-4\mu^2<t<-16\epsilon$ as does the function R. However, complex singularities (in addition to the cut on the positive s axis) will appear in R_0 for those values of s and t for which α_0^2 is positive. R_0 clearly corresponds to the more usual form of the impulse approximation.

Equation (4.19), along with the defining equations, Eqs. (4.1) and (4.18), represents the central result of this paper. The two major assumptions we have made are (a) the neglect of the binding potential U in the expression for the Green's function, Eq. (2.3), is justified in the scattering problem we have considered, and (b) the scattering amplitude R which results from this impulse approximation has the analytic properties expressed by Eq. (4.1). Equation (4.19) may be expected to give the dominant contribution to the absorptive part provided $4\mu^2$ is large compared to 16ϵ, and provided the analysis is confined to scattering in the region where t, the square of the momentum transfer, is small compared to $4\mu^2$. The significant feature of Eq. (4.19) is that it involves the absorptive part of the *physical* two-body scattering amplitude and thus makes possible a quantitative test of the usual procedure of extrapolating nonphysical amplitudes [i.e., the function I in Eq. (2.17)] back on to the energy shell. While we have no quantitative estimates to make at this time, our result suggests that any validity of this extrapolation procedure would be confined to the region of low t, and that for large t, the impulse approximation in its usual form should be subjected to further scrutiny. In fact, the above remark provides an instance of a general feature of the dispersion-theoretic approach, which might be worth emphasizing; this approach has built into it a natural way of assessing the validity of various approximations, namely, in terms of "distant" and "nearby" singularities. Another example of the use of this validity criterion is provided by the question of multiple scattering corrections, which we have ignored in the present paper. It is not difficult to see, however, that multiple scattering contributions to the amplitude involve threshold singularities which lie further out on the negative t axis (in lowest order perturbation theory the nearest of these singularities is at $t=-4\mu^2$) and may therefore be expected to be less important than the contribution considered here. We have ignored contributions to the scattering amplitude arising from intermediate states in which particles 2 and 3 are bound. These terms are singular with a threshold at $t=-64\epsilon$. With the aid of the unitarity relation the absorptive part associated with these terms can be calculated and, in the region $-4\mu^2<t<-64\epsilon$, it is given exactly in terms of the two-body spectral function. This, along with a construction of a unitary impulse approximation which effectively sums an infinite sub-class of diagrams, will be described at a later date.

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[19] A. Klein and C. Zemach, Ann. Phys. (N. Y.) **7**, 440 (1959).