The function f^{-1} satisfies elastic unitarity,

$$\mathrm{Im}[f^{-1}] = -(\nu/\nu+1)] - [\nu/(\nu+1)]^{1/2}, \quad (A4)$$

for $\nu < \nu_T$ and for $\nu > \nu_T$, we have

$$\mathrm{Im}[f^{-1}] = -\left[\left(\frac{\nu}{\nu+1}\right)^{1/2} + \frac{\kappa(\nu-\nu_T)^{1/2}}{\beta^2(\nu_\pi-\nu)^2+\kappa^2(\nu-\nu_T)}\right], \quad (A5)$$

and

$$\mathrm{Re}[f^{-1}] = \frac{A}{\nu}(\nu_\pi-\nu) - \frac{\beta(\nu_\pi-\nu)}{\beta^2(\nu_\pi-\nu)^2+\kappa^2(\nu-\nu_T)}. \quad (A6)$$

In the limit as $\delta_I \to 0$ it follows that

$$\lim_{\kappa\to 0} \mathrm{Re}[f^{-1}] = \frac{A}{\nu}(\nu_\pi-\nu) - \frac{1}{\beta(\nu_\pi-\nu)}, \quad (A7)$$

and

$$\lim_{\kappa\to 0} \mathrm{Im}[f^{-1}] = -\left[\left(\frac{\nu}{\nu+1}\right)^{1/2} + \frac{\pi}{\beta}\delta(\nu-\nu_\pi)\right]. \quad (A8)$$

This demonstrates that the CDD pole in f^{-1} moves onto the real axis as $\delta_I \to 0$.

PHYSICAL REVIEW VOLUME 129, NUMBER 2 15 JANUARY 1963

Quantum-Mechanical Measurement Operator

JAMES ALBERTSON
Department of Physics, Loyola University of Los Angeles, Los Angeles, California
(Received 12 July 1962)

A unitary operator is defined, connecting the states of the measured system and the measuring-instrument system before and after interaction, by means of which the post-interaction values of S in the instrument can be used to calculate the pre-interaction $\langle R\rangle_{\mathrm{av}}$ and $\Delta^2 R$ in the measured system, where R and S are Hermitian operators. The premeasurement state of the instrument need not be known, and the same measurement operator is applicable whether the system to be measured is originally described by a pure case or a mixture. Finally, this theory is contrasted briefly with the measurement theory of von Neumann.

IN this paper a formal theory of measurement for quantum mechanics is developed which seeks to realize, as nearly as possible, the same objectives proposed and attained in classical measurements. To this purpose a brief discussion of the nature of classical measurement and the necessary modifications imposed by quantum mechanics is followed by definition and investigation of a unitary operator which, it is said, successfully fills the role of a measurement operator in quantum mechanics. Because this theory differs in several respects from the well-known theory of von Neumann, some points of contrast are made explicit in an Appendix.

1. MEASUREMENT

The process of measurement, taken in a classical framework, can be conceived schematically as follows.

There is a physical system to be measured, i.e., a physical system with a property to which some numerical value can be assigned, and there is another physical system to act as measuring instrument, i.e., another physical system with a property to which some numerical value also can be assigned, and this value can be ascertained by reading the instrument. Before measurement the system to be measured is in an indefinable state such that the property in question has a definite but unknown value. A measurement is performed by allowing this system to interact for a time with the measuring instrument, and after this interaction the instrument is read, i.e., a numerical value is obtained from it by observation. If the interaction has been of the proper kind, then the numerical value read from the instrument can be correlated with the numerical value of the property to be measured as it existed in the measured system prior to the measurement—"prior to the measurement" because it seems essential to the notion of a measurement that it answer a question about the given situation existing before measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question.

When one applies this concept of the measurement process to the systems encountered in quantum mechanics, however, certain additional refinements must be made.[1]

It is no longer true in the quantum-mechanical case that the property of the system to be measured necessarily has a definite value before (or after) the measurement interaction. If the property is represented by the Hermitian operator R and the premeasurement state of the system by the normalized vector $|\phi\rangle$, then one can say only that in an ensemble of identical systems the property has the average value $\langle R\rangle_{\mathrm{av}} = \langle\phi|R|\phi\rangle$, with the dispersion about this mean given

[1] Of the very extensive literature on measurement in quantum mechanics perhaps the most informative and most provocative article is still that of H. Margenau, Phil. Sci. 4, 337 (1937).

by $\Delta^2 R = \langle R^2\rangle_{av} - \langle R\rangle_{av}^2$, and in general this dispersion will not vanish. Further indefiniteness is introduced if the ensemble of systems is represented not by a pure case but by a mixture with statistical operator U, so that $\langle R\rangle_{av} = \mathrm{Tr}(UR)$. In quantum mechanics, moreover, the interaction with the instrument changes the initial state of the system being measured, so that the result of measurement, as here described, is by the nature of things applicable only to the premeasurement state (or to that subensemble of the original ensemble which did not interact with the instrument). It is an extra dividend, so to speak, that in classical or macroscopic measurements one can assume the measured quantity will either be the same after measurement or else changed to an extent that can be allowed for in the calculation.

On the basis of this brief discussion of the measurement process, the following description of a measurement in quantum mechanics can be proposed: *The system in which quantity R is to be measured interacts with an appropriately chosen measuring instrument, in which quantity S can be read off, in such a way that the values of S after interaction give* $\langle R\rangle_{av}$ *and* $\Delta^2 R$ *as they were before interaction.* No mention is made here of using the measurement to determine $|\phi\rangle$ or U, for since the measurement deals with the directly observable it cannot reasonably be expected to give $|\phi\rangle$ or U, neither of which is immediately an object of experience.[2]

In addition to the system to be measured, which we can assume given, the measurement involves two other elements; namely, the choice of an appropriate system to act as measuring instrument, and the design of a suitable interaction.

As has been indicated, two basically different situations must be distinguished. The premeasurement state of the system on which the measurement is to be performed may be represented by a state vector $|\phi\rangle$, i.e., the system is in a pure case, or it may be represented only by a statistical operator U, i.e., the system is one of an ensemble in a mixture of quantum-mechanical states. As it may not be known which of these alternatives is realized in a given case, it is desirable that the same measurement process work equally well in both situations. For convenience the less general case will be treated first.

2. PURE CASE

Before measurement the unknown state of the system in which R is to be measured will be represented by the state vector $|\phi\rangle$ and the unknown state of the measuring instrument by $|\psi\rangle$. The eigenvectors of R are denoted by $|\phi_\lambda\rangle$ and those of S by $|\psi_\lambda\rangle$, so that

$$R|\phi_\lambda\rangle = r_\lambda|\phi_\lambda\rangle, \quad S|\psi_\lambda\rangle = s_\lambda|\psi_\lambda\rangle. \tag{1}$$

Operator S, representing the quantity read off in the instrument, is chosen so that the two sets of eigenvectors are equal in number. Previous to any interactions the combined state of the two systems will be represented by $|\Phi\rangle^{\text{before}}$, and one writes

$$|\Phi\rangle^{\text{before}} = |\phi\rangle \otimes |\psi\rangle = \sum_{\mu,\nu} a_\mu b_\nu |\phi_\mu\rangle \otimes |\psi_\nu\rangle, \tag{2}$$

where

$$a_\mu \equiv \langle\phi_\mu|\phi\rangle, \quad b_\nu \equiv \langle\psi_\nu|\psi\rangle. \tag{3}$$

In the pre-interaction state the average or expectation value of R is

$$\langle R\rangle_{av}^{\text{before}} = \sum_\mu |a_\mu|^2 r_\mu, \tag{4}$$

and the dispersion is given by

$$\Delta^2 R^{\text{before}} = \sum_\lambda |a_\lambda|^2 r_\lambda (r_\lambda - \sum_\mu |a_\mu|^2 r_\mu). \tag{5}$$

To perform a measurement, therefore, one must be able to determine the set $\{|a_\mu|^2\}$ from repeated readings of the quantity S in the measuring instrument after interaction with individual systems in an ensemble all of whose members are in the state described by $|\phi\rangle$.

The combined state of the two systems prior to the measurement interaction must be related to the combined state after interaction by means of a unitary operator, M, the measurement operator:

$$M|\Phi\rangle^{\text{before}} = |\Phi\rangle^{\text{after}}. \tag{6}$$

At this point, then, the measurement problem consists in construction of a suitable operator M. It will be shown that the desired unitary[3] operator is defined by the expression

$$M \sum_{\mu,\nu} a_\mu b_\nu |\phi_\mu\rangle \otimes |\psi_\nu\rangle = \sum_{\mu,\nu} a_\mu b_\nu |\phi_\nu\rangle \otimes |\psi_\mu\rangle, \tag{7}$$

where Eq. (2) has been used for $|\Phi\rangle^{\text{before}}$.

This operator has the required measurement properties, for it yields the result

$$\langle S\rangle_{av}^{\text{after}} = \sum_\mu |a_\mu|^2 s_\mu. \tag{8}$$

Each reading of the instrument, therefore, will give one of the values s_μ, and by establishing the frequency with which each of these values appears in repeated readings of the instrument it is possible to obtain the set of numbers $\{|a_\mu|^2\}$. Employed in Eqs. (4) and (5) these numbers allow calculation of $\langle R\rangle_{av}^{\text{before}}$ and $\Delta^2 R^{\text{before}}$. From the statistics of the instrument readings one passes to the statistics of the measured quantity, and thus the measurement has been made.

To perform this measurement it is not necessary to have an ensemble of measuring-instrument systems all

[2] The possibility of constructing an "equivalent" state vector is mentioned in reference 7. Note also in the Appendix the comments on von Neumann's projection postulate.

[3] The operator M, one may note, is not only unitary but also Hermitian, so that $M^2=1$. Its eigenvalues of 1 and -1 correspond, respectively, to the multiply degenerate eigenvectors

$$2^{-1/2}(1+\delta_{\mu\nu})^{-1/2}(|\phi_\mu\rangle\otimes|\psi_\nu\rangle + |\phi_\nu\rangle\otimes|\psi_\mu\rangle) \quad (\mu \leq \nu)$$
$$2^{-1/2}(|\phi_\mu\rangle\otimes|\psi_\nu\rangle - |\phi_\nu\rangle\otimes|\psi_\mu\rangle) \quad (\mu < \nu).$$

in the same initial state, since the measurement is independent of the instrument's initial state. Therefore one may use an ensemble of instruments in arbitrary states, or else the same instrument may interact successively with members of the ensemble of systems being measured if it is read between interactions.

An important feature of the measurement interaction defined by the operator M of Eq. (7) is that these interactions can be linked together to form a chain terminating in a measurement. Writing $\bar{a}_\mu$ for $(|a_\mu|^2)^{1/2}$, one can say that the post-interaction state of the instrument is equivalently

$$|\psi\rangle' = \sum_\mu \bar{a}_\mu e^{i\epsilon(\mu)}|\psi_\mu\rangle, \tag{9}$$

where $\epsilon(\mu)$ is an unknown phase factor—"equivalently" in the sense that it reproduces Eq. (8).[4] If now a second interaction takes place using this instrument as the system to be measured and choosing a new instrument whose state is represented by $|\chi\rangle$ and in which the quantity T can be read off, that measurement will be represented by the operator M_2, where

$$\begin{aligned} M_2|\psi\rangle'\otimes|\chi\rangle &= M_2\sum_{\mu,\nu}\bar{a}_\mu e^{i\epsilon(\mu)}c_\nu|\psi_\mu\rangle\otimes|\chi_\nu\rangle \\ &= \sum_{\mu,\nu}\bar{a}_\mu e^{i\epsilon(\mu)}c_\nu|\psi_\nu\rangle\otimes|\chi_\mu\rangle. \end{aligned} \tag{10}$$

After this second interaction one has

$$\langle T\rangle_{\text{av}}^{\text{after}} = \sum_\mu |a_\mu|^2 t_\mu, \tag{11}$$

and determination of the frequencies of the various t_μ through repeated interactions and readings of the new instrument allows one to calculate $\{|a_\mu|^2\}$, and thus the original $\langle R\rangle_{\text{av}}^{\text{before}}$ and $\Delta^2R^{\text{before}}$ are known. This procedure of constructing additional measurement interactions can be carried on indefinitely with no loss of precision in measuring the quantity initially sought. Such additivity is a desirable feature in any measurement process.

If the operators R and S represent the same dynamical quantity, then their sum may be conserved during the interaction. The expectation values are

$$\begin{aligned} \langle R+S\rangle_{\text{av}}^{\text{before}} &= \sum_{\mu,\nu}|a_\mu|^2|b_\nu|^2(r_\mu+s_\nu), \\ \langle R+S\rangle_{\text{av}}^{\text{after}} &= \sum_{\mu,\nu}|a_\mu|^2|b_\nu|^2(r_\nu+s_\mu), \end{aligned} \tag{12}$$

and these two expressions will be equal if the eigenvalues are such that

$$r_\nu - r_\mu = s_\nu - s_\mu. \tag{13}$$

For this case $R+S$ is conserved or, in other words, $R+S$ commutes with the measurement operator[5]:

$$[M, R+S]=0. \tag{14}$$

Finally, it is possible to give an explicit formulation of the measurement operator. Inspection of the definition of M in Eq. (7) shows that one can write[6]

$$M = \sum_{\mu,\nu}|\phi_\mu\rangle\langle\phi_\nu|\otimes|\psi_\nu\rangle\langle\psi_\mu|. \tag{15}$$

This form makes evident the dependence of the measurement interaction on both the quantity being measured and the quantity used as an index in the measuring instrument. Unitarity is apparent, and Eq. (14) is verified under the conditions stated in the discussion leading up to that equation.

3. MIXTURE

Thus far the measurement process has been considered for the pure case in which the system to be measured is found in a quantum-mechanical state represented by $|\phi\rangle$. But one must also allow for the possibility that the system is part of an ensemble which cannot be represented by a single state vector but only by a (positive-definite Hermitian) statistical operator U. This more general case can be treated briefly since it leads to no new difficulties.

Let the statistical operator for the combined systems (the system to be measured and the measuring-instrument system) before interaction be U, where

$$U = U_{\text{I}}\otimes U_{\text{II}}. \tag{16}$$

The operator U_{I} refers to the system to be measured and represents a mixture, while U_{II} refers to the measuring-instrument system and represents the state $|\psi\rangle$. The operators R and S are as before, and they again have the eigenvectors $|\phi_\lambda\rangle$ and $|\psi_\lambda\rangle$ and eigenvalues r_λ and s_λ. If M is the measurement operator defined in Eq. (7), its effect on the statistical operator will be given by

$$U' = MUM^\dagger, \tag{17}$$

where U' is the statistical operator of the combined systems after the interaction. Using these statistical operators one obtains

$$\begin{aligned} \langle R\rangle_{\text{av}}^{\text{before}} &= \text{Tr}(UR) \\ &= \sum_\mu [\sum_\nu w_\nu|a_{\nu\mu}|^2]r_\mu, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \langle S\rangle_{\text{av}}^{\text{after}} &= \text{Tr}(U'R) \\ &= \sum_\mu [\sum_\nu w_\nu|a_{\nu\mu}|^2]s_\mu, \end{aligned} \tag{19}$$

[4] Because of this arbitrary phase factor introduced by reading the instrument it is not possible to reapply the measurement operator after measurement and use the property $M^2=1$ to regain the premeasurement situation.

[5] It is also possible to consider conservation by observing that any quantity commuting with M (i.e., a conserved quantity) will have as eigenvectors the two eigenvector sets of M noted in reference 3. Those vectors are eigenvectors of $R+S$ under the condition of Eq. (13).

[6] This operator could also be obtained by summing the projection operators formed from the individual eigenvectors of M listed in reference 3, each projection operator being weighted with the corresponding eigenvalue (dyadic representation).

where $a_{\nu\mu}\equiv\langle\phi_\mu|u_\nu\rangle$, and w_ν and $|u_\nu\rangle$ are eigenvalues and eigenvectors of U_{I}. For a given statistical operator U_{I} the sum $\sum_\nu w_\nu|a_{\nu\mu}|^2$ is a function of μ, so that the situation represented by Eqs. (18) and (19) corresponds exactly to that in Eqs. (4) and (8). Thus the measurement process is the same in both cases.[7]

4. CONCLUSION

If one accepts the proposition that the function of measurement in quantum mechanics is to determine the average value and the dispersion of some physical quantity in a given system as they are prior to measurement, then it is possible to define a unitary operator which can rightly be called a measurement operator. This operator links the instrument readings after measurement with the premeasurement condition of the measured quantity in just the way that allows the desired calculation, and one need not assume the instrument is in a known state prior to the interaction.

The interaction which gives rise to the measurement can be effectuated through intermediate systems in a series leading up to an ultimate instrument reading and measurement without any loss of precision. In the measurement interaction itself not all quantities can be conserved, of course, but under certain circumstances the sum of the measured quantity and the measurement-index quantity is conserved.

A final property of the measurement is that the same measurement operator is equally effective whether the system on which the measurement is performed is a pure case or a mixture. Because the measurement is concerned only with the average value of a particular quantity it makes no distinction between state vectors and general statistical operators.[7a]

APPENDIX

The following comparison with the measurement theory of von Neumann may be of interest.

According to von Neumann[8] a measurement has been performed only if after interaction the quantities R and S, in the measured system and the instrument, respectively, will simultaneously have the pair of values r_μ and s_ν with probability 0 for $\mu\neq\nu$, and with probability $|\langle\phi_\mu|\phi\rangle|^2$ for $\mu=\nu$, where $|\phi\rangle$ is the unknown pre-measurement state of the measured system. To achieve this result he defines the unitary measurement operator Δ by the relation

$$\Delta\sum_{\mu,\nu=-\infty}^{\infty}x_{\mu\nu}|\phi_\mu\rangle\otimes|\psi_\nu\rangle=\sum_{\mu,\nu=-\infty}^{\infty}x_{\mu\nu}|\phi_\mu\rangle\otimes|\psi_{\mu+\nu}\rangle. \quad \text{(A1)}$$

Written out explicitly, the operator is

$$\Delta=\sum_{\mu,\nu}|\phi_\mu\rangle\langle\phi_\mu|\otimes|\psi_{\mu+\nu}\rangle\langle\psi_\nu|, \quad \text{(A2)}$$

which is unitary, provided $\{|\psi_\nu\rangle\}$ is an infinite set. Before measurement he assumes the instrument is in the known state $|\psi_0\rangle$ so that the combined state is given by

$$|\Phi\rangle=\sum_\mu\langle\phi_\mu|\phi\rangle|\phi_\mu\rangle\otimes|\psi_0\rangle, \quad \text{(A3)}$$

and after the measurement interaction it is

$$|\Phi\rangle'=\Delta|\Phi\rangle=\sum_\mu\langle\phi_\mu|\phi\rangle|\phi_\mu\rangle\otimes|\psi_\mu\rangle. \quad \text{(A4)}$$

By invoking what has been referred to as the "projection postulate,"[9] which states that each measurement puts a system into an eigenstate corresponding to the observed eigenvalue, von Neumann obtains a measurement of r_μ in the measured system through an observation of s_μ in the instrument, and the probability of this measurement is $|\langle\phi_\mu|\phi\rangle|^2$. Stating the result more generally, one can say that von Neumann's measurement operator together with the projection postulate yields the relation

$$\langle P[\phi_\mu]\otimes P[\psi_\nu]\rangle_{\text{av}}{}^{\text{after}}=\delta_{\mu\nu}\langle P[\phi_\mu]\rangle_{\text{av}}{}^{\text{before}}, \quad \text{(A5)}$$

as a statement of the measurement process, where $P[\phi_\mu]$ is the projection operator for $|\phi_\mu\rangle$. Equation (A5) is valid whether the measured system is described by a pure state or a mixture prior to the interaction.

The chief differences between von Neumann's theory of measurement and the theory developed in this paper are two. In the first place, von Neumann must assume the premeasurement state of the instrument is known, whereas the above theory does not make that assumption. In the second place, von Neumann employs the projection postulate to yield measurements which give an exact value to the measured quantity with each single reading of the instrument.[10] The theory of this paper does not use that postulate and produces a more thoroughly statistical type of measurement process.

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[7] In reference to the statement in Sec. 1 about the measurement not allowing calculation of $|\phi\rangle$ or U (here U_{I}), it may be remarked that the measurement does not even tell us whether the measured system was in a pure case or in a mixture. As far as the results of the measurement are concerned it is always possible to reconstruct an equivalent state vector for the measured system, i.e., one which will give Eqs. (4) and (18).

[7a] *Note added in proof*. Another publication will investigate the realization of this formal theory in actual physical process.

[8] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955), Chap. VI, Sec. 3, especially p. 440.

[9] For a recent criticism of this postulate on the basis of its incompatibility with accepted statistical notions, see H. Margenau and R. N. Hill, Progr. Theoret. Phys. (Kyoto) **26**, 722 (1961).

[10] There is clearly a relation between these two differences, since knowledge of the instrument's pre-interaction state is presumably gained by a measurement or observation which produces an eigenstate.