Analytic Approximation to the Low-Energy Solutions of Inverse Amplitude Dispersion Relations

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An analytic approximation to the solution of the inverse amplitude dispersion relations exhibiting two resonances with the same quantum numbers is presented. It is shown that, given a solution exhibiting one resonance, a double resonance solution can be produced without violating crossing symmetry if a CDD (Castillejo, Dalitz, and Dyson) pole is inserted near the original resonance position. In the presence of inelastic scattering the CDD pole must be off the real axis and will occur on an unphysical sheet. For various values of the pion-pion coupling constant in the range $-0.15 \leq \lambda \leq -0.1$ the masses and widths of the ζ and ρ resonances are calculated.

IN a recent paper,[1] we have presented solutions of the inverse amplitude dispersion relations for pion-pion scattering including inelastic intermediate states. When the inelastic cross section is small and slowly varying, there exist solutions exhibiting two P-wave resonances if the solution of the original pure elastic dispersion relations possesses a single sharp resonance in the P wave. The splitting of this single resonance occurs when the real part of the phase shift passes through π in the inelastic region, since then the function $R=1+\sigma^{\text{in}}/\sigma^{\text{el}}$ is peaked at this point, σ^{el} having a minimum, and this peak gives a large contribution.

In this note a simple analytic approximation is presented, which leads to the double resonance solution assuming that one knows the solution of the dispersion relations with only pure elastic scattering.[2] The notation of reference 1 is used throughout this paper. Let the original solution[3] be described in the physical region by

$$\begin{aligned}\mathrm{Re}[\nu f^{-1}(\nu)]&=A(\nu_R-\nu),\\ \mathrm{Im}[f^{-1}(\nu)]&=-[\nu/(\nu+1)]^{1/2},\end{aligned}\tag{1}$$

where ν_R and A^{-1} are the position and reduced width of the single resonance. When inelastic intermediate states are included, (1) is modified to read

$$\mathrm{Re}[\nu f^{-1}(\nu)]=A(\nu_R-\nu)-\frac{\nu}{\pi}P\times\int_{\nu_T}^{\infty}\frac{d\nu'[\nu'/(\nu'+1)]^{1/2}[R(\nu')-1]}{\nu'-\nu}\tag{2}$$

$$\mathrm{Im}[f^{-1}(\nu)]=-[\nu/(\nu+1)]^{1/2}R(\nu).$$

As will be shown later, the splitting of the resonance has little effect on the inverse amplitude left cut and, therefore, the constants ν_R and A in (2) will be the same as in (1), but there is no longer a resonance at ν_R. The complications arising from the use of subtracted dispersion relations are ignored[1] [we have assumed here that $R(\nu)-1\sim\nu^{-2}$ as $|\nu|\to\infty$].

Above the inelastic threshold ν_T, $\delta=\delta_R+i\delta_I$ and let us suppose that δ_R passes through π at $\nu=\nu_\pi$, say, and that $\delta_I(\nu_\pi)$ is small. Then from scattering theory[1] R may be written in the form

$$R\approx\frac{\delta_I}{(\delta_R-\pi)^2+\delta_I{}^2}+1\tag{3}$$

for the region in which $(\delta_R-\pi)^2\ll\delta_I$. As δ_I is small, R has a resonant behavior and this resonance dominates the integral in (2). The analyticity properties of $f(\nu)$ are discussed in Appendix A, where it is shown that f^{-1} has a CDD (Castillejo, Dalitz, and Dyson)[4] pole on the unphysical sheet. Expanding δ_I and $(\delta_R-\pi)$ in powers of $(\nu-\nu_\pi)$ and keeping only the lowest order terms, it follows that

$$\begin{aligned}\delta_I&\approx\alpha,\\ \delta_R-\pi&\approx\beta(\nu-\nu_\pi),\end{aligned}\tag{4}$$

and

$$[\nu/(\nu+1)]^{1/2}[R(\nu)-1]\approx\alpha/[\beta^2(\nu-\nu_\pi)^2+\alpha^2].\tag{5}$$

With this approximation, it will be shown that $\nu_\pi=\nu_R$. From scattering theory,[1] we have

$$\mathrm{Re}[\nu f^{-1}(\nu)]=\left(\frac{\nu^3}{\nu+1}\right)^{1/2}\frac{2\sin 2\delta_R e^{-2\delta_I}}{1-2\cos 2\delta_R e^{-2\delta_I}+e^{-4\delta_I}}.\tag{6}$$

By substituting (5) into the integral in (2), we get

$$\frac{\nu}{\pi}\int_{\nu_T}^{\infty}\frac{d\nu'[\nu'/(\nu'+1)]^{1/2}[R(\nu')-1]}{\nu'-\nu}\approx-\frac{\nu_\pi\beta(\nu-\nu_\pi)}{[\beta^2(\nu-\nu_\pi)^2+\alpha^2]}.\tag{7}$$

Both the expressions (6) and (7) vanish when $\nu=\nu_\pi$ and, therefore, from (2) we find $A(\nu_R-\nu_\pi)=0$, i.e., $\nu_\pi=\nu_R$ (since $A\neq0$).

[1] B. H. Bransden, I. R. Gatland, and J. W. Moffat, Phys. Rev. **128**, 859 (1962).

[2] B. H. Bransden and J. W. Moffat, Nuovo Cimento **21**, 505 (1961).

[3] See in particular Fig. 3 of reference 2.

[4] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

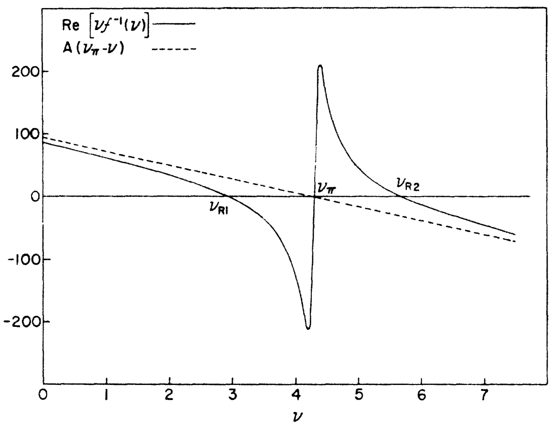

FIG. 1. The functions $\mathrm{Re}[\nu f^{-1}(\nu)]$ (full line) and $A(\nu_\pi - \nu)$ (broken line) for $\lambda = -0.13$, $\alpha = 0.01$, and $\beta = 0.1$.

For pion-pion scattering the values of ν_π and A are determined by the pion-pion coupling constant λ.[2] In the numerical solution of the dispersion relations presented in reference 1 there occur two independent parameters λ and G^2 (where G^2 corresponds to α), and β is determined by the iteration scheme, but in the approximate solution presented here we do not attempt to calculate β, which remains an arbitrary constant. Thus, we have three parameters λ, α, and β at our disposal. We have already assumed that α and β are small and experimental evidence from S-wave scattering gives $\lambda \simeq -0.15$.[5]

From (2) and (7), we obtain

$$\mathrm{Re}[\nu f^{-1}] = A(\nu_\pi - \nu) + \beta\nu_\pi(\nu - \nu_\pi)/[\beta^2(\nu - \nu_\pi)^2 + \alpha^2]. \quad (8)$$

The functions $\mathrm{Re}[\nu f^{-1}]$ and $A(\nu_\pi - \nu)$ are shown in Fig. 1 and $\mathrm{Re}[\nu f^{-1}]$ has three zeros given by

(i) $\nu = \nu_\pi$ corresponding to $\delta_R = \pi$ (a minimum in σ^{el}),

(ii) $\nu = \nu_{R1} \equiv \nu_\pi - (\nu_\pi / A\beta - \alpha^2/\beta^2)^{1/2}$,

and

$$\text{(iii)} \quad \nu = \nu_{R2} \equiv \nu_\pi + (\nu_\pi / A\beta - \alpha^2/\beta^2)^{1/2}. \quad (9)$$

Here (ii) and (iii) correspond to $\delta_R = \pi/2$ and $3\pi/2$, respectively, and give the positions of the two resonances in σ^{el}. At ν_{R1} and ν_{R2}, R is given by[1]

$$R = 2/(1 + e^{-2\delta_I}) \approx 1 + \alpha. \quad (10)$$

If, however, we use (5) and (9), we get

$$R \approx 1 + \alpha A/\beta\nu_\pi, \quad (11)$$

which disagrees with (10) as $\nu_\pi < A^2$ and $\beta < 1$. We expect this disagreement if (11) is used some distance from the peak in R and from (9) this will be the case if

$$(\nu_\pi / A\beta - \alpha^2/\beta^2)^{1/2} \gg \Delta = \alpha/\beta, \quad (12)$$

where Δ is the half-width of the peak in R. If we adopt (12) as a consistency requirement it will place a restriction on the parameter β:

$$\beta \gg 2\alpha^2 A/\nu_\pi. \quad (13)$$

The reduced width γ, which is the same for both resonances in this approximation, is given by

$$\gamma = \left[-\frac{R(\nu)}{(d/d\nu)\,\mathrm{Re}[\nu f^{-1}]} \right]_{\nu=\nu_{R1},\nu_{R2}} \approx (1/2A)(1 + \alpha + \alpha^2 A/\nu_\pi\beta) \approx 1/2A. \quad (14)$$

In the pure elastic case the reduced width is $\gamma_0 = A^{-1}$ so that $\gamma \approx \frac{1}{2}\gamma_0$.

In view of these results, we see that the total area and the center of mass of the two resonances are approximately the same as for the single resonance in the pure elastic case. This justifies the remark made earlier that the splitting of the resonance has little effect on the inverse amplitude left cut, which determines the constants ν_π and A.

The exact numerical solutions of the S waves for $I = 0, 2$ have shown that the effective-range behavior of the S waves is given correctly to within 10% by the formula[2]

$$\left(\frac{\nu}{\nu+1}\right)^{1/2} \cot\delta_0{}^I = \frac{1}{a_I} + \frac{2}{\pi}\left[\left(\frac{\nu}{\nu+1}\right)^{1/2} \times \log[(\nu)^{1/2} + (\nu+1)^{1/2}] - \sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}\right], \quad (15)$$

where $a_0 = -5\lambda$ and $a_2 = -2\lambda$. There is little change in (15) when inelastic intermediate states are included for $I = 0, 2$ provided that the inelastic cross section is not too large.[1]

As an example of the application of the approximate inelastic solution for the P-wave case, we consider the pion-pion scattering solution (without cutoff) given in reference 1. We have $\lambda = -0.1$, $\alpha = 0.025$, $\beta = 0.122$ and the values of ν_π and A corresponding to this value of λ in the pure elastic case are $\nu_\pi = 4.6$, $A = 29$. The results given by the approximate equations and the exact calculation are compared in Table I, the masses and total widths of the resonances being given by

$$M_\zeta = 2\mu(\nu_{R1} + 1)^{1/2},$$
$$M_\rho = 2\mu(\nu_{R2} + 1)^{1/2},$$
$$\Gamma_\zeta = \mu\left[\left(\frac{\nu_{R1}}{\nu_{R1}+1}\right)^{1/2} \nu_{R1}\gamma\right]^{1/2}, \quad (16)$$
$$\Gamma_\rho = \mu\left[\left(\frac{\nu_{R2}}{\nu_{R2}+1}\right)^{1/2} \nu_{R2}\gamma\right]^{1/2},$$

where μ is the pion mass.

[5] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. 128, 1881 (1962).

In Table II are given the values of M_ζ, M_ρ, Γ_ζ, and Γ_ρ for various values of λ, α, and β calculated using the approximate equations.

The techniques for dealing with the double resonances presented in this note are not limited to pion-pion scattering, or to P waves. All that is necessary is that there exists a part of the physical region in which the solution derived from the left cut contributions has the form given by (1) (i.e., $\mathrm{Re}[\nu f^{-1}]$ passes linearly through zero), that δ_I be small there and that the two resonances exhibited when CDD pole effects are added should appear as a single resonance when viewed from the left cut (or any other cuts which may be present).[6] For instance, in pion-hyperon scattering, if the Y_0^* and Y_0^{**} were found to have the same quantum numbers, they could be explained by this theory. In general, for other processes the solution of the elastic dispersion relations is not known so that the single parameter λ, which serves for pion-pion scattering, must be replaced by the two parameters ν_π and A. Also, if α is very small, it can be ignored except insofar as it limits β.

The approximations in this note can be refined [particularly (7)], but in view of the approximate nature of the initial assumption (5), and the possibility of exact calculations on the computer, this hardly seems worthwhile.

In the Appendix, it is shown that in the limit as $\delta_I \to 0$ the amplitude f becomes

$$f(\nu)=\left[\frac{A}{\nu}(\nu_\pi-\nu)+\frac{1}{\beta(\nu-\nu_\pi)}-i\left(\frac{\nu}{\nu+1}\right)^{1/2}-i\frac{\pi}{\beta}\delta(\nu-\nu_\pi)\right]^{-1}. \quad (17)$$

The combination of the inverse amplitude left cut contribution [given by the first term on the right of (17)] and the CDD pole effect (the second term) produces the two resonances in σ^{el}. The phase shift δ_R passes through π (changes sign) between the two resonances and when $\delta_I>0$ this corresponds to an unstable elementary particle.

TABLE I. The values of ν_π, A, M_ζ, M_ρ, and Γ_ζ, Γ_ρ obtained from the approximate equations compared with the results of an exact numerical calculation for $\lambda=-0.1$ and $\alpha=0.025$. The exact calculation gives $\beta=0.122$ and this value has been used in the approximate calculations.

	Approximate calculation	Exact calculation
ν_π	4.6	5.2
A	29	26
M_ζ	590 MeV	640 MeV
M_ρ	725 MeV	730 MeV
Γ_ζ	35 MeV	35 MeV
Γ_ρ	45 MeV	40 MeV

[6] J. Kennedy and T. D. Spearman, Phys. Rev. **126**, 1596 (1962).

TABLE II. Values of M_ζ, M_ρ and Γ_ζ, Γ_ρ in MeV for various values of the parameters λ, α, and β. The values of ν_π and A corresponding to each λ are also given.

	α	β	M_ζ	M_ρ	Γ_ζ	Γ_ρ
$\lambda=-0.10$	0.01	0.10	585	735	30	45
$\nu_\pi=4.6$	0.01	0.04	535	770	30	45
$A=29$	0.04	0.04	550	760	30	50
$\lambda=-0.13$	0.01	0.10	555	725	35	50
$\nu_\pi=4.3$	0.01	0.04	495	765	30	50
$A=22$	0.04	0.04	510	755	35	55
$\lambda=-0.15$	0.01	0.10	535	720	35	50
$\nu_\pi=4.1$	0.01	0.04	570	760	30	55
$A=19$	0.04	0.04	485	755	30	60

The possible existence of a pion-pion scattering solution with a CDD pole has been noted previously,[7] but not investigated in detail. In the present paper the CDD pole is inserted into the known solution of the elastic pion-pion scattering equations, and its effect on the left cut is shown to be small. It was not possible to obtain such a solution with the original numerical iteration scheme for pure elastic scattering,[2] because the CDD pole would have occurred on the real axis and produced an infinity in $\mathrm{Re}[f^{-1}]$. However, when inelastic scattering is included no infinity occurs and the double resonance solution with a CDD pole is generated by a numerical iteration of the inverse amplitude equations as demonstrated in reference 1.

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APPENDIX

Let us consider the analyticity properties of the modified form of the inverse amplitude given by (2). We remark that the phase shift $\delta(\nu)$ is analytic in the cut plane with the cut in the physical region beginning at the inelastic threshold ν_T. In view of this the discontinuity δ_I is written $\delta_I=\alpha=\kappa(\nu-\nu_T)^{1/2}$, where κ is an analytic function of ν and is positive on the real axis. Substituting this expression for δ_I into (2), (5), and (8), we find

$$f^{-1}(\nu)=\frac{A}{\nu}(\nu_\pi-\nu)-i\left(\frac{\nu}{\nu+1}\right)^{1/2}-\frac{1}{\beta(\nu_\pi-\nu)-i\kappa(\nu-\nu_T)^{1/2}}, \quad (\mathrm{A1})$$

which satisfies the reality condition $[f^{-1}(\nu)]^*=f^{-1}(\nu^*)$. The function f^{-1} has a CDD pole on the unphysical sheet in the $\nu-\nu_T$ plane and its position ν_P is given by

$$\beta(\nu_\pi-\nu_P)-i\kappa(\nu_P-\nu_T)^{1/2}=0. \quad (\mathrm{A2})$$

Assuming that $\alpha\ll\beta$ it is found that

$$\nu_P\approx\nu_\pi-i\alpha/\beta. \quad (\mathrm{A3})$$

[7] J. W. Moffat, Phys. Rev. **121**, 926 (1961).

The function f^{-1} satisfies elastic unitarity,

$$\text{Im}[f^{-1}] = -(\nu/\nu+1)] - [\nu/(\nu+1)]^{1/2}, \quad \text{(A4)}$$

for $\nu < \nu_T$ and for $\nu > \nu_T$, we have

$$\text{Im}[f^{-1}] = -\left[\left(\frac{\nu}{\nu+1}\right)^{1/2} + \frac{\kappa(\nu-\nu_T)^{1/2}}{\beta^2(\nu_\pi-\nu)^2+\kappa^2(\nu-\nu_T)}\right], \quad \text{(A5)}$$

and

$$\text{Re}[f^{-1}] = \frac{A}{\nu}(\nu_\pi-\nu) - \frac{\beta(\nu_\pi-\nu)}{\beta^2(\nu_\pi-\nu)^2+\kappa^2(\nu-\nu_T)}. \quad \text{(A6)}$$

In the limit as $\delta_I \to 0$ it follows that

$$\lim_{\kappa\to 0} \text{Re}[f^{-1}] = \frac{A}{\nu}(\nu_\pi-\nu) - \frac{1}{\beta(\nu_\pi-\nu)}, \quad \text{(A7)}$$

and

$$\lim_{\kappa\to 0} \text{Im}[f^{-1}] = -\left[\left(\frac{\nu}{\nu+1}\right)^{1/2} + \frac{\pi}{\beta}\delta(\nu-\nu_\pi)\right]. \quad \text{(A8)}$$

This demonstrates that the CDD pole in f^{-1} moves onto the real axis as $\delta_I \to 0$.

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Quantum-Mechanical Measurement Operator

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A unitary operator is defined, connecting the states of the measured system and the measuring-instrument system before and after interaction, by means of which the post-interaction values of S in the instrument can be used to calculate the pre-interaction $\langle R\rangle_{\text{av}}$ and $\Delta^2 R$ in the measured system, where R and S are Hermitian operators. The premeasurement state of the instrument need not be known, and the same measurement operator is applicable whether the system to be measured is originally described by a pure case or a mixture. Finally, this theory is contrasted briefly with the measurement theory of von Neumann.

IN this paper a formal theory of measurement for quantum mechanics is developed which seeks to realize, as nearly as possible, the same objectives proposed and attained in classical measurements. To this purpose a brief discussion of the nature of classical measurement and the necessary modifications imposed by quantum mechanics is followed by definition and investigation of a unitary operator which, it is said, successfully fills the role of a measurement operator in quantum mechanics. Because this theory differs in several respects from the well-known theory of von Neumann, some points of contrast are made explicit in an Appendix.

1. MEASUREMENT

The process of measurement, taken in a classical framework, can be conceived schematically as follows.

There is a physical system to be measured, i.e., a physical system with a property to which some numerical value can be assigned, and there is another physical system to act as measuring instrument, i.e., another physical system with a property to which some numerical value also can be assigned, and this value can be ascertained by reading the instrument. Before measurement the system to be measured is in an indefinable state such that the property in question has a definite but unknown value. A measurement is performed by allowing this system to interact for a time with the measuring instrument, and after this interaction the instrument is read, i.e., a numerical value is obtained from it by observation. If the interaction has been of the proper kind, then the numerical value read from the instrument can be correlated with the numerical value of the property to be measured as it existed in the measured system prior to the measurement—"prior to the measurement" because it seems essential to the notion of a measurement that it answer a question about the given situation existing before measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question.

When one applies this concept of the measurement process to the systems encountered in quantum mechanics, however, certain additional refinements must be made.[1]

It is no longer true in the quantum-mechanical case that the property of the system to be measured necessarily has a definite value before (or after) the measurement interaction. If the property is represented by the Hermitian operator R and the premeasurement state of the system by the normalized vector $|\phi\rangle$, then one can say only that in an ensemble of identical systems the property has the average value $\langle R\rangle_{\text{av}} = \langle\phi|R|\phi\rangle$, with the dispersion about this mean given

[1] Of the very extensive literature on measurement in quantum mechanics perhaps the most informative and most provocative article is still that of H. Margenau, Phil. Sci. 4, 337 (1937).