Sampson-Seitz Procedures and the Static Responses of Normal Fermion Systems

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It is shown that a differential characterization of Sampson-Seitz methods for calculating the static responses of a normal fermion system at zero temperature yields expressions for these quantities which are identical with those deduced by Landau from a semiphenomenological basis and by the author, in several instances, from spherical, time-independent, many-body perturbation theory. This connection is explicitly demonstrated for the Galilean invariance, magnetic susceptibility, and compressibility of a normal fermion system with translation-invariant interactions. The calculation of the magnetic susceptibility and compressibility of a dense electron gas is examined from both points of view. In the case of the magnetic susceptibility, it is shown that the contribution to α_c from graphs with two virtual excitations vanishes to $O(r_s)$ in agreement with the Sampson-Seitz calculation of Brueckner and Sawada. Although this demonstration is trivial by Sampson-Seitz methods, it is only a consequence of detailed calculation in Landau's formulation. It is found that $(1/K)(2/\rho R)=\alpha_c-0.0676\ln r_s+0.255$, where K is the compressibility, ρ is the density, and R is the rydberg.

I. INTRODUCTION

OUR primary objective in this paper is the reconcilation of Sampson-Seitz[1] procedures for the calculation of the static responses of normal fermion systems[2] at zero temperature with the canonical relations for the same quantities due to Landau.[3] An example of the latter is the Landau expression[3] for the magnetic susceptibility,[4]

$$\chi=\frac{\mu^2k_F}{\pi^2}\left[\frac{1}{M_{(\text{exact})}{}^*}+\frac{k_F}{2(2\pi)^3}\int_{(k,k'=k_F)}d\Omega_{k'}\,f_{(\text{exact})}{}^{\text{ex}}(\mathbf{k},\mathbf{k}')\right]^{-1}. \quad (1.1)$$

$M_{(\text{exact})}{}^*$, the effective mass of a quasi-particle to all orders of coupling, has also been related by Landau[3] through the principle of Galilean invariance[3] to the P-wave part of the "ordinary"[5] forward scattering amplitude[6] of quasi-particles evaluated at the Fermi surface,[3]

$$\frac{1}{M_{(\text{exact})}{}^*}=\frac{1}{M}-\frac{2k_F}{(2\pi)^3}\int_{(k,k'=k_F)}d\Omega_{k'}\,(\hat{k}\cdot\hat{k}')\times f_{(\text{exact})}{}^0(\mathbf{k},\mathbf{k}'). \quad (1.2)$$

$f_{(\text{exact})}{}^{\text{ex}}$ denotes the forward (exchange) scattering amplitude of quasi-particles[6] exact to all orders of particle-particle coupling. We remark that expression (1.1) has recently been rigorously derived from spherical, time-independent, many-body perturbation theory[7]; Eq. (1.2) was similarly derived[6] from perturbation theory in a discussion of the static response known as the cranking moment[6] in the case of a periodic system with the additional weakly restrictive assumption of a translation-invariant particle-particle interaction. Far from being "phenomenological" statements, relations (1.1) and (1.2), as well as the Landau expression for the compressibility,[3]

$$K=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T=\frac{k_F}{\rho^2\pi^2}\left[\frac{1}{M_{(\text{exact})}{}^*}+\frac{2k_F}{(2\pi)^3}\int_{(k,k'=k_F)}d\Omega_k\,f_{(\text{exact})}{}^0(\mathbf{k},\mathbf{k}')\right]^{-1}, \quad (1.3)$$

ought to be properly regarded as consequences of the structure of many-body perturbation theory; the "Landau decomposition" of the static responses of normal fermion systems into two-legged effective-mass graphs and four-legged scattering graphs[6] (with these legs evaluated on the Fermi surface) follows directly from the "opening" of internal lines in closed ground-state energy graphs *on the Fermi surface.*

In view of the extraordinary simplicity of the zero-temperature results of Fermi liquid theory,[3] it is of some interest to find them again in the Sampson-Seitz[1] procedures which take a seemingly different standpoint.

[1] J. B. Sampson and F. Seitz, Phys. Rev. **58**, 633 (1940), hereafter referred to as SS. A detailed application of this method may also be found in Sec. 9 of D. Pines, in *Solid-State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1955), Vol. **1**, p. 367.

[2] The appellation "normal fermion system" denotes a fermion system amenable to ordinary linked-cluster perturbation theory.

[3] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 1058 (1956) [translation: Soviet Phys.—JETP **3**, 920 (1956)]; L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, (1958), pp. 207–213; A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22, p. 329.

[4] We have taken $\hbar=1$.

[5] For spin-independent potentials, the Hamiltonian H is invariant under rotations about the axis of quantization; that is, $\exp(-iS_z\alpha)H\exp(-iS_z\alpha)=H$, where $S_z=\frac{1}{2}\int d\mathbf{r}\psi^\dagger(\mathbf{r})\sigma_z\psi(\mathbf{r})$. Hence the forward scattering amplitude regarded as a matrix in spin space has a spin dependence no more complicated than $\mathbf{f}=f^0(\mathbf{p},\mathbf{p}')\mathbf{I}+\frac{1}{4}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'f^{\text{ex}}(\mathbf{p},\mathbf{p}')$.[3]

[6] The forward scattering amplitude $\mathbf{f}$ is to be understood as the limit $q\to0$ of the forward scattering amplitude of pairs with small net-momentum $\mathbf{q}$ in the irreducible sense. (One omits scattering graphs constructed of subgraphs connected by pairs with small net-momentum $\mathbf{q}$.) [See R. M. Rockmore, Phys. Rev. **124**, 27 (1961).]

[7] R. M. Rockmore, Phys. Rev. **125**, 1778 (1962).

We will show in Sec. II that the explicit demonstration of the equivalence of these points of view requires a differential characterization of the SS procedures. For purposes of illustration, we consider, later in this paper, the calculation of the magnetic susceptibility and compressibility of a dense electron gas. The former problem has previously been treated by Brueckner and Sawada[8] according to the SS method. We will find it instructive to review that calculation from both Landau and Sampton-Seitz (in our present interpretation) points of view. These considerations also furnish the basis for a calculation, by both methods, of the compressibility of such a system to $O(r_s)$ in the remainder of the paper.

II. GENERATOR PROPERTIES OF GENERALIZED SAMPSON-SEITZ PROCEDURES

To exhibit the proper connection between Sampson-Seitz procedures and the Landau results for the various static responses of a normal fermion system at zero temperature, it will not be necessary to use a formalism any more sophisticated than that employed by Landau himself, namely, the method of functional variation.[3] Since the SS procedure was originally[1] formulated to deal with the calculation of the magnetic susceptibility, we discuss that response first. According to SS, we are instructed to calculate the change in the ground-state energy of the system as its spins are polarized, i.e., as the population of electrons of each spin varies, to $O[(\delta p)^2]$, where δp is the polarization parameter,[9] adding to this the interaction energy (spin-field energy), $O(\delta p)$, and then minimize the resulting *total* change in energy with respect to δp. δp, so variationally determined, is then substituted in $\delta_{(\mathrm{tot})}E$, yielding the susceptibility per unit volume χ, through the familiar relationship

$$\delta_{(\mathrm{tot})}E=\tfrac{1}{2}N\chi H^2. \tag{2.1}$$

The change in the ground-state energy due to a *virtual* change in the spin population of the system may be obtained through the introduction of two fictive "perturbed" Fermi surfaces,[10] one for spin up and one for the spin down. Thus one introduces

$$k_F{}^\sigma=[1+\zeta(\sigma)\delta p]^{1/3}k_F, \tag{2.2}$$

where $\zeta(\sigma)$ is the sign function for spin,

$$\zeta(\sigma)=\pm 1,\quad \sigma=\pm 1. \tag{2.3}$$

Rather than replace the unperturbed distribution functions,[11]

$$\theta(k_F-p)=n(\mathbf{p},\sigma), \tag{2.4}$$

by the corresponding perturbed ones,

$$n'(\mathbf{p},\sigma)=\theta(k_F{}^\sigma-p), \tag{2.5}$$

wherever they occur in E, as is usually done in the conventional interpretation of the Sampson-Seitz procedure,[8] we shall instead expand the perturbed distribution function, $n'(\mathbf{p},\sigma)$, *directly* in powers of the virtual polarization, δp, so that

$$n'(\mathbf{p},\sigma)\to n(\mathbf{p},\sigma)+\delta n(\mathbf{p},\sigma), \tag{2.6}$$

where[12]

$$\delta n(\mathbf{p},\sigma)=\delta p\left[\frac{\partial k_F{}^\sigma}{\partial(\delta p)}\right]_0\delta(p-k_F)+\frac{1}{2!}(\delta p)^2\left\{\left[\frac{\partial^2 k_F{}^\sigma}{\partial(\delta p)^2}\right]_0\delta(p-k_F)+\left[\frac{\partial k_F{}^\sigma}{\partial(\delta p)}\right]_0^2\frac{d}{dk_F}\delta(p-k_F)\right\}, \tag{2.7}$$

with

$$\left[\frac{\partial k_F{}^\sigma}{\partial(\delta p)}\right]_0=\tfrac{1}{3}\zeta(\sigma)k_F, \tag{2.8}$$

and

$$\left[\frac{\partial^2 k_F{}^\sigma}{\partial(\delta p)^2}\right]_0=-(2/9)k_F. \tag{2.9}$$

After Landau,[3] we write the virtual change in the ground-state energy density due to a virtual change in $n(\mathbf{p},\sigma)$ as

$$\begin{aligned}\delta E&=E[n(\mathbf{p},\sigma)+\delta n(\mathbf{p},\sigma)]-E[n(\mathbf{p},\sigma)]\\&=\sum_\sigma\int\frac{\delta E}{\delta n(\mathbf{p},\sigma)}\delta n(\mathbf{p},\sigma)d\tau\\&\quad+\tfrac{1}{2}\sum_{\sigma\sigma'}\int\int\frac{\delta^2 E}{\delta n(\mathbf{p},\sigma)\delta n(\mathbf{p}',\sigma')}\\&\qquad\times\delta n(\mathbf{p},\sigma)\delta n(\mathbf{p}',\sigma')d\tau d\tau',\end{aligned} \tag{2.10}$$

where[3]

$$d\tau=d\mathbf{p}/(2\pi)^3. \tag{2.11}$$

We make, after Landau,[3] the usual identifications,

$$\frac{1}{V}\frac{\delta E}{\delta n(\mathbf{p},\sigma)}=\epsilon(\mathbf{p},\sigma)=\epsilon(\mathbf{p})-\zeta(\sigma)\mu H, \tag{2.12}$$

and

$$\frac{1}{V}\frac{\delta^2 E}{\delta n(\mathbf{p},\sigma)\delta n(\mathbf{p}',\sigma')}=f(\mathbf{p},\sigma;\,\mathbf{p}',\sigma'). \tag{2.13}$$

On direct substitution of Eqs. (2.7) and (2.11)–(2.13)

[8] K. A. Brueckner and K. Sawada, Phys. Rev. **112**, 328 (1958).

[9] See below and reference 8.

[10] See also J. J. Quinn and R. A. Ferrell, J. Nuclear Energy **2**, 18 (1961), for a more heuristic treatment.

[11] $\theta(x)$ is unity for positive x and zero for negative x.

[12] It is not necessary to carry this expansion further than $O[(\delta p)^2]$.

into expression (2.10), one finds

$$\delta E=\sum_{\sigma}\int\frac{d\mathbf{p}\,V}{(2\pi)^3}[\epsilon(\mathbf{p})-\zeta(\sigma)\mu H]\left\{\tfrac{1}{3}\delta p\zeta(\sigma)k_F\delta(p-k_F)+\tfrac{1}{9}(\delta p)^2k_F\left[-\delta(p-k_F)+\frac{k_F}{2}\frac{d}{dk_F}\delta(p-k_F)\right]\right\}$$

$$+\tfrac{1}{2}\sum_{\sigma\sigma'}(\delta p)^2\int\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{p}'\,V}{(2\pi)^3}f(\mathbf{p},\sigma;\,\mathbf{p}',\sigma')\frac{k_F^2}{9}\zeta(\sigma)\zeta(\sigma')\delta(p-k_F)\delta(p'-k_F)$$

$$=-\frac{Vk_F^3}{3\pi^2}\mu H\delta p-\frac{1}{9}\frac{k_F}{(2\pi)^3}V(\delta p)^2\int d\mathbf{p}\,\epsilon(\mathbf{p})\left[k_F\frac{d}{dp}\delta(p-k_F)+2\delta(p-k_F)\right]$$

$$+V\left[\frac{k_F^3}{3(2\pi)^3}\right]^2\tfrac{1}{2}(\delta p)^2\sum_{\sigma\sigma'}\int_{(p,p'=k_F)}d\Omega_pd\Omega_{p'}\,\zeta(\sigma)f(\mathbf{p},\sigma;\,\mathbf{p}',\sigma')\zeta(\sigma').\quad(2.14)$$

The subsidiary relations,

$$N=Vk_F^3/3\pi^2,\quad(2.15)$$

$$(d\epsilon/dp)_{p=k_F}=k_F/M^*,\quad(2.16)$$

$$\int_{(p,p'=k_F)}d\Omega_pd\Omega_{p'}\,f(\mathbf{p},\sigma;\,\mathbf{p}',\sigma')=4\pi\int d\Omega\,f_{\sigma\sigma'}(\cos\theta),\quad(2.17)$$

with[5]

$$\sum_{\sigma\sigma'}\zeta(\sigma)f_{\sigma\sigma'}\zeta(\sigma')=f^{\text{ex}},\quad(2.18)$$

produce the further simplification,

$$\delta E=-N\mu H\delta p+\tfrac{1}{3}(\delta p)^2\frac{k_F^2}{2M^*}N+(\delta p)^2\int d\Omega\,f^{\text{ex}}.\quad(2.19)$$

On minimizing Eq. (2.19) with respect to δp, one finds

$$(\delta p)_{\min}=-\frac{3N\mu H}{k_F^2}\left[\frac{1}{M^*}+\frac{k_F}{2(2\pi)^3}\int d\Omega\,f^{\text{ex}}(\cos\theta)\right]^{-1}.\quad(2.20)$$

Thus the requirement that

$$\delta E[(\delta p)_{\min}]=\tfrac{1}{2}N\chi H^2\quad(2.21)$$

yields for χ, the magnetic susceptibility per unit volume, the canonical result of Landau,[3]

$$\chi=\frac{3N\mu^2}{k_F^2}\left[\frac{1}{M^*}+\frac{k_F}{2(2\pi)^3}\int d\Omega\,f^{\text{ex}}(\cos\theta)\right]^{-1}.\quad(2.22)$$

One may, of course, consider the ground-state energy, E, simply as a *parametric function* of k_F^σ. Then[13]

$$E\{k_F[1+\zeta(\sigma_i)\delta p]^{1/3}\}-E(k_F)\simeq\tfrac{1}{2}(\delta p)^2\sum_{\substack{(i)\\ \text{distinct closed loops}}}k_{F(i)}\times\left(-\frac{2}{9}\frac{\partial E}{\partial k_{F(i)}}+\tfrac{1}{9}k_{F(i)}\frac{\partial^2E}{\partial k_{F(i)}{}^2}\right),\quad(2.23)$$

where the *parameter* $k_{F(i)}$ appears only in the distribution functions,

$$n^{(i)}(\mathbf{p})=\theta(k_{F(i)}-p),$$

which refer to the ith closed loop. Because of the presence of the sign-function, $\zeta(\sigma_i)$, the contribution from the parametric differentiation of two "elements"[14] in *distinct* closed loops vanishes, so that the derivative operation, $-2\partial/\partial k_{F(i)}+k_{F(i)}\partial^2/\partial k_{F(i)}{}^2$ acts only on "elements" of the same closed loop [hence the additional label (i) on the parameter k_F on the right-hand side of (2.23)].[15] The independent sums over spin, $\sum_{(i)}\sum_{(j)}\times\sum_{\sigma_i}\sum_{\sigma_j}\zeta(\sigma_i)\zeta(\sigma_j)$, then go over into a sum over the "loop-spins" of distinct loops, $\sum_{(i)}\sum_{\sigma_i}$.

In the case of the compressibility, it will be sufficient merely to connect the appropriate SS procedure to the proper Landau expression.[3] Thus we are led to consider[10]

$$n'(\mathbf{p})=\theta(k_F(1+\delta_p)^{1/3}-p)\rightarrow n(\mathbf{p})+\delta n(\mathbf{p}),\quad(2.24)$$

where

$$\delta n(\mathbf{p})=\delta p\left[\frac{\partial}{\partial(\delta p)}k_F(1+\delta p)^{1/3}\right]\delta(p-k_F)+\frac{1}{2!}(\delta p)^2\left\{\left[\frac{\partial^2}{\partial(\delta p)^2}k_F(1+\delta p)^{1/3}\right]_0\times\delta(p-k_F)+\left[\frac{\partial}{\partial(\delta p)}k_F(1+\delta p)^{1/3}\right]_0^2\frac{d}{dk_F}\delta(p-k_F)\right\}.\quad(2.25)$$

[13] The spin-field term is omitted; thus no term linear in δp appears. We must introduce an additional index i on the spin label in order to distinguish the ith closed loop, all segments of which have the same value of $\zeta(\sigma_i)$.

[14] By "elements" we mean the internal lines; these are represented by the step functions, $n(\mathbf{p})$ (holes), $1-n(\mathbf{p})$ (particles). See also Sec. III A.

[15] *With these "rules of interpretation,"* the relation

$$\sum_{\substack{(i)\\ \text{distinct closed loops}}}\tfrac{1}{2}k_{F(i)}\left(-\frac{2}{9}\frac{\partial E}{\partial k_{F(i)}}+\frac{1}{9}k_{F(i)}\frac{\partial^2E}{\partial k_{F(i)}{}^2}\right)=\tfrac{1}{3}N\epsilon(k_F)+\int d\Omega\,f^{\text{ex}}(\cos\theta)$$

is implied.

On substituting (2.25), together with (2.11)–(2.13), into (2.10), one finds

$$\delta E=\delta p\, N\epsilon(k_F)+2\int\frac{d\mathbf{p}}{(2\pi)^3}V\epsilon(\mathbf{p})\tfrac{1}{9}(\delta p)^2k_F\left[-\delta(p-k_F)+\frac{k_F}{2}\frac{d}{dk_F}\delta(p-k_F)\right]$$
$$+2(\delta p)^2\frac{k_F^2}{9}\int\int\frac{d\mathbf{p}d\mathbf{p}'}{(2\pi)^6}Vf_{\mathbf{p},\mathbf{p}'}{}^0\delta(p-k_F)\delta(p'-k_F)$$
$$=\delta p\, N\epsilon(k_F)+\tfrac{1}{3}(\delta p)^2\left[N\frac{k_F^2}{2M^*}+\frac{Nk_F^3}{(2\pi)^3}\int d\Omega\, f^0(\cos\theta)\right]. \tag{2.26}$$

From the Landau relation,[3]

$$\frac{\partial\epsilon(k_F)}{\partial N}=\frac{1}{16\pi V}\int d\Omega\, f^0(\cos\theta)+\frac{2(2\pi)^3}{16\pi Vk_F}\frac{1}{M^*}, \tag{2.27}$$

and the identification

$$\delta N=N\delta p \tag{2.28}$$

one finds

$$\delta E=\epsilon(k_F)\delta N+\tfrac{1}{2}(\delta N)^2\partial\epsilon(k_F)/\partial N \tag{2.29}$$

as expected. Further, consideration of E as a parametric function of k_F, leads to the result,

$$E[k_F(1+\delta p)^{1/3}]-E(k_F)$$
$$\simeq\tfrac{1}{3}\delta p\, k_F\frac{\partial E}{\partial k_F}+\tfrac{1}{2}(\delta p)^2k_F\left(-\frac{2}{9}\frac{\partial E}{\partial k_F}+\frac{1}{9}k_F\frac{\partial^2E}{\partial k_F^2}\right). \tag{2.30}$$

However, in the case of the compressibility, we are to understand the parametric operator,

$$-2\partial/\partial k_F+k_F\partial^2/\partial k_F^2,$$

to act on *all* the elements of all distinct closed loops in a given graph.[16]

We consider now the SS procedure which yields Landau's Galilean invariance relation, Eq. (1.2). For a *virtual* displacement, $\delta\boldsymbol{\kappa}$, of the Fermi sphere, one has[17]

$$n'(\mathbf{p})=\theta(k_F-|\mathbf{p}-\delta\boldsymbol{\kappa}|)\rightarrow n(\mathbf{p})+\delta n(\mathbf{p}), \tag{2.31}$$

where

$$\delta n(\mathbf{p})=\left\{\mathbf{n}\cdot\delta\boldsymbol{\kappa}-\frac{1}{2k_F}[(\delta\boldsymbol{\kappa})^2-(\mathbf{n}\cdot\delta\boldsymbol{\kappa})^2]\right\}\delta(k_F-p)+\tfrac{1}{2}(\mathbf{n}\cdot\delta\boldsymbol{\kappa})^2\frac{d}{dk_F}\delta(k_F-p), \tag{2.32}$$

with

$$\mathbf{n}=\mathbf{p}/p. \tag{2.33}$$

The substitution of (2.32) into (2.10) then yields

$$\delta E=\sum_\sigma\int\frac{d\mathbf{p}\,V}{(2\pi)^3}\epsilon(\mathbf{p})\left[\left\{\mathbf{n}\cdot\delta\boldsymbol{\kappa}-\frac{1}{2k_F}[(\delta\boldsymbol{\kappa})^2-(\mathbf{n}\cdot\delta\boldsymbol{\kappa})^2]\right\}\delta(p-k_F)+\tfrac{1}{2}(\mathbf{n}\cdot\delta\boldsymbol{\kappa})^2\frac{d}{dk_F}\delta(p-k_F)\right]$$
$$+\tfrac{1}{2}\sum_{\sigma\sigma'}\int\int\frac{d\mathbf{p}d\mathbf{p}'\,V}{(2\pi)^6}f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}')\mathbf{n}\cdot\delta\boldsymbol{\kappa}\delta\boldsymbol{\kappa}\cdot\mathbf{n}'\delta(p-k_F)\delta(p'-k_F)$$
$$=N\frac{(\delta\boldsymbol{\kappa})^2}{2M^*}+\frac{Nk_F(\delta\boldsymbol{\kappa})^2}{(2\pi)^3}\int d\Omega_\gamma\cos\gamma f^0(\cos\gamma). \tag{2.34}$$

For translation-invariant potentials,[18] the interaction energy, $E-E_0$, where

$$E_0=\sum\nolimits_{p\sigma}(\mathbf{p}^2/2M)n(\mathbf{p}), \tag{2.35}$$

is invariant under the simultaneous displacement of all internal momenta,[19]

$$\mathbf{p}\rightarrow\mathbf{p}-\delta\boldsymbol{\kappa}. \tag{2.36}$$

Thus,

$$\delta E_{(\mathbf{p}\to\mathbf{p}-\delta\kappa)}=\delta(E_{(\mathbf{p}\to\mathbf{p}-\delta\kappa)}-E_{0(\mathbf{p}\to\mathbf{p}-\delta\kappa)})+\delta E_{0(\mathbf{p}\to\mathbf{p}-\delta\kappa)}$$
$$=N(\delta\boldsymbol{\kappa})^2/2M, \tag{2.37}$$

and the Landau relation (1.2) follows on equating (2.37) and (2.34).

The three cases treated above are illustrative of the "generator" property of SS procedures.[20] It is clear that

[16] It follows that in every order of perturbation theory the set of scattering graphs which contributes to the compressibility contains the set of scattering graphs appropriate to the magnetic susceptibility. See also Sec. IV of this paper.

[17] There is no compelling reason other than that of simplicity for terminating the expansion of δE in variational derivatives at second order in this case. The continued expansion in variational derivatives here will, in fact, produce innumerable identities.

[18] The relation $V_{\mathbf{k}',\mathbf{k}}=V(\mathbf{k}'-\mathbf{k})$ is meant.

[19] This follows from footnote 18 and the invariance of the denominators in perturbation theory under such a displacement by virtue of momentum and particle-number conservation. By way of illustration, one has in second-order perturbation theory, the denominator

$$\Delta_{\mathbf{pq};\mathbf{lm}}=\mathbf{p}^2+\mathbf{q}^2-\mathbf{l}^2-\mathbf{m}^2\rightarrow\Delta_{\mathbf{pq};\mathbf{lm}}-2\delta\boldsymbol{\kappa}\cdot(\mathbf{p}+\mathbf{q}-\mathbf{l}-\mathbf{m})+2(\delta\boldsymbol{\kappa})^2-2(\delta\boldsymbol{\kappa})^2=\Delta_{\mathbf{pq};\mathbf{lm}}$$

under $\mathbf{p}\rightarrow\mathbf{p}-\delta\boldsymbol{\kappa}$, etc.

[20] D. Pines (private communication) finds it "paradoxical" that the usual application of Sampson-Seitz to $\epsilon_{a-||}{}^{(2)}=A\ln\beta$, the cutoff, second-order contribution of antiparallel spins to the correlation energy (β is the minimum momentum transfer considered) does not yield the corresponding term $O[(\delta p)^2]$ in the

the proper generalization of the SS method to the calculation of any static response of a normal fermion system presupposes the judicious choice of a fictive or "trial" Fermi surface.[21] We hasten to remark that one is not suprised, in the context of field-theoretic techniques, to find the physical ground state of a quantum mechanical system to be the generator of its static responses.

III. MAGNETIC SUSCEPTIBILITY OF A DENSE ELECTRON GAS

A. Perturbation Theory

Preliminary to our review of the calculation of the magnetic susceptibility of a dense electron gas to $O(r_s)$, we consider for the sake of example, the second-order Coulomb exchange correlation energy,[22] $\epsilon_b^{(2)}$ [which we take to be a typical term in the ordinary (linked) Rayleigh-Schrödinger perturbation-theoretic expansion of the physical ground-state energy of this system[22]] from both SS and Landau points of view. It will rapidly become clear that the Landau expression is generally far less useful as a basis for calculation than that expression to which the SS procedure leads.

The exchange correlation energy in lowest order is given by the expression,

$$\epsilon_b^{(2)} = -\frac{3}{16\pi^5}\int\frac{d\mathbf{q}_1}{q_1^2}\int\frac{d\mathbf{q}_2}{q_2^2}\int\frac{d\mathbf{p}}{\mathbf{q}_1\cdot\mathbf{q}_2} \times\theta(k_F-|\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2|)\theta(|\mathbf{p}+\mathbf{q}_2|-k_F) \times\theta(k_F-p)\theta(|\mathbf{p}+\mathbf{q}_1|-k_F)|_{(k_F=1)}, \quad (3.1)$$

and is shown graphically in Fig. 1(a). Note the parametric dependence of $\epsilon_b^{(2)}$ on k_F. By our previous discussion,

$$\delta\epsilon_b^{(2)} = \tfrac{1}{2}(\delta p)^2\left\{-\frac{2}{9}\frac{\partial\epsilon_b^{(2)}}{\partial k_F}+\frac{1}{9}k_F\frac{\partial^2\epsilon_b^{(2)}}{\partial k_F^2}\right\}\Bigg|_{(k_F=1)}, \quad (3.2)$$

since the parametric differentiation with respect to k_F operates on the elements of a single closed loop which is $\epsilon_b^{(2)}$.

As a result of the operation, $\partial^2\epsilon_b^{(2)}/\partial k_F^2$, one finds two types of terms: those where a parametric derivative of a δ function appears in the integrand and those where a pair of δ functions appear. [In the former case, the derivative, say $(d/dk_F)\delta(k_F-p)$, when rewritten as $-(d/dp)\delta(k_F-p)$, will yield a term (by partial integration), which, through the differentiation of the momentum weight p^2, just cancels the operation $-(2/k_F)\times\partial\epsilon_b^{(2)}/\partial k_F$.] The double δ-function terms may be identified with contributions to $\int d\Omega\, f^{\text{ex}}(\cos\theta)$, the δ' terms with effective mass contributions $\propto 1/M^*$, so that parametric differentiation may be equated to the opening up of internal lines on the Fermi surface. Explicitly,

$$\delta\epsilon_b^{(2)} = \tfrac{1}{4}(\alpha_b^{(2)})(\delta p)^2 \quad (3.3)$$

with

$$\alpha_b^{(2)} = \textstyle\sum_{j=1}^3 [\alpha_b^{(2)}]_j, \quad (3.4)$$

where

$$[\alpha_b^{(2)}]_1 = -\frac{1}{12\pi^5}\int\frac{d\mathbf{q}_1}{q_1^2}\int\frac{d\mathbf{q}_2}{q_2^2}\int\frac{d\mathbf{p}}{\mathbf{q}_1\cdot\mathbf{q}_2}\delta(p-1)\delta(1-|\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2|) \times\theta[1-|\mathbf{p}+\mathbf{q}_1|)\theta(1-|\mathbf{p}+\mathbf{q}_2|)[\mathbf{p}\cdot(\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)-1], \quad (3.5a)$$

$$[\alpha_b^{(2)}]_2 = -\frac{1}{12\pi^5}\int\frac{d\mathbf{q}_1}{q_1^2}\int\frac{d\mathbf{q}_2}{q_2^2}\int\frac{d\mathbf{p}}{\mathbf{q}_1\cdot\mathbf{q}_2}\delta(p-1)\delta(1-|\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2|) \times\theta(|\mathbf{p}+\mathbf{q}_1|-1)\theta(|\mathbf{p}+\mathbf{q}_2|-1)[\mathbf{p}\cdot(\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2)-1], \quad (3.5b)$$

$$[\alpha_b^{(2)}]_3 = \frac{1}{3\pi^5}\int\frac{d\mathbf{q}_1}{q_1^2}\int\frac{d\mathbf{q}_2}{q_2^2}\int\frac{d\mathbf{p}}{\mathbf{q}_1\cdot\mathbf{q}_2}\delta(p-1)\delta(1-|\mathbf{p}+\mathbf{q}_1|)\theta(1-|\mathbf{p}+\mathbf{q}_2|)\theta(|\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2|-1)[\mathbf{p}\cdot(\mathbf{p}+\mathbf{q}_1)-1], \quad (3.5c)$$

after some rearrangement. The P-wave parts of the forward ordinary scattering amplitude which appear in (3.5) above, result from the partial integration of the δ' terms as expected from the principle of Galilean invariance (1.2). We have used the customary relation,

$$\delta\epsilon = \tfrac{1}{4}\alpha(\delta p)^2, \quad (3.6)$$

with[23]

$$\alpha = \frac{2(N/V)\mu^2}{\chi R} = \frac{4}{3\beta^2 r_s^2}\left[\frac{M}{M_{(\text{exact})}^*} + \frac{\beta r_s}{(2\pi)^2}\int d\Omega\, F_{(\text{exact})}^{\text{ex}}(\mathbf{n}\cdot\mathbf{n}')\right], \quad (3.7)$$

$$f_{(\text{exact})}^{\text{ex}} = (4\pi e^2/k_F^2)F_{(\text{exact})}^{\text{ex}}, \quad (3.8)$$

polarized ground-state energy, $\epsilon_{a-||}^{(2)}(\delta p) = A'\ln\beta(\delta p)^2$, where the latter expression is the result of direct calculation. This is most likely due to the approximate nature of cutoff calculations; that is, the momentum cutoff, β, only approximately simulates the effects of screening. (See the Appendix of this paper for an example.)

[21] For example, the fictive Fermi surface,

$$n'(\mathbf{p}) = \theta(k_F - p[1-2\delta\eta\, P_l(\cos\theta)]^{1/2}), \quad (l>0)$$

with

$$\delta n(\mathbf{p}) = \delta\eta\, k_F P_l(\mu_p)\delta(k_F-p) + \tfrac{1}{2}(\delta\eta)^2[P_l(\mu_p)]^2 \times\left[k_F\delta(k_F-p)+k_F^2\frac{d}{dk_F}\delta(k_F-p)\right]$$

is associated with

$$\delta E = \tfrac{3}{2}k_F^2(\delta\eta)^2\left[\frac{5N}{2M^*}\frac{1}{2l+1}+\frac{2k_F}{(2\pi)^3}\int d\Omega_\gamma\, f^0(\cos\gamma)P_l(\cos\gamma)\right].$$

[22] M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 369 (1957).

[23] β is $(4/9\pi)^{1/3}$; R is the rydberg.

and

$$1/M=2/\beta^2 r_s^2. \qquad (3.9)$$

Further, for the exchange correlation graphs, the relation[24]

$$f_{\pm\frac{1}{2},\mp\frac{1}{2}}=f^0-\tfrac{1}{4}f^{\text{ex}}=0 \qquad (3.10)$$

holds; this implies

$$(\alpha-\alpha_{\text{free}})_{\text{ex.corr.}}=\frac{1}{3\pi^2\beta r_s}\int d\Omega_n[F_{(\text{exact})}{}^{\text{ex}}(\mathbf{n}\cdot\mathbf{n}') -4\mathbf{n}\cdot\mathbf{n}'F_{(\text{exact})}{}^0(\mathbf{n}\cdot\mathbf{n}')]_{\text{ex.corr.}}$$
$$=-\frac{4}{3\pi^2\beta r_s}\int d\Omega_n(\mathbf{n}\cdot\mathbf{n}'-1)\times F_{(\text{exact})}{}^0(\mathbf{n}\cdot\mathbf{n}')|_{\text{ex.corr.}}, \qquad (3.11)$$

which form, one notes Eqs. (3.5) take. The scattering graphs associated with the terms $[\alpha_b{}^{(2)}]_j$ ($j=1$, 2, 3) are represented graphically in Figs. 1(b)–1(d); they are the "exchange-ladder" corrections ($j=1$, 2) and the vertex correction to scattering ($j=3$) in lowest order (static limit). Because of their divergence in this limit, one would be obliged to supply the appropriate "renormalization of interaction" and calculate their contribution to α_c in the Landau theory. However, we can show quite easily that $\alpha_b{}^{(2)}$ vanishes identically [and hence, $(\alpha_c)_{\text{ex. corr.}}=0$ to $O(r_s)$] by the SS method[8]: One simply makes the scale transformation,

$$\mathbf{q}_1\to\mathbf{q}_1k_F,\quad \mathbf{q}_2\to\mathbf{q}_2k_F,\quad \mathbf{p}\to\mathbf{p}k_F, \qquad (3.12)$$

in (3.1). Then it follows that

$$\epsilon_b{}^{(2)}(k_F)=k_F{}^3\epsilon_b{}^{(2)}(1), \qquad (3.13)$$

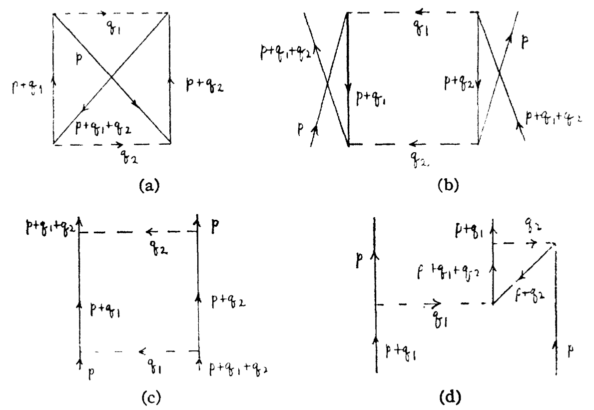

FIG. 1. Exchange correlation energy in second order (a) and derived contributions to $f(\mathbf{p}\sigma;\mathbf{p}'\boldsymbol{\sigma}')$ [(b)–(d)]. (d) represents a vertex correction to scattering in the static limit.

and hence

$$-2\partial\epsilon_b{}^{(2)}/\partial k_F+k_F\partial^2\epsilon_b{}^{(2)}/\partial^2k_F{}^2=0. \qquad (3.14)$$

Since this relation is not apparent in the Landau formulation, we may conclude that that formulation is not generally useful as a calculational tool.

B. Cancellation of Exchange Correlation Contribution to α_c

In this subsection we explicitly obtain the result

$$(\alpha_e)_{\text{ex}}=0, \qquad (3.15)$$

to $O(r_s)$, by doing the appropriate calculation according to the Landau scheme. Thus, following the Feynman rules for many-body perturbation theory at zero temperature,[25] one has

$$(\alpha_e)_{\text{ex}}=\tfrac{4}{3}(K_{11}-L_{11})+\tfrac{2}{3}(K_{21}-L_{21}), \qquad (3.16)$$

with

$$K_{11}=-\frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{dw}{2\pi i}\int d\mathbf{q}_1\int d\mathbf{q}_2\frac{\mathbf{n}\cdot(\mathbf{n}+\mathbf{q}_1)\delta(|\mathbf{n}+\mathbf{q}_1|-1)}{q_1{}^2+(\beta r_s/\pi^2)Q_{r_s}(\mathbf{q}_1,0)}S(\mathbf{n}+\mathbf{q}_1+\mathbf{q}_2,1+w)P_{r_s}(\mathbf{q}_2,w)S(\mathbf{n}+\mathbf{q}_2,1+w), \qquad (3.17\text{a})$$

$$K_{21}=\frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{dw}{2\pi i}\int_{-\infty}^{\infty}dw'\int d\mathbf{q}_1\int d\mathbf{q}_2\, S(\mathbf{n}+\mathbf{q}_1,1+w)P_{r_s}(\mathbf{q}_1,w)\mathbf{n}\cdot(\mathbf{n}+\mathbf{q}_1+\mathbf{q}_2)$$
$$\times\delta(|\mathbf{n}+\mathbf{q}_1+\mathbf{q}_2|-1)\delta(w+w')S(\mathbf{n}+\mathbf{q}_2,1+w')P_{r_s}(\mathbf{q}_2,w'), \qquad (3.17\text{b})$$

and

$$L_{11}=K_{11}[\mathbf{n}\cdot(\mathbf{n}+\mathbf{q}_1)\to 1], \qquad (3.17\text{c})$$

$$L_{21}=K_{21}[\mathbf{n}\cdot(\mathbf{n}+\mathbf{q}_1+\mathbf{q}_2)\to 1]. \qquad (3.17\text{d})$$

To $O(r_s)$,

$$K_{11}-L_{11}=\frac{1}{2\pi^3}\int_{-1}^{1}dx\, I_{r_s}(x), \qquad (3.18)$$

with

$$I_{r_s}(x)=A(x)\ln\frac{4\beta r_s}{\pi}+B(x)+\Delta(x), \qquad (3.19)$$

[24] All legs have the same z component of spin.

[25] D. F. DuBois, Ann. Phys. (New York) 7, 174 (1959); 8, 24 (1959). Our notation follows that of this and references 8 and 22 closely.

where[26]

$$A(x)=-\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\int d\Omega_q\frac{1}{(iu-\mathbf{n}\cdot\hat{q})(iu-\mathbf{n}'\cdot\hat{q})}=\pi[2(1-x)]^{-1/2}\ln\left|\frac{1+[\frac{1}{2}(1-x)]^{1/2}}{1-[\frac{1}{2}(1-x)]^{1/2}}\right|, \quad (3.20)$$

$$B(x)=2\pi[\tfrac{1}{2}(1-x)]^{-1/2}\int_{-\infty}^{\infty}\frac{du}{2\pi}\ln R(u)[1+u^2-\tfrac{1}{2}(1-x)]^{-1/2}\tan^{-1}\left[\frac{\frac{1}{2}(1-x)}{1+u^2+\frac{1}{2}(1-x)}\right]^{1/2}, \quad (3.21)$$

$$\Delta(x)=\lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{du}{2\pi}\left[\int_{q>\delta}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\frac{1}{2}q-\mathbf{n}\cdot\hat{q})(iu+\frac{1}{2}q-\mathbf{n}'\cdot\hat{q})}-\int_{\delta<q<1}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\mathbf{n}\cdot\hat{q})(iu-\mathbf{n}'\cdot\hat{q})}\right]. \quad (3.22)$$

Since

$$\int_{-1}^{1}dx\,A(x)=4\pi\ln 2, \quad (3.23)$$

$$\int_{-1}^{1}dx\,B(x)=2\int_{-\infty}^{\infty}du\ln R(u)[\tan^{-1}(1/u)]^2=-15.3, \quad (3.24)$$

$$\int_{-1}^{1}dx\,\Delta(x)=-\pi[8\ln 2(1-\ln 2)+\pi^2/3], \quad (3.25)$$

one obtains

$$K_{11}-L_{11}=0.141\ln r_s-0.369. \quad (3.26)$$

Further,

$$K_{21}-L_{21}=\frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{du}{2\pi}\int d\Omega_n\,(\mathbf{n}\cdot\mathbf{n}'-1)\int_{q<1}\frac{d\mathbf{q}}{q}\frac{1}{(iu-\mathbf{n}\cdot\hat{q})(iu-\mathbf{n}'\cdot\hat{q})}\times\frac{1}{q^2+4(\beta r_s/\pi)R(u)}\frac{1}{(\mathbf{n}-\mathbf{n}')^2+4\beta r_s/\pi}+\Delta E(r_s), \quad (3.27)$$

where the correction integral,

$$\Delta E(r_s)=\frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{du}{2\pi}\int d\Omega_n\,(\mathbf{n}\cdot\mathbf{n}'-1)\lim_{\epsilon\to 0}\left[\int_{q>\epsilon}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\frac{1}{2}q-\mathbf{n}\cdot\hat{q})(iu+\frac{1}{2}q-\mathbf{n}'\cdot\hat{q})}\times\frac{1}{(\mathbf{n}-\mathbf{n}'-\mathbf{q})^2+4\beta r_s/\pi}-\int_{1>q>\epsilon}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\mathbf{n}\cdot\hat{q})(iu-\mathbf{n}'\cdot\hat{q})}\frac{1}{(\mathbf{n}-\mathbf{n}')^2+4\beta r_s/\pi}\right] \quad (3.28)$$

$$=-0.141\ln r_s+0.448, \quad (3.29)$$

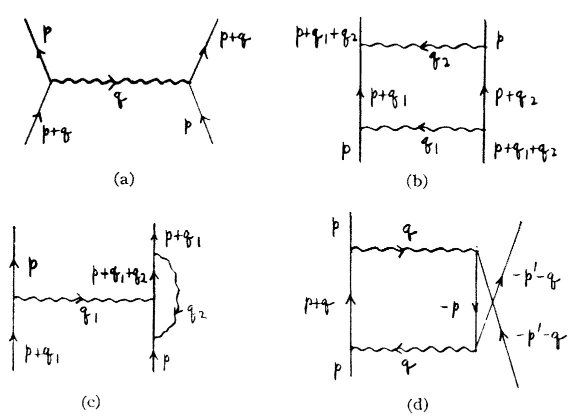

FIG. 2. (a) One-virtual-excitation exchange scattering graph; (b), (c) are graphs with two virtual excitations occurring. (d) is one of four two-excitation exchange scattering graphs by which f^0 differs from f^{ex} to $O(r_s)$.

[26] Du Bois' evaluation of $A(x)$ [(Eq. (4.42) of the second of references (25)] is not correct. It is not necessary to evaluate $\Delta(x)$ to do the integral in Eq. (3.25) (this paper). [Du Bois' evaluation of $\Delta(x)$ is also not correct; indeed one may find a closed form for it via the introduction of Spence functions.]

has been left to an appendix. Since to $O(r_s)$,

$$(K_{21}-L_{21})-\Delta E(r_s)=-(K_{11}-L_{11})+\frac{1}{2\pi^3}\int_{-1}^{1}dx\,\Delta(x)=-0.141\ln r_s+0.290, \tag{3.30}$$

we have, collecting terms,

$$(\alpha_c)_{\text{ex}}=0,$$

the result so trivially obtained in (3.13)-(3.14). As we have shown that the contribution to α_c to $O(r_s)$ from graphs with two virtual excitations [see Figs. 2(b) and 2(c)] vanishes, we turn now to a consideration of the contribution from graphs with only one virtual excitation occurring [see Fig. 2(a)].

C. α_c as a One-Virtual-Excitation Calculation

It will be sufficient for our purposes merely to extablish the identity of the two approaches in the case of the nonexchange (or one virtual excitation) contribution to α_c. Thus, we apply Sampson-Seitz to the expression for the nonexchange correlation energy in the high-density limit,[8,22]

$$\epsilon(k_F)|_{\text{nonex. corr.}}=-\frac{3}{4\pi}\int_0^\infty q^3dq\int_{-\infty}^{\infty}du\left\{\ln\left[1+\frac{\beta r_s}{\pi^2q^2}Q_q(u;k_F)\right]-\frac{\beta r_s}{\pi^2q^2}Q_q(u;k_F)\right\}\frac{1}{\beta^2r_s^2}, \tag{3.31}$$

which expression results from the parametrization,[8]

$$Q_q(u;k_F)=\int d\mathbf{p}\,\theta(|\mathbf{p}+\mathbf{q}|-k_F)\theta(k_F-p)\int_{-\infty}^{\infty}\exp[ituq-|t|(\tfrac{1}{2}q^2-\mathbf{q}\cdot\mathbf{p})]dt=k_FQ_{q/k_F}(u/k_F) \tag{3.32a}$$

with

$$Q_0(u;k_F)=k_FQ_0(u/k_F)=4\pi k_FR(u/k_F). \tag{3.33}$$

According to our previous discussion, we are to consider

$$\delta\epsilon=\tfrac{1}{2}(\delta p)^2\sum_{\substack{(i)\\ \text{distinct closed loops}}}\left(-\frac{2}{9}\frac{\partial\epsilon}{\partial k_{F(i)}}+\frac{1}{9}k_{F(i)}\frac{\partial^2\epsilon}{\partial k_{F(i)}{}^2}\right)\bigg|_{(k_F=1)}=\tfrac{1}{4}(\delta p)^2\alpha, \tag{3.34}$$

where the parametric derivatives are to be taken on the same closed loop [the factor, $k_FQ_{q/k_F}(u/k_F)$, represents such a loop]. Thus, one has

$$\alpha=\frac{1}{6\pi^5}\int_0^\infty\frac{dq}{q}\int_{-\infty}^{\infty}du\frac{Q_q(u;k_F)\{(-2\partial/\partial k_F+k_F\partial^2/\partial k_F{}^2)Q_q(u;k_F)\}}{1+(\beta r_s/\pi^2q^2)Q_q(u;k_F)}\bigg|_{(k_F=1)}. \tag{3.35}$$

The SS result of Brueckner and Sawada then follows from the parametrization of (3.32b), since the relation,

$$\left(-2\frac{\partial}{\partial k_F}+k_F\frac{\partial^2}{\partial k_F{}^2}\right)Q_0(u;k_F)\bigg|_{(k_F=1)}=\left(-2\frac{\partial}{\partial k_F}+k_F\frac{\partial^2}{\partial k_F{}^2}\right)[k_FQ_0(u/k_F)]\bigg|_{(k_F=1)} \tag{3.36}$$

$$=u^2Q_0''(u)+2uQ_0'(u)-2Q_0(u)=-8\pi/(1+u^2)^2, \tag{3.37}$$

enables us to write[8]

$$\alpha=-\frac{12}{\pi^4}\int_{-\infty}^{\infty}du\int_0^1 qdq\frac{Q_0(u)g(u)}{q^2+(\beta r_s/\pi^2)Q_0(u)}+\Delta\alpha, \tag{3.38}$$

where

$$\Delta\alpha=-\frac{1}{6\pi^5}\lim_{\beta\to0}\left[\int_\beta^\infty\frac{dq}{q}\int_{-\infty}^{\infty}du\,Q_q(u)\left\{\left(-2\frac{\partial}{\partial k_F}+\frac{\partial^2}{\partial k_F{}^2}\right)Q_q(u;k_F)\right\}\right.$$
$$\left.-\int_\beta^1\frac{dq}{q}\int_{-\infty}^{\infty}du\,Q_0(u)\left\{\left(-2\frac{\partial}{\partial k_F}+\frac{\partial^2}{\partial k_F{}^2}\right)Q_q(u;k_F)_{(q\to0)}\right\}\right]\bigg|_{(k_F=1)} \tag{3.39}$$

$$=-\frac{2}{3\pi^4}\lim_{\beta\to0}\left\{\int_\beta^\infty\frac{dq}{q^2}\int_{|\mathbf{p}'+\mathbf{q}|>1,p'<1}d\mathbf{p}'\left(-2\frac{\partial}{\partial k_F}+\frac{\partial^2}{\partial k_F{}^2}\right)\int_{|\mathbf{p}+\mathbf{q}|>k_F,p<k_F}d\mathbf{p}\frac{1}{q^2+\mathbf{q}\cdot(\mathbf{p}+\mathbf{p}')}\right.$$
$$\left.-\int_\beta^1\frac{dq}{q}\lim_{q\to0}\frac{1}{q}\int_{|\mathbf{p}'+\mathbf{q}|>1,p'<1}d\mathbf{p}'\left(-2\frac{\partial}{\partial k_F}+\frac{\partial^2}{\partial k_F{}^2}\right)\int_{|\mathbf{p}+\mathbf{q}|>k_F,p<k_F}d\mathbf{p}\frac{1}{q^2+\mathbf{q}\cdot(\mathbf{p}+\mathbf{p}')}\right\}\bigg|_{(k_F=1)}. \tag{3.40}$$

On the other hand, the choice of the parametrization of (3.32a) will yield the terms appropriate to the Landau scheme. In particular, one has

$$\left(-2\frac{\partial}{\partial k_F}+\frac{\partial^2}{\partial k_F{}^2}\right)Q_q(u;k_F)\bigg|_{(k_F=1)}=-\int d\Omega_n\,\delta(|\mathbf{n}+\mathbf{q}|-1)[1-\mathbf{n}\cdot(\mathbf{n}+\mathbf{q})]2\pi\delta(uq)+2\int d\Omega_n\,\mathbf{q}\cdot\mathbf{n}\frac{u^2q^2-(\frac{1}{2}q^2+\mathbf{q}\cdot\mathbf{n})^2}{[(\frac{1}{2}q^2+\mathbf{q}\cdot\mathbf{n})^2+u^2q^2]^2},\tag{3.41}$$

with

$$\alpha=\frac{1}{3\pi^4}\int_0^\infty dq\int d\Omega_n\,\delta(|\mathbf{n}+\mathbf{q}|-1)[1-\mathbf{n}\cdot(\mathbf{n}+\mathbf{q})]\frac{Q_q(0)}{q^2+(\beta r_s/\pi^2)Q_q(0)}-\frac{1}{3\pi^5}\int_0^\infty qdq\int_{-\infty}^\infty du\frac{\mathbf{q}\cdot\mathbf{n}Q_q(u)}{q^2+(\beta r_s/\pi^2)Q_q(u)}\frac{[u^2q^2-(\frac{1}{2}q^2+\mathbf{q}\cdot\mathbf{n})^2]}{[(\frac{1}{2}q^2+\mathbf{q}\cdot\mathbf{n})^2+u^2q^2]^2}.\tag{3.42}$$

The P-wave parts of these integrals may be identified (within a constant factor) as DuBois' calculation[25] of the contribution to the effective mass from one virtual excitation. Note the division into contributions from transitions on and off the Fermi surface.

IV. COMPRESSIBILITY OF A DENSE ELECTRON GAS

As a result of our preceding analysis, one finds that the auxiliary expression,

$$\frac{1}{M^*}+\frac{2k_F}{(2\pi)^3}\int d\Omega\, f^0(\cos\theta),\tag{4.1}$$

which relates to the compressibility, differs [to $O(r_s)$] from that appropriate to the susceptibility,

$$\frac{1}{M^*}+\frac{k_F}{2(2\pi)^3}\int d\Omega\, f^{\mathrm{ex}}(\cos\theta),\tag{4.2}$$

by a contribution from the nonexchange correlation energy and, in particular, from elements in *disjoint* closed loops. We may write this additional contribution as

$$\delta\alpha\equiv\left\{-\frac{4}{9}\frac{\partial}{\partial k_F}+\frac{2}{9}k_F\frac{\partial^2}{\partial k_F{}^2}\right\}\{[\epsilon(k_F)]_{\text{nonex. corr.}}\}_{\text{(disjoint loops)}}\bigg|_{(k_F=1)}=\frac{1}{6\pi^5}\int_0^\infty\frac{dq}{q}\int_{-\infty}^\infty du\frac{1}{[1+(\beta r_s/\pi^2q^2)Q_q(u)]^2}\left[\frac{\partial}{\partial k_F}Q_q(u;k_F)\right]^2\bigg|_{(k_F=1)},\tag{4.3}$$

which to $O(r_s)$ reduces to

$$\delta\alpha=-\frac{12}{\pi^3}\int_0^1 d(q^2)\int_{-\infty}^\infty du\frac{g(u)}{q^2+(4\beta r_s/\pi)R(u)}+\delta,\tag{4.4}$$

where

$$\delta=\frac{1}{6\pi^5}\lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty\frac{dq}{q}\left[\frac{\partial}{\partial k_F}Q_q(u;k_F)\right]^2-\int_\epsilon^1\frac{dq}{q}\lim_{q\to 0}\left[\frac{\partial}{\partial k_F}Q_q(u;k_F)\right]^2\right\}\bigg|_{(k_F=1)}.\tag{4.5}$$

The integrals required to evaluate δ have all been previously calculated by Brueckner and Sawada;[8] in particular, the identification

$$\delta=-\frac{4}{9}\left[\frac{\partial^2}{\partial x\partial y}f(x,y)\right]_{x=y=1}\tag{4.6}$$

yields

$$\delta=(4/3\pi^2)(\ln 2+\tfrac{1}{2}).\tag{4.7}$$

Since to $O(r_s)$,

$$\delta\alpha-\delta=\frac{12}{\pi^3}\int_{-\infty}^{\infty}du\,g(u)\ln\left[\frac{4\beta r_s}{\pi}R(u)\right]$$
$$=\frac{12}{\pi^3}\int_{-\infty}^{\infty}du\,g(u)\ln r_s+\frac{12}{\pi^3}\int_{-\infty}^{\infty}du\,g(u)\ln\left[\frac{4\beta}{\pi}R(u)\right]$$
$$=-\frac{2}{3\pi^2}\ln r_s+\left[\tfrac{1}{3}(0.199)-\frac{2}{3\pi^2}\ln\frac{4\beta}{\pi}\right], \tag{4.8}$$

it follows that[27]

$$\frac{1}{K}\left(\frac{2}{\rho R}\right)=\alpha_c-0.0676\ln r_s+0.255. \tag{4.9}$$

This calculation could have been dealt with as well in the Landau scheme. There the contribution $\delta\alpha$ arises as an ordinary ladder correction resulting from the exchange of two virtual excitations [see Fig. 2(d)]. Without recourse to the Feynman rules, we may obtain it in Landau form by the parametric differentiation of $Q_q(u;k_F)$ in Eq. (4.3). Thus, in that scheme, one finds

$$\delta\alpha=\frac{1}{12\pi^5}\int d\mathbf{q}\int_{-\infty}^{\infty}q\frac{du}{2\pi}\int d\Omega_{\mathbf{n}}\left[\frac{1}{\frac{1}{2}q^2+\mathbf{n}\cdot\mathbf{q}+iuq}\,\frac{1}{\frac{1}{2}q^2+\mathbf{n}\cdot\mathbf{q}-iuq}\right]$$
$$\times\int d\Omega_{\mathbf{n}'}\left[\frac{1}{\frac{1}{2}q^2+\mathbf{n}'\cdot\mathbf{q}+iuq}+\frac{1}{\frac{1}{2}q^2+\mathbf{n}'\cdot\mathbf{q}-iuq}\right]\frac{1}{[q^2+(\beta r_s/\pi^2)Q_q(u)]^2}$$
$$=\frac{4}{3\pi^2}\int_0^1 qdq\int_{-\infty}^{\infty}\frac{du}{2\pi}\iint_{-1}^{1}dxdy\,\frac{(u^2-x^2)(u^2-y^2)}{(x^2+u^2)^2(y^2+u^2)^2}\,\frac{1}{[q^2+(4\beta r_s/\pi)R(u)]}+\delta, \tag{4.10}$$

with the resulting calculated correction, $\delta\alpha$, in agreement with (4.9).

ACKNOWLEDGMENTS

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APPENDIX

We consider here the evaluation of the correction integral of Sec. III. We write it as

$$\Delta E(r_s)=\frac{1}{2\pi^4}\int_{-\infty}^{\infty}\frac{du}{2\pi}\int d\Omega_{\mathbf{n}'}\lim_{\epsilon\to0}\left\{(\mathbf{n}\cdot\mathbf{n}'-1)\int_{\epsilon<q}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\frac{1}{2}q-\mathbf{n}\cdot\hat{q})(iu+\frac{1}{2}q-\mathbf{n}'\cdot\hat{q})}\right.$$
$$\left.\times\frac{1}{(\mathbf{n}'-\mathbf{n}-\mathbf{q})^2+4\beta r_s/\pi}+\frac{1}{2}\int_{\epsilon<q<1}\frac{d\mathbf{q}}{q^3}\frac{1}{(iu-\mathbf{n}\cdot\hat{q})(iu-\mathbf{n}'\cdot\hat{q})}\right\}, \tag{A1}$$

where the limit $r_s\to0$ has been taken in the second integral of (3.28) for simplification. Since the second integral has the value $(4/\pi^2)\ln2\ln\epsilon$, we expect a like divergence in the first. That integral becomes, on performing the integration over the variable u,

$$J=\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{\epsilon<q}\frac{d\mathbf{q}}{q^2}\frac{1}{(\mathbf{n}'-\mathbf{n}-\mathbf{q})^2+(4\beta r_s/\pi)}\,\frac{1}{\mathbf{q}\cdot(\mathbf{q}+\mathbf{n}-\mathbf{n}')}[\theta(-\tfrac{1}{2}q-\mathbf{n}\cdot\hat{k})-\theta(\tfrac{1}{2}q-\mathbf{n}'\cdot\hat{k})]. \tag{A2}$$

We now make the transformation

$$\mathbf{q}+\mathbf{n}-\mathbf{n}'=\mathbf{k}$$

[27] We have used the relation $1/K=N\rho(\partial\mu/\partial N)$.

in the $\mathbf{q}$ integrand of (A2) and further split the integral into contributions from $k>2$ and $k<2$,

$$J=-\frac{1}{2\pi^4}\int_{-1}^{1}2\pi dx_1\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{k>2}d\mathbf{k}\frac{1}{|\mathbf{k}+\mathbf{n}'-\mathbf{n}|^2}\frac{1}{\mathbf{k}\cdot(\mathbf{k}+\mathbf{n}-\mathbf{n}')}$$
$$+\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{k<2}\frac{d\mathbf{k}}{k^2+(4\beta r_s/\pi)}\frac{1}{|\mathbf{k}+\mathbf{n}'-\mathbf{n}|^2}\frac{\theta(|\mathbf{k}+\mathbf{n}'-\mathbf{n}|-\epsilon)}{\mathbf{k}\cdot(\mathbf{k}+\mathbf{n}'-\mathbf{n})}[\theta(-k^2-2\mathbf{k}\cdot\mathbf{n}')-\theta(k^2-2\mathbf{k}\cdot\mathbf{n})]. \quad \text{(A3)}$$

The further rearrangement,

$$J=\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{k<2,\epsilon<|\mathbf{n}-\mathbf{n}'|}\frac{d\mathbf{k}}{k^2+(4\beta r_s/\pi)}\frac{1}{2(1-\mathbf{n}\cdot\mathbf{n}')}\frac{[\theta(-\hat{k}\cdot\mathbf{n}')-\theta(-\hat{k}\cdot\mathbf{n})]}{\mathbf{k}\cdot(\mathbf{n}'-\mathbf{n})}+\Delta J, \quad \text{(A4)}$$

with

$$\Delta J=\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{\epsilon<|k+\mathbf{n}-\mathbf{n}'|}\frac{d\mathbf{k}}{k^2}\frac{1}{|\mathbf{k}+\mathbf{n}'-\mathbf{n}|^2}\frac{[\theta(-k^2-2\mathbf{k}\cdot\mathbf{n}')-\theta(k^2-2\mathbf{k}\cdot\mathbf{n})]}{\mathbf{k}\cdot(\mathbf{k}+\mathbf{n}'-\mathbf{n})}$$
$$-\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{k<2,\epsilon<|\mathbf{n}-\mathbf{n}'|}\frac{d\mathbf{k}}{k^2}\frac{1}{2(1-\mathbf{n}\cdot\mathbf{n}')}\frac{[\theta(-\hat{k}\cdot\mathbf{n}')-\theta(-\hat{k}\cdot\mathbf{n})]}{\mathbf{k}\cdot(\mathbf{n}'-\mathbf{n})}, \quad \text{(A5)}$$

serves to separate out the term $O(\ln r_s)$, so that one finds

$$J=-(2/\pi^2)\ln(\beta r_s/\pi)\ln 2+\Delta J. \quad \text{(A6)}$$

We now observe that the $\mathbf{k}$ integrand of the first integral in ΔJ is invariant under the transformation

$$\mathbf{k}+\mathbf{n}-\mathbf{n}'\rightarrow\mathbf{k} \quad \text{(A7)}$$

except for the factor $\theta(|\mathbf{k}+\mathbf{n}-\mathbf{n}'|-\epsilon)$ which goes to $\theta(k-\epsilon)$ under that transformation. This implies that our specification of divergence in this integral (and hence in the subtraction integral as well) is inadequate; a factor $\theta(k-\epsilon)$ is missing. Indeed the divergence $\alpha\ln\epsilon$ has its basis in the logarithmic divergence of the k integral. On the other hand *no* divergence arises from dropping the factor $\theta(|\mathbf{k}+\mathbf{n}-\mathbf{n}'|-\epsilon)$. Thus,

$$\Delta J=\frac{1}{2\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{\epsilon<k}\frac{d\mathbf{k}}{k^2}\frac{1}{|\mathbf{k}+\mathbf{n}-\mathbf{n}'|^2}\frac{[\theta(-k^2-2\mathbf{k}\cdot\mathbf{n}')-\theta(k^2-2\mathbf{k}\cdot\mathbf{n})]}{\mathbf{k}\cdot(\mathbf{k}+\mathbf{n}'+\mathbf{n})}$$
$$+\frac{1}{4\pi^4}\int d\Omega_{\mathbf{n}}(\mathbf{n}\cdot\mathbf{n}'-1)\int_{\epsilon<k,k<2}\frac{d\mathbf{k}}{k^2}\frac{[\theta(-\hat{k}\cdot\mathbf{n}')-\theta(-\hat{k}\cdot\mathbf{n})]}{\mathbf{k}\cdot(\mathbf{n}'-\mathbf{n})}, \quad \text{(A8)}$$

where the latter integral has the value $-(4/\pi^2)\ln 2\ln(2/\epsilon)$. The former integral may be split at $k=2$ so that

$$\Delta J=\frac{2}{\pi^3}\int_0^{\pi}d\varphi\int_{-1}^{1}dx_1\int_{-1}^{1}dx_2[x_1x_2+(1-x_1^2)^{1/2}(1-x_2^2)^{1/2}\cos\varphi-1]\int_{\epsilon}^{2}\frac{dk}{k}\frac{1}{(k+x_1-x_2)}[k^2+2kx_1-2kx_2$$
$$+2(1-x_1x_2-(1-x_1^2)^{1/2}(1-x_2^2)^{1/2}\cos\varphi)]^{-1}[\theta(-k-2x_1)-\theta(k-2x_2)]+(4/\pi^2)\ln 2[3+\ln(\epsilon/2)]-\tfrac{1}{2}$$
$$=-(4/\pi^2)\ln 2\ln(\epsilon/2). \quad \text{(A9)}$$

It follows that

$$\Delta E(r_s)=-(2/\pi^2)\ln(\beta r_s/\pi)\ln 2+(4/\pi^2)(\ln 2)^2=-0.141\ln r_s+0.448. \quad \text{(A10)}$$