Semi-Phenomenological Model for Meson Production in High-Energy Nucleon-Nucleon Collisions

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A "two-center" model is proposed for the description of meson production in high-energy nucleon-nucleon collisions. In this model the effect of one nucleon on the other is replaced by an equivalent interaction which frees pions "bound" in the second nucleon. The model takes into account "peripheral" collisions only and does not include mesons emanating from core-core collisions. A transverse momentum (p_T) distribution is obtained as a function of only the impact parameter and core radius. Our distributions agree well with experiment, the most probable p_T being 0.3–0.4 BeV/c. The angular distribution obtained does not deviate much from an isotropic distribution in the center-of-mass system of the emitting nucleon.

I. INTRODUCTION

SEVERAL models have been proposed to explain the various properties of the pions formed in very high energy ($\geqslant 10^{11}$ eV) nucleon-nucleon collisions. Koba and Takagi[1] have given a review of these together with relevant experimental results. In most observed high-energy nucleon-nucleon collisions the angular distribution is decidedly anisotropic in the center-of-mass system, being peaked heavily in the forward and backward directions. This led Cocconi,[2] Ciok *et al.*[3,4] and Niu[5] to the so-called "fireball" model in which the result of the collision is the formation of two centers of emission, the pions being emitted isotropically relative to each center. As proposed, the model was purely empirical and the authors, in general, interpreted the emitting centers to be moving more slowly than the outgoing nucleons.

Another two-center model is the "isobar" or "excited nucleons" model where particles are emitted from the moving nucleons which have become excited during the interaction.[6,7] A recent article discussing the relationship between the isobar and fireball models is that of Pernegr *et al.*[8] In the earlier forms of the theory of the "isobar" model (e.g., Takagi[9]) the theories of Fermi[10] or Heisenberg[11] were used for describing the phenomena of emission of the secondaries from both centers. Recently, some authors have considered the excitation as being caused by exchange of pions (cf., Romanov and Chernavskii).[12]

Much attention has been given experimentally to the distribution of p_T, the transverse momentum of the secondary particles.[13–16] It has been found that there is a peak in the p_T distribution at approximately 0.3–0.4 BeV/c. However little has been done on the basis of a two-center model to derive a theoretical p_T distribution. Most of the models by their very nature are unable to give a p_T distribution independent of arbitrary parameters (see, for example, the recent article by Gramenitskii *et al.*[17] for nucleon-nucleon interactions at 9 BeV). The model we wish to introduce gives p_T distributions at very high energies independent of arbitrary parameters, together with angular distributions in the center-of-mass system of each nucleon. These results are in good agreement with experimental observations.

The nucleon-nucleon cross section for inelastic collisions is approximately 30 mb in the range 10–50 BeV[18,19]; however, McCusker and Roesler[20] and Brisbout *et al.*[21] contend that a higher energies (3500 BeV) the cross section for meson production has become

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approximately geometric. It can be seen then that the majority of collisions will be peripheral collisions, that is, involving the meson clouds. It will be shown that such collisions produce a reasonable proportion of the mesons produced in all nucleon-nucleon collisions. For example, in a diametrical transversal of a silver nucleus by a high-energy nucleon, the average number of mesons produced may be at least six to eight (see Sec. III for further discussion of this), half of them being of high energy. (This number moreover makes no allowance for additional mesons produced by cascading effects—it is the number produced directly by the initial primary alone.)

The model we wish to introduce is for pion production from high-energy nucleon-nucleon peripheral collisions, that is, collisions which do not involve the nucleon "cores." In this model the effect of one nucleon on the other is replaced by an equivalent potential which frees pions "bound" in the second nucleon (see Sec. II.2).

The agreement which we obtain with the experimental transverse momentum distribution can be understood qualitatively in the following manner. When a meson is "freed" by the interaction with the other nucleon it still sees an absorbing core of radius, say, r_0. Since the meson is in a p state it is to be expected that its momentum distribution relative to its parent core show a maximum for $kr_0 \sim 1$, or $k \sim 1/r_0$.

The angular distribution of both incident and struck nucleons tends to be close to the incident direction, in view of the high forward momentum brought in by the incident nucleon. Thus the transverse momentum (p_T) distribution of all emitted mesons, in any frame of reference, tends to be peaked around $p_T \sim \hbar/r_0$. With choice of r_0 in the vicinity of 0.5 F, we would anticipate a p_T distribution peaked a little below 0.4 BeV/c; with smaller values of r_0 the p_T maximum would be expected to occur at correspondingly higher values.

In this paper we carry through the calculation in detail for the simple model in which we assume only one type of meson (zero isotopic spin). The p_T distributions are essentially as anticipated from the above arguments. It can, of course, be shown that the p_T distributions are not dependent whatsoever on the model of isoscalar mesons, and also follow when the three isotopic spin states for the mesons are included; the results regarding multiplicities, however, would become slightly modified, but not so as to greatly change our conclusions in this regard.

II. FORMULATION OF THE MODEL

1. Nucleon Wave Function

As we are only considering π production due to peripheral collisions we take our nucleon wave function as a function of the coordinates of the pions. These pions may be either inside or outside the core. For the initial symmetrical wave function of the nucleon we put

$$\psi_s(N) \equiv \psi_s(\mathbf{r}_1, \cdots, \mathbf{r}_N), \tag{1}$$

where $s = +$ or $-$ denotes the spin of the nucleon as $\pm 1/2$ and $\mathbf{r}_1, \cdots, \mathbf{r}_N$ are the coordinates of the N pions present at any instant. N is taken to be constant during the short period of the nucleon-nucleon interaction.

$\psi_s(N)$ is then expanded in terms of states of the remaining nucleon when one pion (say the ith) has been extracted and single-particle wave functions for the ith pion, i.e.,

$$\psi_s(N) = \sum_{s_1 t_1} a_{s_1 t_1}{}^i \psi_{s_1}(N-i) g_{t_1}(\mathbf{r}_i), \tag{2}$$

where $(N-i)$ represents all coordinates of the N pions apart from the ith pion, the $\psi_{s_1}(N-i)$ are normalized and symmetrical and the $g_{t_1}(\mathbf{r}_i)$ are normalized wave functions of states t_1 of the ith pion in the potential field $V_N'(\mathbf{r}_i)$ (see next paragraph). Because of the symmetry of $\psi_s(N)$ we have

$$a_{s_1 t_1}{}^i = a_{s_1 t_1}{}^j \equiv a_{s_1 t_1}. \tag{3}$$

We may also expand $\psi_{s_1}(N-i)$ as

$$\psi_{s_1}(N-i) = \sum_{s_2 t_2} a_{s_2 t_2}' \psi_{s_2}(N-i-j) g_{t_2}'(\mathbf{r}_j), \tag{4}$$

where $g_{t_2}'(\mathbf{r}_j)$ represents a normalized wave function of the state t_2 of the jth pion in the potential field $V_{N-i}'(\mathbf{r}_j)$. We think of $V_N'(\mathbf{r}_i)$ as being a good average of the interaction of the ith pion in the nucleon and $V_{N-i}'(\mathbf{r}_j)$ of the jth pion in the nucleon with the ith pion removed. Since N is large,

$$V_{N-i}'(\mathbf{r}) = V_N'(\mathbf{r}) = V'(\mathbf{r}) \tag{5}$$

so that, substituting Eq. (4) in Eq. (2),

$$\psi_s(N) = \sum_{s_1 t_1 s_2 t_2} a_{s_1 t_1} a_{s_2 t_2} \psi_{s_2}(N-i-j) g_{t_1}(\mathbf{r}_i) g_{t_2}(\mathbf{r}_j). \tag{6}$$

It will be noted here that we have neglected the interaction between the ith and jth pions. This is reasonable outside the core where the effect of the interaction is small compared to the effect of $V'(\mathbf{r})$.

In the region outside the core we approximate to $\psi_s(N)$ as

$$\psi_s(N) = \sum_{s_1 m_1} C_{\frac{1}{2}1}(\tfrac{1}{2}, s; s_1, m_1) \psi_{s_1}(N-i) B h_1(i\kappa r_i) \times Y_{1m_1}(\theta_i, \varphi_i), \tag{7}$$

where $\psi_{s_1}(N-i)$ represents the ground state of the nucleon (in our model this is the only state with all mesons bound), $r_i > r_0 =$ core radius, C is the appropriate Clebsch-Gordan coefficient, h_1 is the spherical Hankel function, Y_{1m_1} is a spherical harmonic, and B is a constant such that $\psi_s(N)$ is normalized; that is, B^2 measures the probability of any given meson being in the cloud.

The ith pion is bound to the nucleon with zero binding energy, that is,

$$0=\hbar^2(i\kappa)^2c^2+m_\pi^2c^4, \tag{8}$$

thus

$$\kappa=m_\pi c/\hbar, \tag{9}$$

where m_π is the pion rest mass. We only have terms h_1Y_{1m} since the nucleon remains in an $l=0$ state (although its spin may change) and the pion is pseudoscalar. Other terms, h_lY_{lm} for $l>1$, are taken to be negligible outside the core. We shall measure momentum in units of $m_\pi c/\hbar$ (140 MeV/c) and lengths in units of $\hbar/m_\pi c$ (1.4 F).

As we are only to be concerned with terms outside the core, we define the function $g_m(i)$ as

$$g_m(i)=h_1(i\kappa r_i)Y_{1m}(\theta_i,\varphi_i) \tag{10}$$

for $r_i>r_0$ and a smooth function fitting to this form for $r_i<r_0$ such that

$$\int g_{m_1}{}^*(i)g_{m_2}(i)d\mathbf{r}_i=B^{-2}\delta_{m_1m_2}. \tag{11}$$

Then Eq. (7) may be written

$$\psi_\pm(N)=\pm3^{-1/2}B[\psi_\pm(N-i)g_0(i) -2^{1/2}\psi_\mp(N-i)g_{\pm1}(i)]. \tag{12}$$

Continuing expanding using Eqs. (2), (6), (12), for n even,

$$\psi_+(N)=\frac{B^n}{(n+1)^{1/2}n!}P_{ij\cdots klm}\{\psi_+(N-i-\cdots-m)[g_0(i)g_0(j)-2g_1(i)g_{-1}(j)]\cdots[g_0(l)g_0(m)-2g_1(l)g_{-1}(m)]\}, \tag{13}$$

and for n odd,

$$\psi_+(N)=\frac{B^n}{(n+2)^{1/2}n!}P_{ij\cdots klm}\{[\psi_+(N-i-\cdots-m)g_0(m)-2^{1/2}\psi_-(N-i-\cdots-m)g_1(m)] \times[g_0(i)g_0(j)-2g_1(i)g_{-1}(j)]\cdots[g_0(k)g_0(l)-2g_1(k)g_{-1}(l)]\}, \tag{14}$$

where $P_{ij\cdots klm}$ = sum over all permutations of the n pions i, j, $\cdots k$, l, m. Terms which give zero on performing $P_{ij\cdots klm}$ have been eliminated. The $\psi_+(N)$ are normalized.

It is quite obvious, of course, that the wave function expressed by Eqs. (13) and (14) will only be a good approximation when $n\ll N$. It is only reasonable when there are few mesons outside the core radius. As we shall see later, however, we do indeed obtain maximum contributions from terms which involve few mesons only outside the core, consistent with the idea that the average number of mesons in the nucleon meson cloud is only a little above unity.

2. Nucleon-Nucleon Interaction

We consider one nucleon, n_1, as being at rest and the other, n_2, approaching with impact parameter b. The effect of n_2 is to free some pions from n_1, the interaction between n_2 and a pion in the cloud of n_1 being replaced by an equivalent potential $V(\mathbf{r})$ which is symmetrical about a line parallel to the z axis (defined by direction of motion of n_2) and at a distance b from it (Fig. 1). This neglects any elastic scattering of n_2 by n_1. Strictly this potential depends on both z and t. However, at high energies we can take $V(\mathbf{r})$ as a constant potential applied for a time t where ct is approximately the thickness of n_2 in the direction of motion.

We shall put

$$V(\mathbf{r})=Vv(r'), \tag{15}$$

where

$$r'=(b^2+r^2\sin^2\theta-2br\sin\theta\cos\varphi)^{1/2}. \tag{16}$$

The question now arises as to what form we shall take for $v(r')$. This involves more detailed knowledge of the π-nucleon interaction than is available at the present. However, taking into account the effect of the Lorentz contraction of n_2 and the range of the π-nucleon interaction it does not seem unreasonable to take

$$v(r')=e^{-r'}, \tag{17}$$

where lengths are in units of the pion Compton wavelength. It will be seen that our results are reasonably unchanged when other shapes of $v(r')$ are taken.

3. Final Wave Function

The final wave function will be

$$\Psi=\sum_m b_m u_m, \tag{18}$$

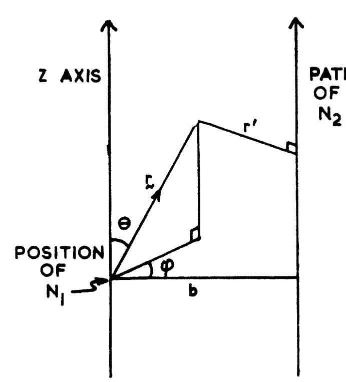

FIG. 1. Relationship between r', b, and $\mathbf{r}$.

where the u_m's are given by

$$\begin{aligned} u_0 &= \psi_{s'}(N), \\ u_{\mathbf{k}_1} &= N^{-1/2} \textstyle\sum_i \psi_{s'}(N-i) C_{\mathbf{k}_1}(\mathbf{r}_i), \\ u_{\mathbf{k}_1\mathbf{k}_2} &= [N(N-1)]^{-1/2} \textstyle\sum_{ij} \psi_{s'}(N-i-j) \\ &\qquad \times C_{\mathbf{k}_1}(\mathbf{r}_i) C_{\mathbf{k}_2}(\mathbf{r}_j), \text{ etc.} \end{aligned} \tag{19}$$

Here $C_{\mathbf{k}}(\mathbf{r})$ is the normalized wave function of a free pion which has the asymptotic form $e^{i\mathbf{k}\cdot\mathbf{r}}$, and $\psi_{s'}$ is a normalized symmetrical wave function of the remaining nucleon which we shall assume is the same as the corresponding $\psi_{s'}$ in the initial nucleon wave function, Eqs. (13), (14), with possible spin change of the nucleon.

The u_m's are orthonormal, that is,

$$\int u^*_{\mathbf{k}_1\cdots\mathbf{k}_r} u_{\mathbf{k}_1'\cdots\mathbf{k}_s'} d\mathbf{r}_1 \cdots d\mathbf{r}_N = \delta_{rs} \prod_{i=1}^{r} \sum_{j=1}^{r} \delta(\mathbf{k}_i - \mathbf{k}_j'). \tag{20}$$

The effect of the core on our model is to absorb any "free" mesons which may pass through it, and we therefore approximate $C_{\mathbf{k}}(\mathbf{r})$ as

$$\begin{aligned} C_{\mathbf{k}}(\mathbf{r}) &= (2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (r > r_0) \\ &= 0, \qquad\qquad (r < r_0). \end{aligned} \tag{21}$$

In this form the $C_{\mathbf{k}}(\mathbf{r})$ are not exactly orthonormal. However Eq. (21) is only used in evaluating integrals.

In Appendix A we derive the perturbation formula used to calculate b_m.

4. $P_n(\mathbf{k},b)$, $P_n(b)$

We will now define $P_n(\mathbf{k},b)d\mathbf{k}$ as the probability of n mesons being emitted, one of these having momentum $\mathbf{k}$, for a given impact parameter b, that is,

$$\begin{aligned} P_n(\mathbf{k},b) &= (n!)^{-1} \int |b_{\mathbf{k}_1\cdots\mathbf{k}_n}|^2 \delta(\mathbf{k}-\mathbf{k}_1) d\mathbf{k}_1 \cdots d\mathbf{k}_n \\ &\equiv (n!)^{-1} \int |b_{\mathbf{k}_1\cdots\mathbf{k}_n}|^2 \delta(\mathbf{k}-\mathbf{k}_r) d\mathbf{k}_1 \cdots d\mathbf{k}_n. \end{aligned} \tag{22}$$

$P_n(b)$ is the probability of n mesons being emitted at a given impact parameter b, that is

$$\begin{aligned} P_n(b) &= \int P_n(\mathbf{k},b) d\mathbf{k} \\ &= (n!)^{-1} \int |b_{\mathbf{k}_1\cdots\mathbf{k}_n}|^2 d\mathbf{k}_1 \cdots d\mathbf{k}_n. \end{aligned} \tag{23}$$

The factor $(n!)^{-1}$ is needed since by definition of u_m by Eq. (19) we have

$$b_{\mathbf{k}_1\cdots\mathbf{k}_n} \equiv b_{P(\mathbf{k}_1\cdots\mathbf{k}_n)}, \tag{24}$$

where P is any permutation of $\mathbf{k}_1, \cdots, \mathbf{k}_n$.

If we define

$$I_m(\mathbf{k}) = (2\pi)^{3/2} (4\pi/3)^{1/2} \int C_{\mathbf{k}}^*(\mathbf{r}) \exp[-itVv(r')/\hbar] g_m(\mathbf{r}) d\mathbf{r}, \tag{25}$$

$$L_{rss'} = \int \psi_s^*(N-i-\cdots-l) \prod_{j=1 (j\neq i,\cdots,l)}^{N} \{\exp[-itVv(r_j')/\hbar]\} \psi_{s'}(N-i-\cdots-l) d\tau_{N-i-\cdots-l}, \tag{26}$$

where $i, \cdots, l$ are r pions and put

$$f = (3/2)^{1/2} B / 4\pi^2, \tag{27}$$

$$d = tV/\hbar, \tag{28}$$

we have, in general,

$$b_{\mathbf{k}_1\cdots\mathbf{k}_n} = f^n ({}^N C_n / n!)^{1/2} (n+1)^{-1/2} L_{ns'+} P_{12\cdots n} \prod_{i=1}^{n/2} [I_0(\mathbf{k}_{2i-1}) I_0(\mathbf{k}_{2i}) - 2 I_1(\mathbf{k}_{2i-1}) I_{-1}(\mathbf{k}_{2i})] \tag{29a}$$

for n even, and

$$\begin{aligned} b_{\mathbf{k}_1\cdots\mathbf{k}_n} = f^n ({}^N C_n / n!)^{1/2} (n+2)^{-1/2} P_{12\cdots n} \{ \prod_{i=1}^{(n-1)/2} [I_0(\mathbf{k}_{2i-1}) I_0(\mathbf{k}_{2i}) - 2 I_1(\mathbf{k}_{2i-1}) I_{-1}(\mathbf{k}_{2i})] \\ \times [I_0(\mathbf{k}_n) L_{ns'+} - 2^{1/2} I_1(\mathbf{k}_n) L_{ns'-}] \} \end{aligned} \tag{29b}$$

for n odd.

If we define

$$A_m = (2\pi)^{-3} \int I_{m_1}^*(\mathbf{k}) I_{m_2}(\mathbf{k}) d\mathbf{k}, \tag{30}$$

where $m = m_1 m_2$ and $|m_1| = |m_2|$ (the integral is zero for $|m_1| \neq |m_2|$), and perform the sum over final and average over initial spin states, we find for the n-even case

$$P_n(\mathbf{k},b)=(\tfrac{1}{2}n)!(\tfrac{1}{2}n-1)![(n+1)!]^{-1}\,{}^NC_nf^{2n}(2\pi)^{3(n-1)}(|L_{n+}|^2+|L_{n-}|^2)$$
$$\times\sum_{l=0}^{\frac{1}{2}n-1}\sum_{m=0}^{l}({}^lC_mA_0^{\frac{1}{2}n-l-1}A_1^mA_{-1}^{l-m}2^l)^2(\tfrac{1}{2}n-l)\;{}^{n-2l}C_{\frac{1}{2}n-l}$$
$$\times\left\{A_0|I_0|^2+\frac{(l+1)^2}{(n-2l-1)}\left[\frac{A_1(|I_1|^2+|I_{-1}|^2)}{m+1}+\frac{2A_{-1}\,\mathrm{Re}I_1^*I_{-1}}{l-m+1}\right]\right\} \quad (31a)$$

and

$$P_n(b)=[(\tfrac{1}{2}n)!]^2[(n+1)!]^{-1}\,{}^NC_n(3B^2/4\pi)^n(|L_{n+}|^2+|L_{n-}|^2)\sum_{l=0}^{\frac{1}{2}n}\sum_{m=0}^{l}{}^{n-2l}C_{\frac{1}{2}n-l}[{}^lC_mA_0^{\frac{1}{2}n-l}A_1^mA_{-1}^{l-m}]^2\,2^{2l}. \quad (32)$$

In Eqs. (31) and (32) we have slightly altered our notation by putting

$$L_{nss'}=L_{n+},\quad (s=s')$$
$$=L_{n-},\quad (s=-s'). \quad (33)$$

If as a first approximation we put

$$A_0=A_1=A,$$
$$A_{-1}=0, \quad (34)$$

(for justification see Appendix B), we have

$$P_n(\mathbf{k},b)=\tfrac{1}{3}\;{}^NC_n(3B^2/4\pi)^nA^{n-1}(2\pi)^{-3}(|L_{n+}|^2+|L_{n-}|^2)(|I_0|^2+|I_1|^2+|I_{-1}|^2), \quad (35)$$

$$P_n(b)={}^NC_n(3B^2A/4\pi)^n(|L_{n+}|^2+|L_{n-}|^2). \quad (36)$$

It should be noted that we have used the relation

$$\sum_{l=0}^{\frac{1}{2}n}{}^{n-2l}C_{\frac{1}{2}n-l}2^{2l}=(n+1)![(\tfrac{1}{2}n)!]^{-2}. \quad (37)$$

For n odd and using the approximation (34) we find that $P_n(b)$ is the same but there is a slight change to $P_n(\mathbf{k},b)$:

$$P_n(\mathbf{k},b)=\tfrac{1}{3}(3B^2A/4\pi)^nA^{n-1}(2\pi)^{-3}\;{}^NC_n\{(|L_{n+}|^2+|L_{n-}|^2)(|I_0|^2+|I_1|^2+|I_{-1}|^2)$$
$$-(2^{3/2}/n)\,\mathrm{Re}(L_{n+}^*L_{n-})\,\mathrm{Re}[I_0^*(I_1+I_{-1})]\}. \quad (31b)$$

In Appendix C we show that

$$|L_{n+}|^2+|L_{n-}|^2=(1-3B^2A/4\pi)^{N-n}, \quad (38)$$

so that

$$P_n(b)={}^NC_n(3B^2A/4\pi)^n(1-3B^2A/4\pi)^{N-n}. \quad (39)$$

This is the result we would have obtained if we had let x be the probability of freeing any given meson so that

$$P_n={}^NC_nx^n(1-x)^{N-n}. \quad (40)$$

Hence we see that x is given by $3B^2A/4\pi$.

We now define $I_{mn}(\mathbf{k})$ by

$$I_m(\mathbf{k})=\sum_{n=1}^{\infty}[(-id)^n/n!]I_{mn}(\mathbf{k}), \quad (41)$$

that is, I_{mn} is given by expanding $\exp(-idv)$ in powers of d. Then making the approximations given by Eqs. (7), (10), and (21), we have from Eqs. (25) and (41),

$$I_{mn}(\mathbf{k})=\int_{r_0}^{\infty}dr\int_0^{\pi}d\theta\int_{-\pi}^{\pi}d\varphi\,\exp\{ikr[\cos\chi\cos\theta+\sin\chi\sin\theta\cos(\sigma-\varphi)]\}v^n(\{b^2+r^2\sin^2\theta-2br\sin\theta\cos\varphi\}^{1/2})$$
$$\times(1+r)e^{-r}(\delta_{m0}\cos\theta+\delta_{|m|1}2^{-1/2}\sin\theta e^{im\varphi})\sin\theta, \quad (42)$$

where

$$\mathbf{k}=(k,\chi,\sigma).$$

On making use of various relationships between the $I_{mn}(\mathbf{k})$ for related χ and σ, for example

$$I_{mn}^*(k,\chi,\sigma)=I_{-mn}(k,\pi-\chi,\sigma\pm\pi), \quad (43)$$

we find that

$$\int_{-\pi}^{\pi}|I_m|^2d\sigma=4\int_0^{\frac{1}{2}\pi}d\sigma\sum_{n=1}^{\infty}d^{2n}\sum_{r=1}^{2n-1}\frac{(-1)^{r+n}}{r!(2n-r)!}I_{mr}{}^*I_{m,2n-r}\,, \tag{44}$$

$$\int_{-\pi}^{\pi}\mathrm{Re}I_0{}^*(I_1+I_{-1})d\sigma=4\int_0^{\frac{1}{2}\pi}d\sigma\sum_{n=1}^{\infty}d^{2n+1}\sum_{r=1}^{2n}\frac{(-1)^{r+n}}{r!(2n+1-r)!}\mathrm{Im}[I_{0r}{}^*(I_{1,2n+1-r}+I_{-1,2n+1-r})]. \tag{45}$$

That is,

$$\int_{-\pi}^{\pi}|I_m|^2d\sigma=4\int_0^{\frac{1}{2}\pi}d\sigma[d^2|I_{m1}|^2+O(d^4)], \tag{44a}$$

$$\int_{-\pi}^{\pi}\mathrm{Re}I_0{}^*(I_1+I_{-1})d\sigma=4\int_0^{\frac{1}{2}\pi}d\sigma[(d^3/2!)\,\mathrm{Im}\{I_{01}{}^*(I_{12}+I_{-12})-I_{02}{}^*(I_{11}+I_{-11})\}+O(d^5)]. \tag{45a}$$

Thus for n even and odd we have, neglecting higher orders than d^2,

$$\int_{-\pi}^{\pi}d\sigma\,P_n(\mathbf{k},b)\propto\int_0^{\frac{1}{2}\pi}(|I_{01}|^2+|I_{11}|^2+|I_{-11}|^2)d\sigma. \tag{46}$$

In Figs. 2–7 we have plotted this integral multiplied by $k^2\sin\chi$. Thus, in Figs. 2–7 the curves plotted give the distribution of k and p_T, where p_T is the transverse momentum $k\sin\chi$. In Sec. III we discuss the multiplicity distribution and the magnitude of d. It will be seen that d is approximately unity so that our neglect of terms of higher order than d^2 requires justification. Detailed numerical calculation, however, checks what is apparent intuitively, that the coefficients of d^n become small rapidly as n increases. It is thus found to be quite accurate to ignore orders higher than d^2.

The integrals involved in calculating I_m and A_m were performed numerically on SILLIAC, the digital electronic computer of the Basser Computing Department, School of Physics, University of Sydney. Details are given in Appendix D.

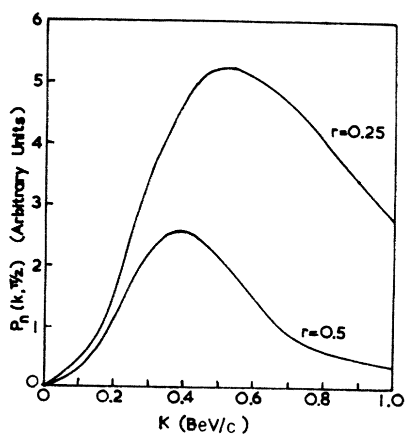

FIG. 2. Momentum distribution of pions in center-of-mass system of the emitting nucleon for impact parameter 0.6 and angle of emission $\frac{1}{2}\pi$. Distributions are shown for core radii r of 0.25 and 0.5, and averaged over azimuthal angles of emission. Lengths in each case are units of 1.4 F.

5. Cross Sections

In any collision pions may be produced from either nucleon. $P_n(b)$ gives the probability of n pions being produced from one nucleon, so we now introduce $\mathcal{P}_n(b)$, the probability of n pions being produced in a single collision. For simplification we take N, the number of pions in the nucleon at the time of collision, to be the same for each nucleon. It is clear that such a simplification should not affect our results in any way. Then we will have

$$\mathcal{P}_n(b)=\sum_{i=0}^{n}P_{n-i}(b)P_i(b). \tag{47}$$

Using Eq. (40) we have

$$\mathcal{P}_n(b)={}^{2N}C_nx^n(1-x)^{2N-n}. \tag{48}$$

To obtain the cross section for pion production from peripheral collisions we assume that no pions are pro-

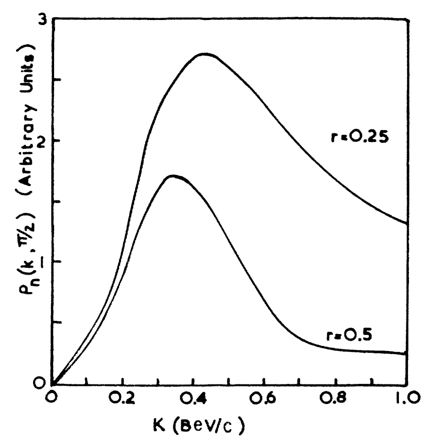

FIG. 3. As in Fig. 2 except for impact parameter 1.0, same angle of emission.

duced for an impact parameter less than b_0, where b_0 is approximately twice the core radius. Then we have that σ_n, the cross section for production of n mesons, is given by

$$\sigma_n = 2\pi \int_{b_0}^{\infty} \mathcal{P}_n(b) b db. \tag{49}$$

From Appendix B and Fig. 10 we see that for a core radius of 0.25,

$$A \simeq 6e^{-2b}d^2, \tag{50}$$

so that x in Eq. (48) will be given by

$$x = (18B^2d^2/4\pi)e^{-2b}. \tag{51}$$

We put

$$\alpha = 18B^2d^2/4\pi \tag{52}$$

and have calculated σ_n for $n=1, 2, 3$ and various values

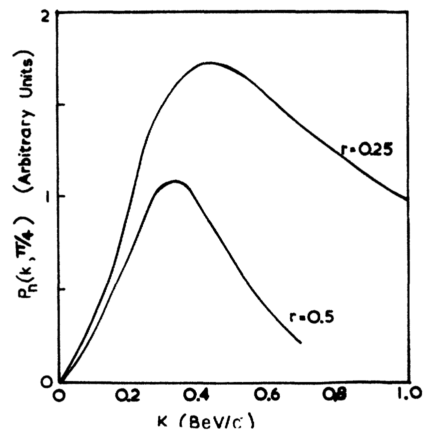

FIG. 4. As in Fig. 2 except for impact parameter 1.0 and angle of emission $\frac{1}{4}\pi$.

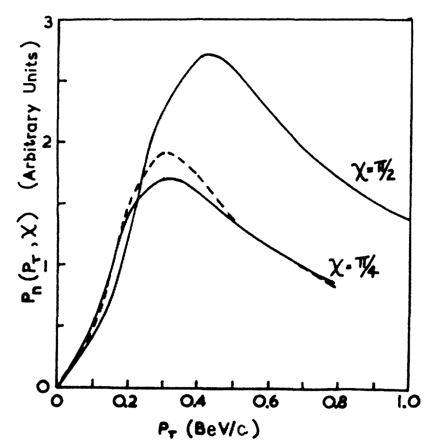

FIG. 5. Transverse momentum distribution of pions for angles of emission χ equal to $\frac{1}{2}\pi$ and $\frac{1}{4}\pi$ in the center-of-mass system of the emitting nucleons and for a core radius of 0.25 and impact parameter of 1.0. The dashed line shows what the distribution for $\chi=\frac{1}{4}\pi$ would be on the basis of the distribution for $\chi=\frac{1}{2}\pi$ assuming isotropy. The distributions have been averaged over all azimuthal angles.

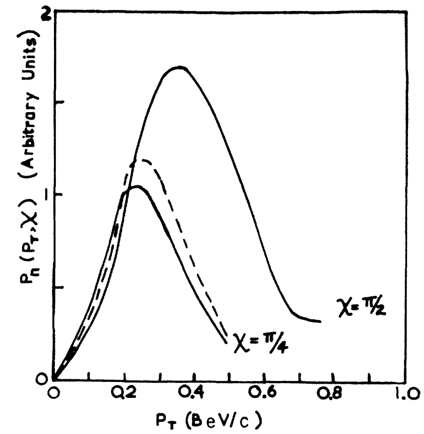

FIG. 6. As for Fig. 5, except for core radius of 0.5.

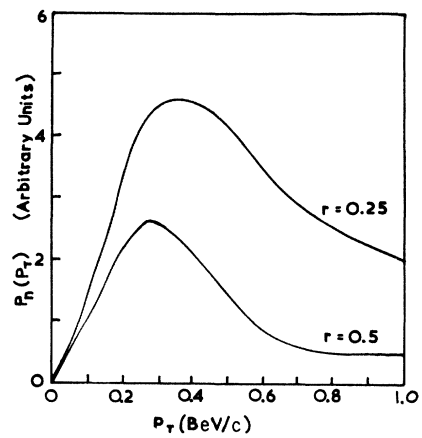

FIG. 7. Transverse momentum distribution of pions averaged over all angles of emission for impact parameter 1.0 and core radii r of 0.25 and 0.5.

TABLE I. Cross sections for 1, 2, 3 pion production for different values of $N\alpha$ (Eq. 52) and minimum impact parameter b_0. The cross sections are given in millibarns and b_0 in units of 1.4 F.

$N\alpha$	b_0	σ_1	σ_2	σ_3
0.5	0	24.6	2.52	0.33
	0.2	23.7	2.18	0.22
	0.5	19.8	1.25	0.08
	1.0	11.8	0.32	0.009
1.0	0	40 6	7.00	1.61
	0.2	40.0	6.36	1.28
	0.5	35.1	4.11	0.54
	1.0	22.4	1.18	0.06
2.0	0	60.6	15.2	5.46
	0.2	60.4	14.7	4.90
	0.5	56.7	11.3	2.70
	1.0	40.4	4.1	0.42
5.0	0	89.0	29.1	14.3
	0.2	88.7	28.9	14.2
	0.5	88.0	27.7	12.2
	1.0	76.9	16.2	3.9

of N and α. It was found that the results depended only on the product $N\alpha$ (within 1%) and in Table I we give the results for different values of b_0 and $N\alpha$.

III. DISCUSSION OF RESULTS

This model gives a transverse momentum distribution and energy and momentum distributions in the center-of-mass system of each nucleon, as a function only of the impact parameter and the radius of the core of the nucleon. For an impact parameter of 1.0 (i.e., 1.4 F) and core radii of 0.25 and 0.50, the most probable p_T are 350 MeV/c and 280 MeV/c, respectively. With an impact parameter of 0.6 and the same core radii, the most probable p_T are 450 MeV/c and 340 MeV/c, respectively. These values are averaged over all angles of emission.

These results are in agreement with the experimental results[13–16] except that the "tail" in our results is longer. This is to be expected as we have not introduced any energy limitations in the pion production. Such an energy limitation would shorten the tail and also decrease slightly the value of the most probable p_T. This is suggested by experimental observations at lower energies. Blue *et al.*[22] obtained a most probable p_T of 90 MeV/c for 4.2-BeV proton-proton collisions. This is an energy where we expect energy restrictions to be very important. In comparing our results with experiment we also need to remember that our model is based on nucleon-nucleon collisions and most experiments involve nucleon-nucleus collisions. However, we expect the p_T distributions from such collisions (particularly distributions for the high-energy pions) not to be much different. This has been shown experimentally by Matsumoto.[23]

The angular distribution of the emitted pions is almost isotropic (Figs. 5 and 6) with respect to the center of mass of the emitting nucleon. Such a distribution could not be distinguished from an isotropic distribution with present experimental techniques.

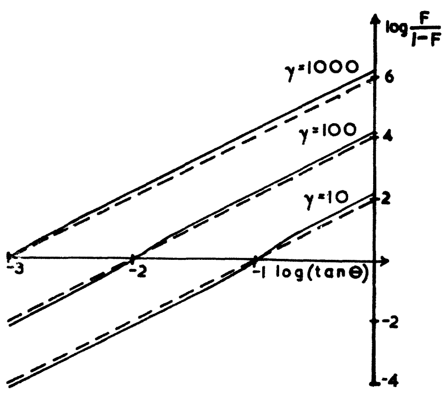

FIG. 8. Walker-Duller plot for an isotropic distribution $f(\theta)=1$ (shown by dashed lines) and for the distribution $f(\theta)=\frac{1}{2}(1+\sin\theta)$ (shown by solid lines). Lines are shown for $\gamma=10$, 100, and 1000 where γ is the Lorentz factor for emitting center in the laboratory system.

TABLE II. Average multiplicity $\bar{n}$ as a function of Nx [see Eq. (53)].

Nx	0.1	0.2	0.5	1.0	5.0
$\bar{n}$	1.04	1.21	1.58	2.31	10.01

This can be seen directly by comparing the Walker-Duller plot[24] (Fig. 8) for an isotropic distribution, $f(\theta)=1$, and for the distribution, $f(\theta)=\frac{1}{2}(1+\sin\theta)$, to which our calculated distribution approximates. In the center-of-mass system of both nucleons the distribution is, of course, strongly peaked in the forward and backward directions.

We now turn to the multiplicity distribution, that is, $\mathcal{P}_n$ as a function of n. Using Eq. (48), the average value of n is $2Nx$; on correcting $\mathcal{P}_n$ so that $n=0$ is not included, we have

$$\bar{n}=2Nx/[1-(1-x)^{2N}]. \tag{53}$$

In Table II we have $\bar{n}$ as a function of Nx. It is seen that the multiplicity distribution is determined only by Nx, that is, the product B^2NA. To determine B^2N we note that with our approximation in Eq. (7), the average number of mesons outside the core is N_0, where $N_0=2.73B^2N$ for core radius of 0.25, and $N_0=0.92B^2N$ for core radius of 0.5. Thus, using Eq. (51) we have, for a core radius of 0.25,

$$Nx\simeq 0.52d^2e^{-2b}N_0. \tag{54}$$

The experimental observations at very high energies suggest that the inelastic nucleon-nucleon cross section is geometric.[20,21] Allowing for a minimum impact parameter of 0.5 (twice core radius 0.25) and a cross section for inelastic core-core collisions of no more than one-quarter geometric, we see on comparison with Table I that $N\alpha\simeq 1.5$. We will have then, from Eq. (54), $d^2N_0\simeq 3$. Thus, we have that d is approximately unity.

To relate d to V, the strength of the effective perturbing potential, Eq. (28), we have that $ct\simeq\gamma^{-1}\times 1.4$ F, where γ is the Lorentz contraction factor for one nucleon relative to the other. Hence

$$V\simeq 140\gamma d\text{ MeV}. \tag{55}$$

Thus $140d$ MeV is approximately the average strength of the effective π-nucleon potential at low energies.

For $N\alpha=1.5$ and an impact parameter of 0.5 we have, from Eq. (54), $Nx\simeq 0.55$. From Table II, we see this gives an average multiplicity for a collision with such an impact parameter of 1.6. An approximate calculation for a diametral collision with a silver nucleus gives an average of six pions, 3 of high energy and 3 of low, produced according to this model by the incident primary nucleon; to this would have to be added all

[22] M. H. Blue, J. J. Lord, J. G. Parks, and C. H Tsao, Nuovo Cimento **20**, 274 (1961).
[23] S. Matsumoto, J. Phys. Soc. Japan **17**, 1 (1962).
[24] N. M. Duller and W. D. Walker, Phys. Rev. **93**, 215 (1954).

additional pions produced by cascading effects within the nucleus.

While our model is unable to produce the high multiplicities observed in experiment without introducing core-core collisions, yet it is seen that it does account for a significant proportion of the mesons produced. We would contend, from a study of Table I, that most of the low-multiplicity nucleon-nucleon collisions are described by this model. Moreover, if there is any cascading within the nucleus, the pions formed by these secondary collisions would have the same p_T distribution.

In order to determine the sensitivity of our model to the original shape of the effective potential $v(r')$ we have also considered the square-well case $v(r')=1$ for $r'<R$ and $v(r')=0$ for $r'>R$. Results for the function $(|I_{01}|^2+|I_{11}|^2+|I_{-11}|^2)k^2\sin\chi$ are plotted in Fig. 9 for $b=1$, $\chi=\frac{1}{2}\pi$, $\sigma=0$, and for the two ranges $R=0.25$ and 0.5. We see that the over-all behavior of the results is again achieved and that there is again a peak in the transverse momentum distribution—this time, however, at a somewhat higher value. Thus, for detailed comparison with experiment, the shape $v(r')=e^{-r'}$ is preferable to the sharp square well.

The present model is clearly in the category of an "isobar" model, in the sense that pions are emitted independently from a parent excited nucleon. It has been suggested that a "fireball" model is applicable rather than an "isobar" model,[8] although this has not been demonstrated conclusively owing to experimental uncertainties involved in determining energies. The present model ignores outgoing pion-pion correlations due to pion resonances; while important at the particular resonance center-of-mass energies, the total contribution of the resonances is considered not to be of dominant importance after integration over all outgoing energies is performed.

Thus, in proposing the present model, we are able to obtain momentum and angular distributions essentially independent of arbitrary parameters; although not yielding the observed multiplicities, the model is applicable to a significant proportion of the pions produced.

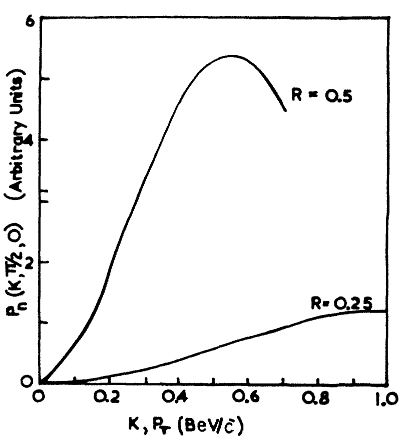

FIG. 9. Transverse momentum, p_T, distribution for a square-well effective potential for $\sigma=0$, $\chi=\frac{1}{2}\pi$, $b=1$, and $R=0.25$ and 0.5 (see text).

ACKNOWLEDGMENTS

The authors wish to thank Professor H. Messel for providing the excellent research facilities which have been necessary for the present investigation, and members of the Basser Computing Department for assistance given in carrying out the various numerical calculations. One of us (J.W.O.) also wishes to thank the Australian Atomic Energy Commission for the award of a studentship.

APPENDIX A. PERTURBATION FORMULA

The following derivation is an extension of that given by Schiff.[25] Suppose that

$$\begin{aligned} H&=H_0, \quad t<0,\ t>t_0\\ &=H_0+h, \quad 0<t<t_0\\ H_0u_n&=E_nu_n,\\ (H_0+h)w_k&=E_kw_k, \end{aligned} \tag{A1}$$

where u_n, w_k are a complete orthonormal set of functions

$$\begin{aligned} t<0:&\quad \psi=S_na_nu_n\exp(-iE_nt/\hbar),\\ 0<t<t_0:&\quad \psi=S_kc_kw_k\exp(-iE_kt/\hbar),\\ t>t_0:&\quad \psi=S_mb_mu_m\exp(-iE_mt/\hbar), \end{aligned} \tag{A2}$$

where S_n denotes sum of discrete n and integration of the continuous part of its range.

Continuity conditions at $t=0$, $t=t_0$ give

$$b_m=S_na_n\int\int u_m'^*\{S_kw_k'w_k^*\times\exp[-i(E_k-E_m)t_0/\hbar]\}u_nd\tau d\tau', \tag{A3}$$

where the prime denotes a different set of coordinate variables of integration. If the system is initially in a state u_0, $a_n=\delta_{n0}$ and

$$b_m=\int u_m^*\exp[-(it_0/\hbar)h]u_0d\tau, \tag{A4}$$

where we first use the property

$$f(H_0+h)w_k=f(E_k)w_k$$

and then the closure property

$$S_kw_k(\mathbf{r}')w_k^*(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}').$$

In our model,

$$\begin{aligned} h&=\sum_{i=1}^{N}Vv(r_i'),\\ t_0&=t, \end{aligned} \tag{A5}$$

[25] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., pp. 217–8.

so that Eq. (A4) gives

$$b_m=\int u_m^* \exp[(-it_0/\hbar)\sum v(r_i')]u_0 d\tau_N. \quad (A6)$$

APPENDIX B. SIMPLICATION OF A_m

Combining Eqs. (25) and (30) we have

$$A_m=(4\pi/3)\int C_{\mathbf{k}}(\mathbf{r}_1)C_{\mathbf{k}}^*(\mathbf{r}_2) \times\exp\{id[v(r_1')-v(r_2')]\}g_{m_1}^*(\mathbf{r}_1) \times g_{m_2}(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{k}. \quad (B1)$$

Let us perform the $\mathbf{k}$ integration first.

The $C_{\mathbf{k}}(\mathbf{r})$ and the bound-state wave functions form a complete orthonormal set so that

$$\int C_{\mathbf{k}}(\mathbf{r}_1)C_{\mathbf{k}}^*(\mathbf{r}_2)d\mathbf{k}\simeq\delta(\mathbf{r}_1-\mathbf{r}_2)-\sum_m B^2 g_m(\mathbf{r}_1)g_m^*(\mathbf{r}_2), \quad (m=0,\pm1). \quad (B2)$$

Hence,

$$A_m\simeq(4\pi/3)\left\{B^{-2}\delta_{m_1m_2}-\sum_m B^2\int g_m(\mathbf{r}_1)g_m^*(\mathbf{r}_2) \times\exp\{id[v(r_1')-v(r_2')]\}g_{m_1}^*(\mathbf{r}_1) \times g_{m_2}(\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2\right\}. \quad (B3)$$

On putting

$$g_m(\mathbf{r})=R(r)Y_{1m}(\theta,\varphi)$$

we see that the integral is zero if $|m_1|\neq|m_2|$ and depends only on the relative signs of m_1 and m_2. Thus, if

$$E_m=(4\pi/3)\int g_{m_1}^*(1-e^{-idv})g_{m_2}d\mathbf{r}, \quad (B4)$$

where $m=m_1m_2$, $|m_1|=|m_2|$, we have that

$$\begin{aligned} A_0&=2\,\mathrm{Re}E_0-(3B^2/4\pi)|E_0|^2,\\ A_1&=2\,\mathrm{Re}E_1-(3B^2/4\pi)\{|E_1|^2+|E_{-1}|^2\},\\ A_{-1}&=2\,\mathrm{Re}E_{-1}-(3B^2/4\pi)\{E_1E_{-1}^*+E_{-1}E_1^*\}. \end{aligned} \quad (B5)$$

We now put [compare Eqs. (41) and (42)]

$$E_m=\sum_{n=1}^{\infty}-\frac{(-id)^n}{n!}\mathcal{E}_{mn}, \quad (B6)$$

that is,

$$\mathcal{E}_{mn}=(4\pi/3)\int g_{m_1}^* v^n g_{m_2}d\mathbf{r}, \quad (B6a)$$

and observe that $\mathcal{E}_{mn}$ is real. We could similarly put

$$A_m=\sum_{n=1}^{\infty}2(-1)^{n+1}\frac{d^{2n}}{(2n)!}\mathcal{A}_{m,2n} \quad (B7)$$

so that

$$\mathcal{A}_{m,2n}=\mathcal{E}_{m,2n}-O(B^2). \quad (B8)$$

In Figs. 10 and 11 we have plotted $\mathcal{E}_{m2}$ and $\mathcal{E}_{m4}$ as a function of the impact parameter b. We see that it is a reasonable approximation to put

$$\begin{aligned} A_1&=A_0=A,\\ A_{-1}&=0, \end{aligned} \quad (B9)$$

even for d approximately unity.

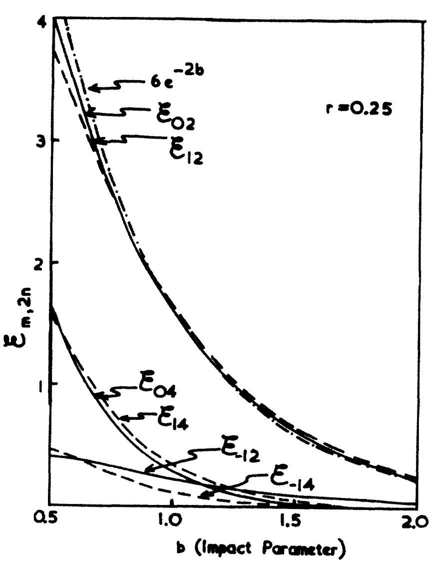

FIG. 10. $\mathcal{E}_{m,2n}$ [see Eq. (B6a)] as a function of impact parameter b, and for core radius of 0.25. Curves are drawn for $m=0,\pm1$; $n=1, 2$. For comparison, the curve given by $6e^{-2b}$ is also drawn. b is given in units of 1.4 F.

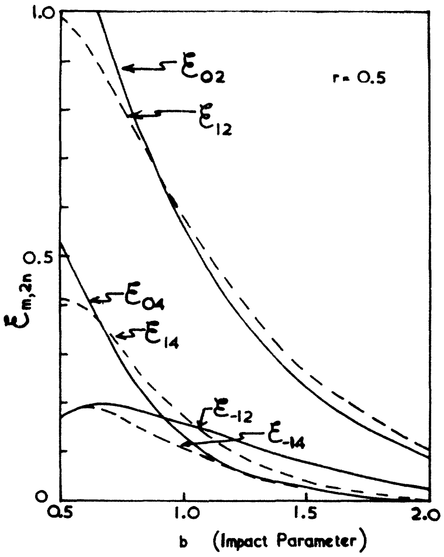

FIG. 11. As for Fig. 10 except for core radius of 0.5.

APPENDIX C. DERIVATION OF EQUATION (38)

We have given in Eq. (26) the definition of $L_{rss'}$. Let us consider the case when $(N-r)$ is even; then

$$L_{rss'}=B^{2(N-r)}[(N-r+1)!(N-r)!]^{-1}\int\phi_s{}^*\phi_{s'}\{P_{1\cdots(N-r)}\prod_{i=1}^{\frac{1}{2}(N-r)}[g_0(2i-1)g_0(2i)-2g_1(2i-1)g_{-1}(2i)]\}^*$$
$$\times\prod_{j=1}^{N-r}\exp[-idv(r_j')]P_{1\cdots(N-r)}\prod_{i=1}^{\frac{1}{2}(N-r)}[g_0(2i-1)g_0(2i)-2g_1(2i-1)g_{-1}(2i)]d\mathbf{r}_1\cdots d\mathbf{r}_{N-r}d\chi,\quad \text{(C1)}$$

where we have introduced an extra term ϕ_s into the expansion of ψ_s to represent the nucleon stripped of all pions and χ represents the variables describing this function. If we are to allow for possible spin change in this "bare" nucleon we would have to include some extra term, say, $v(\chi)$, in the perturbing potential.

Let

$$\int\phi_s{}^*e^{-idv(\chi)}\phi_{s'}d\chi=\alpha_+,\quad (s=s')$$
$$=\alpha_-,\quad (s=-s')\quad \text{(C2)}$$

so that

$$|\alpha_-|^2+|\alpha_+|^2=1.\quad \text{(C3)}$$

Then in a way similar to that used to obtain $P_n(b)$, Eq. (32), we have

$$L_{r\pm}=\alpha_\pm B^{2(N-r)}[(N-r+1)\ {}^{N-r}C_{\frac{1}{2}(N-r)}]^{-1}\sum_{l=0}^{\frac{1}{2}(N-r)}\sum_{m=0}^{l}{}^{N-r-2l}C_{\frac{1}{2}(N-r)-l}[{}^lC_mD_0{}^{\frac{1}{2}(N-r)-l}D_1{}^mD_{-1}{}^{l-m}2^l]^2,\quad \text{(C4)}$$

where

$$D_m=\int g_{m_1}{}^*e^{-idv}g_{m_2}d\mathbf{r},\quad (m=m_1m_2;\ |m_1|=|m_2|).\quad \text{(C5)}$$

From Eq. (B4) we see that

$$D_m=\delta_{m_1m_2}B^{-2}-(3/4\pi)E_m,\quad \text{(C6)}$$

so that, approximately,

$$D_0=D_1=D;\ D_{-1}=0.\quad \text{(C7)}$$

With this approximation,

$$|L_{r+}|^2+|L_{r-}|^2=|(B^2D)^{N-r}|^2.\quad \text{(C8)}$$

Also combining Eqs. (C6) and (B5) we have

$$B^4|D|^2=1-3B^2A/4\pi\quad \text{(C9)}$$

so that

$$|L_{r+}|^2+|L_{r-}|^2=(1-3B^2A/4\pi)^{N-r}.\quad \text{(C10)}$$

APPENDIX D. NUMERICAL INTEGRATION

The numerical method used to calculate $\mathcal{E}_{mn}$ [Eq. (B6)] and I_{mn} [Eq. (42)] was an extension of the Newton-Cotes 9-point formula[26] to three dimensions. This is equivalent to fitting a polynomial of the eight order in each of the three variables r, θ, φ through 9^3 points. The mesh sizes were chosen after various tests and gave $\mathcal{E}_{m2}$ and I_{m1} to an accuracy of better than 3%.

For $\mathcal{E}_{mn}$ we note that

$$\mathcal{E}_{mn}=4\int_{r_0}^{\infty}dr\int_0^{\frac{1}{2}\pi}d\theta\int_0^{\pi}d\varphi\exp(-nr'-2r)(1+r^{-1})^2(\delta_{m0}\cos^2\theta+\tfrac{1}{2}\delta_{m1}\sin^2\theta+\tfrac{1}{2}\delta_{m,-1}\sin^2\theta\cos2\varphi)\sin\theta.\quad \text{(D1)}$$

For I_{mn} we first calculated J_{mn} given by

$$J_{mn}(k,\chi,\sigma)=\int_{r_0}^{\infty}dr\int_0^{\pi}d\theta\int_0^{\pi}d\varphi[\exp(i\mathbf{k}\cdot\mathbf{r}-nr'-r)](1+r)(\delta_{m0}\cos\theta+\delta_{|m|1}2^{-1/2}\sin\theta e^{im\varphi})\sin\theta,\quad \text{(D2)}$$

so that

$$I_{mn}(k,\chi,\sigma)=J_{mn}(k,\chi,\sigma)+J_{-mn}(k,\chi,-\sigma).\quad \text{(D3)}$$

$J_{mn}(k,\chi,\sigma)$ was calculated for $\sigma=0$, $\pm\frac{1}{8}\pi$, $\pm\frac{1}{4}\pi$, $\pm\frac{3}{8}\pi$, $\frac{1}{2}\pi$. These values were used to give $|I_{01}|^2+|I_{11}|^2+|I_{-11}|^2$ for $\sigma=0$, $\frac{1}{8}\pi$, $\frac{1}{4}\pi$, $\frac{3}{8}\pi$, $\frac{1}{2}\pi$. The integration over σ, Eqs. (44a) and (45a), was performed noting that

$$\int_{-\frac{1}{2}\pi}^{0}d\sigma\cdots=\int_0^{\frac{1}{2}\pi}d\sigma\cdots=\int_{\frac{1}{2}\pi}^{\pi}d\sigma\cdots.\quad \text{(D4)}$$

Hence we used the Newton Cotes 9-point formula in two ways:

$$\int_0^{\frac{1}{2}\pi}d\sigma\cdots=\frac{1}{2}\int_0^{\pi}d\sigma\cdots,\quad \text{(D5a)}$$

$$\int_0^{\frac{1}{2}\pi}d\sigma\cdots=\frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}d\sigma\cdots.\quad \text{(D5b)}$$

[26] See, e.g., H. Mineur, *Techniques de Calcul Numérique* (C. Bèranger, Paris, 1952), p. 243.

Comparison of the results from (D5a) and (D5b) gave an idea of the accuracy with which we had calculated the integrand. Agreement was found to be better than 1%.

The results plotted in Fig. 7 (integration over σ and χ) were obtained from the results shown in Figs. 5 and 6 by noting that the p_T distribution for $\chi=\frac{1}{4}\pi$ is not far different from what would have been predicted from the distribution for $\chi=\frac{1}{2}\pi$ assuming isotropy. Curves for $=\chi$ $\frac{1}{12}\pi$, $\frac{1}{6}\pi$, $\frac{1}{3}\pi$, $(5/12)\pi$ were plotted using this fact and the calculated distributions for $\chi=\frac{1}{4}\pi$ and $\frac{1}{2}\pi$ and then the integration performed numerically from the graph. Errors involved in this procedure are negligible compared to other approximations already made.

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Production of Tritons, Deuterons, Nucleons, and Mesons by 30-GeV Protons on Al, Be, and Fe Targets*

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The momentum spectra of particles emerging at 30° to a 30-GeV proton beam impinging upon various targets were measured using time-of-flight techniques. Intensities of protons, antiprotons, π mesons, K mesons, deuterons, and tritons in the range 1 to 3 GeV/c are given. Particular attention is given to the tritons and deuterons emitted from the different targets. Possible mechanisms for their production are discussed.

I. INTRODUCTION

DURING the initial investigations of the composition of secondary particle beams emitted at various angles from internal targets in the 33-GeV alternating gradient synchrotron (AGS) at Brookhaven National Laboratory, we analyzed the beam emerging at 30° by measuring the time of flight of particles after momentum selection by magnetic deflection. The results of the beam surveys at other angles were performed by other groups and have been reported.[1,2] Our results on the intensities of emerging beams of pions, protons, antiprotons, and K mesons are presented mainly for the practical interest in these investigations for the design of future experiments at the AGS.

The copious production of deuterons and mass-three nuclei, discovered at CERN[3] during the observation of forward secondary beams, was also observed at 30° with very little change in intensity relative to pions and protons. If these particles were produced in nucleon-nucleon collisions, one would expect, on the basis of kinematical arguments, that their yield would decrease rapidly at the larger laboratory angles. The large observed yield suggests strongly that the production of these particles involves cooperative phenomena involving several nucleons of the target nucleus. We have studied the momentum distributions from ~1 to 3 GeV/c of these particles (at 30°) from various target nuclei. The main subject of this paper is a report of these measurements and a discussion of the results in terms of existing models.

II. EXPERIMENTAL TECHNIQUE AND RESULTS

1. Counter Arrangements and Electronics

A schematic diagram of the beam layout is given in Fig. 1. The beam of secondary particles emerging from the internal target at 30° from the AGS beam passed through a hole (~6-in.×8-in. cross section) in the main machine shielding wall. Thirty-eight feet from the target the beam passed through a lead collimator 30 in. long with a 1 in. wide and 2 in. high aperture. A 35-in. variable field magnet immediately following the collimator analyzed the particles with respect to their momentum. The two scintillation counters used to determine the time of flight were placed on a line making an 8° angle with the collimated beam. The back counter position was fixed at ~33 ft from the center of the bending magnet. The forward counter position was varied from 6 to 20 ft from the back counter according to the desired resolution.

The first scintillator was $\frac{1}{2}$ in.×$\frac{1}{2}$ in.×$\frac{1}{4}$ in. Pilot B mounted directly on one of its smaller surfaces to an Amperex 56 AVP photomultiplier placed perpendicular

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