Quantum Fluctuations and Noise in Parametric Processes. II

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(Received 14 June 1962)

In this paper we consider further the quantum-statistical properties of radiation in nondegenerate parametric-type amplifiers. In particular we find moment-generating functions and probability distribution functions for the field variables at the output of an amplifier for various input conditions. We find that a classical description of the input fields and of the amplification process is completely valid provided we take correctly into account the response of the amplifier to the input zero-point noise fields. This result is valid for inputs of arbitrarily small power.

I. INTRODUCTION

IN a previous paper (I)[1] the effects of quantum fluctuations on the sensitivity of parametric amplifiers were studied. The equations of motion for the Heisenberg operators representing the electromagnetic field were found, and from their solution, information about the quantum fluctuations in the amplifier output was obtained under various input conditions. It was shown that the parametric amplifier constitutes an ideal amplifier, in the sense that measurement of the amplitude and phase of its output gives as good a measure of the amplitude and phase of its input as is allowed by the uncertainty principle.

In order to understand more completely the nature of the inherent quantum noise in such amplifiers, it seemed important to find the complete probability distributions for the fields at the output. In spite of the considerable literature on quantum noise in linear amplifiers,[2] the probability distributions for the fields have, to the authors knowledge, never been found. As it turns out, our work allows a valuable comparison to be made between the quantum and the classical theories.

II. THE MODEL

In this analysis we use the same model for the parametric amplifier as was used in I, i.e., we consider radiation in a lossless cavity resonant at a signal frequency ω_1 and an idler frequency ω_2. The dielectric constant is modulated by a pump field at frequency $\omega=\omega_1+\omega_2$ which causes the signal and idler modes to be coupled. The Hamiltonian for the system[3] is taken as

$$H=\hbar\omega_1 a_1^\dagger(t)a_1(t)+\hbar\omega_2 a_2^\dagger(t)a_2(t) -\hbar\kappa[a_1^\dagger(t)a_2^\dagger(t)e^{-i(\omega t+\varphi)}+a_1(t)a_2(t)e^{i(\omega t+\varphi)}], \quad (1)$$

where ω_1 and ω_2 are the frequencies of the signal and idler modes, respectively, and $a_i^\dagger(t)$ and $a_i(t)$ are the photon creation and annihilation operators, respectively, for the ω_i mode ($i=1,2$). Throughout this paper we use Heisenberg's form of the equations of motion; thus the operators are time dependent. The coupling constant κ is assumed to be small compared to both ω_1 and ω_2. The angular frequency of the pump wave is ω and is equal to $\omega_1+\omega_2$. The phase of the pump wave is specified by φ.

The operators $a_i(t)$ and $a_i^\dagger(t)$ always satisfy the commutation relations

$$[a_i(t),a_j^\dagger(t)]=\delta_{ij};\quad [a_i(t),a_j(t)] =[a_i^\dagger(t),a_j^\dagger(t)]=0. \quad (2)$$

The operators representing the electric and magnetic fields in mode i are given, respectively, by

$$p_i(t)=i(\hbar\omega_i/2)^{1/2}[a_i^\dagger(t)-a_i(t)]$$
and
$$q_i(t)=(\hbar/2\omega_i)^{1/2}[a_i^\dagger(t)+a_i(t)]. \quad (3)$$

Note that these are canonically conjugate variables, in that they satisfy the commutator

$$[q_i(t),p_i(t)]=i\hbar. \quad (4)$$

Also we note that the operator representing the number of photons in mode i is

$$a_i^\dagger(t)a_i(t)=(1/2\hbar\omega_1)[p_i^2(t)+\omega_i^2q_i^2(t)-\hbar\omega_i]. \quad (5)$$

The solutions to the equations of motion for the operators pertaining to the signal mode were found in I; they are

$$p_1(t)=A_p^*(t)a_1^\dagger+A_p(t)a_1+B_p^*(t)a_2^\dagger+B_p(t)a_2,$$
$$q_1(t)=A_q^*(t)a_1^\dagger+A_q(t)a_1+B_q^*(t)a_2^\dagger+B_q(t)a_2, \quad (6)$$

where

$$A_p(t)=-i\omega_1A_q(t)=-i(\hbar\omega_1K/2)^{1/2}e^{-i\omega_1t},$$
$$B_p(t)=i\omega_1B_q(t)=[\hbar\omega_1(K-1)/2]^{1/2}e^{i(\omega_1t+\varphi)}; \quad (7)$$

here $K=\cosh^2\kappa t$ is the power gain of mode 1 of the amplifier, and where the operators on the right sides of (6) are the operators at $t=0$, i.e., we have written $a_1(0)$ as a_1, etc., for convenience. At $t=0$ the Heisenberg operators are identical with the corresponding Schrödinger operators, so the a_i may be considered alternatively as the Schrödinger operators for the fields.

[1] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1646 (1961).

[2] See, e.g., K. Shimoda, H. Takahasi, and C. H. Townes, J. Phys. Soc. Japan **12**, 686 (1957).

[3] A valid objection has been raised to the "derivation" of the Hamiltonian in I where it was implied that Eq. (15) which gives the interaction Hamiltonian, was exact. In fact, its validity depends on the assumption that the coupling κ is sufficiently small that the fractional growth of the wave per cycle is small compared to unity.

III. THE CHARACTERISTIC FUNCTION

We are interested in the statistical characteristics of the fields at the amplifier output, and we have at hand the solutions to the Heisenberg equations of motion. We will deal only with the output of mode 1; of course, a corresponding analysis holds for mode 2.

Consider, then, the properties of the quantum characteristic functions[4]

$$\begin{aligned} C_p(\xi,t) &\equiv \mathrm{Trace}\{\rho e^{i\xi p_1(t)}\} = \langle\psi|e^{i\xi p_1(t)}|\psi\rangle, \\ C_q(\xi,t) &\equiv \mathrm{Trace}\{\rho e^{i\xi q_1(t)}\} = \langle\psi|e^{i\xi q_1(t)}|\psi\rangle \end{aligned} \quad (8)$$

where ρ is the normalized density matrix (a constant in the Heisenberg picture) which represents the state of the fields, and $|\psi\rangle$ the corresponding state vector (we use Dirac's notation throughout). C_p and C_q are thus the expectation or average values of the operators $\exp[i\xi p_1(t)]$ and $\exp[i\xi q_1(t)]$, respectively. ξ is an arbitrary real constant.

It may be easily seen that C_p and C_q are generating functions for the moments of the distributions of $p_1(t)$ and $q_1(t)$, respectively. The rth moment of $p_1(t)$ or $q_1(t)$ is obtained from the characteristic functions by differentiation, i.e.,

$$\begin{aligned} \langle p_1^r(t)\rangle &= [\partial^r C_p(t)/\partial(i\xi)^r]_{\xi=0}, \\ \langle q_1^r(t)\rangle &= [\partial^r C_q(t)/\partial(i\xi)^r]_{\xi=0}. \end{aligned} \quad (9)$$

Here we have used the common simplified notation $\langle\ \rangle$ for the expectation value of the enclosed operator.

Further insight into the value of the quantum characteristic functions is obtained if we take the indicated traces in a representation in which the pertinent field operator is instantaneously diagonal. For example, let us expand C_p in a representation in which $p_1(t)$ is diagonal. The representative kets of the expansion are labeled $|p_1'(t)\rangle$, and we recall that

$$f(p_1(t))|p_1'(t)\rangle = |p_1'(t)\rangle f(p_1'(t)),$$

i.e., that the operation of any function f of $p_1(t)$ on the ket $|p_1'(t)\rangle$ yields the value of the function at the point $p_1'(t)$. The expansion of the $C_p(\xi,t)$ is then

$$\begin{aligned} C_p(\xi,t) &= \int_{-\infty}^{\infty} \langle p_1'(t)|\rho e^{i\xi p_1(t)}|p_1'(t)\rangle dp_1'(t) \\ &= \int_{-\infty}^{\infty} \langle p_1'(t)|\rho|p_1'(t)\rangle e^{i\xi p_1'(t)} dp_1'(t) \end{aligned} \quad (10)$$

Now we note that

$$\langle p_1'(t)|\rho|p_1'(t)\rangle dp_1'(t)$$

is just the probability that a measurement of p_1 at time t will yield a value between $p_1'(t)$ and $p_1'(t)+dp_1'(t)$. Thus

$$\langle p_1'(t)|\rho|p_1'(t)\rangle$$

is the probability distribution function of p_1 at time t, and the characteristic function $C_p(\xi,t)$ is therefore the Fourier transform of this probability distribution function. The probability distribution function is obtained from the characteristic function by the inverse transformation,

$$\langle p_1'(t)|\rho|p_1'(t)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_p(\xi,t) e^{-i\xi p_1'(t)} d\xi. \quad (11)$$

Note that the density matrix ρ may be written in terms of the state vector $|\psi\rangle$ by the identity

$$\rho = |\psi\rangle\langle\psi|,$$

from which we note that the probability distribution function may take the alternate form

$$\langle p_1'(t)|\rho|p_1'(t)\rangle = |\langle p_1'(t)|\psi\rangle|^2. \quad (12)$$

From the above discussion we see that from the quantum characteristic function we can obtain the probability distribution functions for the field variables and also we can obtain all of their moments. Let us now go on to evaluate these functions for the problem at hand.

Before the coupling is turned on (i.e., at the amplifier input), the two modes of the amplifier are independent. The state vector $|\psi\rangle$ and the density matrix ρ which are time independent and determined from the initial conditions, are therefore, in general, separable into products of two terms, each relating to one mode alone. Thus

$$\begin{aligned} |\psi\rangle &= |\psi_1\rangle|\psi_2\rangle, \\ \rho = |\psi\rangle\langle\psi| &= \{|\psi_1\rangle\langle\psi_1|\}\{|\psi_2\rangle\langle\psi_2|\} = \rho_1\rho_2, \end{aligned}$$

where operators relating to mode 1 at $t=0$ operate only on $|\psi_1\rangle$ and commute with ρ_2, etc., and ρ_2 commutes with ρ_1. Using the solutions to the equations of motion (6) the characteristic functions may be expressed as

$$\begin{aligned} C(\xi,t) &= \langle \exp\{i\xi[A^*(t)a_1^\dagger + A(t)a_1 + B^*(t)a_2^\dagger + B(t)a_2]\}\rangle \\ &= \langle\psi_1|\exp\{i\xi[A^*(t)a_1^\dagger + A(t)a_1]\}|\psi_1\rangle \\ &\qquad \times\langle\psi_2|\exp\{i\xi[B^*(t)a_2^\dagger + B(t)a_2]\}|\psi_2\rangle \\ &= \mathrm{Trace}[\rho_1 \exp\{i\xi[A^*(t)a_1^\dagger + A(t)a_1]\}] \\ &\qquad \times\mathrm{Trace}[\rho_2 \exp\{i\xi[B^*(t)a_2^\dagger + B(t)a_2]\}], \end{aligned} \quad (13)$$

where A and B are given by (7) for the electric and magnetic fields as the case may be. To obtain the last two expressions in (13) we have used the fact that a_1 and $a_1^\dagger$ commute with a_2 and $a_2^\dagger$. By making use of the solutions to the equations of motion, we have thus been able to separate the characteristic function into a product of two terms, each pertaining to only one of the modes.

Further computations are considerably simplified if we write the operators of (13) in normal product form,

[4] Quantum characteristic functions have been applied in the past to situations involving thermal equilibrium. See, for example, M. Born and K. Sarginson, Proc. Roy. Soc. (London) **A179**, 69 (1941/42); A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. 1, p. 448.

wherein all of the creation operators occur to the left of their corresponding annihilation operators. This may be done using the following identity,[5] which results from the commutation relations (2):

$$e^{i\xi(\alpha^* a_i^\dagger+\alpha a_i)}=e^{-\frac{1}{2}\xi^2|\alpha|^2}e^{i\xi\alpha^* a_i^\dagger}e^{i\xi\alpha a_i}, \tag{14}$$

where α is any complex number. Utilizing (14), the characteristic function may be expressed as

$$C(\xi,t)=\exp\{-\tfrac{1}{2}\xi^2[|A|^2+|B|^2]\}\langle\psi_1|e^{i\xi A^* a_1^\dagger}e^{i\xi A a_1}|\psi_1\rangle \times\langle\psi_2|e^{i\xi B^* a_2^\dagger}e^{i\xi B a_2}|\psi_2\rangle. \tag{15}$$

Equation (15) is our basic result, and it is as far as we can go without specifying the input conditions. However, since all of the operators in (15) are those for $t=0$, and A and B are known functions of time, it is clear that if we are given the input conditions we can immediately evaluate the characteristic functions for the output fields. Let us now proceed to examine various cases.

IV. NO INPUT

For the case of no input, the wave function for each mode is simply the vacuum state, i.e.,

$$|\psi_1\rangle=|0_1\rangle; \qquad |\psi_2\rangle=|0_2\rangle.$$

Now any power of an annihilation operator operating on the vacuum state gives zero, whence

$$e^{i\xi\alpha a_i}|0_i\rangle=(1+i\xi\alpha a_i+\cdots)|0_i\rangle =|0_i\rangle.$$

The conjugate equation is

$$\langle 0_i|e^{i\xi\alpha^* a_i^\dagger}=\langle 0_i|.$$

Since $\langle 0_i|0_i\rangle=1$, we have immediately from (15) the result for this case:

$$C(\xi,t)=\exp\{-\tfrac{1}{2}\xi^2[|A|^2+|B|^2]\}. \tag{16}$$

Putting in the values of A and B for C_p and C_q, respectively, we have

$$\begin{aligned}C_p(\xi,t)&=\exp\{-\tfrac{1}{2}\xi^2\hbar\omega_1(K-\tfrac{1}{2})\},\\ C_q(\xi,t)&=\exp\{-\tfrac{1}{2}\xi^2(\hbar/\omega_1)(K-\tfrac{1}{2})\}.\end{aligned} \tag{17}$$

These characteristic functions are typical of Gaussian noise.[6] Taking the Fourier transforms to get the distribution functions, we have

$$\begin{aligned}\langle q_1'(t)|\rho|q_1'(t)\rangle&=\frac{1}{(2\pi)^{1/2}(\Delta q_1)}\exp\left(-\frac{(q_1')^2}{2(\Delta q_1)^2}\right),\\ \langle p_1'(t)|\rho|p_1'(t)\rangle&=\frac{1}{(2\pi)^{1/2}(\Delta p_1)}\exp\left(-\frac{(p_1')^2}{2(\Delta p_1)^2}\right),\end{aligned} \tag{18}$$

where

$$\begin{aligned}(\Delta q_1)^2&=\langle q_1^2(t)\rangle-\langle q_1(t)\rangle^2=(\hbar/\omega_1)(K-\tfrac{1}{2}),\\ (\Delta p_1)^2&=\langle p_1^2(t)\rangle-\langle p_1(t)\rangle^2=\hbar\omega_1(K-\tfrac{1}{2}).\end{aligned} \tag{19}$$

Note that $\langle q_1(t)\rangle=\langle p_1(t)\rangle=0$. It is of interest to compute the expected number of photons in the output. This is, from (5), and (19),

$$\langle a_1^\dagger(t)a_1(t)\rangle=(1/2\hbar\omega_1)\langle p_1^2(t)+\omega_1^2q_1^2(t)\rangle -\tfrac{1}{2}=K-1. \tag{20}$$

Thus with no input to the amplifier there is an output of $K-1$ photons on the average, and the output fields always have Gaussian probability distributions. Note that if the gain is unity, there are no photons out and yet there are still statistical field fluctuations, which may of course be identified with the zero-point fields.

If the gain is large the output noise corresponds to an effective input noise of one photon and may be considered as the response of the amplifier to the zero-point input fields; as we shall see it is appropriate to consider that an effective $\frac{1}{2}$ photon enters each mode.

V. NOISE INPUTS TO BOTH CHANNELS

For the case of noise inputs it is convenient to use the formalism of the density matrix. The density matrix for mode i for a noise input is

$$\rho_i=[1-\tau_i^{-1}]\tau_i^{-a_i^\dagger a_i}, \tag{21}$$

where $\tau_i=\exp(\hbar\omega_i/kT_i)$, T_i being the blackbody temperature corresponding to the noise. Note that the probability of finding n_i photons at the input to mode i is

$$P(n_i)=\langle n_i|\rho_i|n_i\rangle=[1-\tau_i^{-1}]\tau_i^{-n_i},$$

where $|n_i\rangle$ is the eigenket of the number operator $a_i^\dagger a_i$ having the eigenvalue n_i. This is the exponential photon distribution which we know is characteristic of Gaussian noise. The average number of input photons in mode i is

$$\bar{n}_{i0}=\sum n_iP(n_i)=(\tau_i-1)^{-1}. \tag{22}$$

Let us now evaluate the characteristic function. From (15), written in terms of the density matrices, we have

$$C(\xi,t)=\exp\{-\tfrac{1}{2}|\xi|^2(|A|^2+|B|^2)\} \times\mathrm{Trace}\{\rho_1e^{i\xi A^* a_1^\dagger}e^{i\xi A a_1}\}\times\mathrm{Trace}\{\rho_2e^{i\xi B^* a_2^\dagger}e^{i\xi B a_2}\}.$$

Consider the term referring to mode 1; making use of (21), it is

$$Q_1\equiv(1-\tau_1^{-1})\,\mathrm{Trace}\{\tau_1^{-a_1^\dagger a_1}\exp(i\xi A^* a_1^\dagger) \times\exp(i\xi A a_1)\}.$$

This expression is shown in the appendix to evaluate to

$$Q_1=\exp[-\xi^2|A|^2\bar{n}_{10}].$$

A similar expression is obtained for the term pertaining to mode 2, and so we have for the characteristic func-

[5] See, for example, Eq. (39) of R. J. Glauber, Phys. Rev. 84, 395 (1951).
[6] W. R. Bennett, Proc. I. R. E. 44, 609 (1956).

tion:

$$C(\xi,t)=\exp\{-\xi^2[|A|^2(\bar{n}_{10}+\tfrac{1}{2})+|B|^2(\bar{n}_{20}+\tfrac{1}{2})]\}. \quad (23)$$

As in the case of no input, this characteristic function represents a Gaussian noise output, and in this form we see clearly evidenced the fact that the quantum or spontaneous emission noise appears as though an additional $\frac{1}{2}$ photon entered each mode. Putting in the values for A and B, we have the following results:

$$\langle q_1(t)\rangle=\langle p_1(t)\rangle=0,$$

$$\langle q_1^2(t)\rangle=\frac{\hbar}{\omega_1}[K(\bar{n}_{10}+\tfrac{1}{2})+(K-1)(\bar{n}_{20}+\tfrac{1}{2})],$$

$$\langle p_1^2(t)\rangle=\hbar\omega_1[K(\bar{n}_{10}+\tfrac{1}{2})+(K-1)(\bar{n}_{20}+\tfrac{1}{2})],$$

$$\langle n_1(t)\rangle=\langle a_1^\dagger(t)a_1(t)\rangle=K(\bar{n}_{10}+\tfrac{1}{2})+(K-1)(\bar{n}_{20}+\tfrac{1}{2})-\tfrac{1}{2}.$$

If the gain K is large compared to unity, we have

$$\langle n_1(t)\rangle\approx K(\bar{n}_{10}+\bar{n}_{20}+1).$$

From this and from the Gaussian distribution of the output noise, it is clear that the amplification of the Gaussian noise input may be considered to have proceeded in a perfectly classical manner provided that we include the extra effective input photon to account for the response of the amplifier to the input zero-point fields. This result is valid for *arbitrarily small* input noise.

VI. MINIMUM UNCERTAINTY SIGNALS ENTER EACH MODE[7]

The wavefunction which represents a minimum uncertainty wave packet entering each mode was found in I. We have again

$$|\psi\rangle=|\psi_1\rangle|\psi_2\rangle,$$

where

$$|\psi_i\rangle=\exp(-\tfrac{1}{2}\bar{n}_{i0})\exp(x_i^*a_i^\dagger)|0_i\rangle. \quad (25)$$

In this expression $\bar{n}_{i0}$ is the average number of photons at the input of mode i, $|0_i\rangle$ is the vacuum state for mode i and

$$x_i=(\bar{n}_{i0})^{1/2}e^{i\varphi_i},$$

where φ_i defines the phase of the input to mode i.

Consider now the characteristic function of the output, Eq. (15). To evaluate it, we make use of the following identity[8]:

$$f(a_i)e^{\alpha^*a_i\dagger}|0_i\rangle=f(\alpha^*)e^{\alpha^*a_i\dagger}|0_i\rangle, \quad (26)$$

where f may be any function of the operator a_i, and α^* may be any complex constant. From (26) and (25) it follows that

$$e^{i\xi Aa_1}|\psi_1\rangle=e^{i\xi Ax_1^*}|\psi_1\rangle,$$
$$e^{i\xi Ba_2}|\psi_2\rangle=e^{i\xi Bx_2^*}|\psi_2\rangle.$$

With these equations and their conjugates, and remembering that $\langle\psi_1|\psi_1\rangle=\langle\psi_2|\psi_2\rangle=1$, we find that

$$C(\xi,t)=\exp\{-\tfrac{1}{2}\xi^2[|A|^2+|B|^2]\}e^{i\xi(A^*x_1+Ax_1^*)}\times e^{i\xi(B^*x_2+Bx_2^*)}.$$

Before discussing this characteristic function let us find the first few moments and the probability distribution functions. From (9), and putting in the appropriate values for A and B from (6), we have

$$\langle q_1(t)\rangle=A_q^*x_1+A_qx_1^*+B_q^*x_2+B_qx_2^*$$
$$=(2\hbar/\omega_1)^{1/2}\{(\bar{n}_{10}K)^{1/2}\cos(\omega_1t+\varphi_1)+[\bar{n}_{20}(K-1)]^{1/2}\sin(\omega_1t+\varphi-\varphi_2)\}; \quad (27)$$

likewise

$$\langle p_1(t)\rangle=(2\hbar\omega_1)^{1/2}\{-(\bar{n}_{10}K)^{1/2}\sin(\omega_1t+\varphi_1)+[\bar{n}_{20}(K-1)]^{1/2}\cos(\omega_1t+\varphi-\varphi_2)\}.$$

For the variances, we find that

$$(\Delta q_1)^2\equiv\langle q_1^2(t)\rangle-\langle q_1(t)\rangle^2=[|A_q|^2+|B_q|^2]=(\hbar/\omega_1)(K-\tfrac{1}{2}); \quad (28)$$

likewise

$$(\Delta p_1)^2\equiv\langle p_1^2(t)\rangle-\langle p_1(t)\rangle^2=\hbar\omega_1(K-\tfrac{1}{2}).$$

In terms of these first two moments, we see that we can express the characteristic functions as

$$C_q(\xi,t)=e^{-\frac{1}{2}\xi^2(\Delta q_1)^2}e^{i\xi\langle q_1(t)\rangle},$$
$$C_p(\xi,t)=e^{-\frac{1}{2}\xi^2(\Delta p_1)^2}e^{i\xi\langle p_1(t)\rangle}. \quad (29)$$

These characteristic functions represent fields which have Gaussian distributions about their mean values, and it is noteworthy that the variances (and, thus, the noise) have the same values they had when the input was zero. This result is, in fact, not unreasonable, since the statistical field fluctuations inherent in a minimum-uncertainty wave packet may be attributed completely to the zero-point field. Note from (27) that the expectation values of the field operators behave just as do the corresponding classical fields. Thus, again we may consider the amplification to have proceeded classically if we consider that the minimum-uncertainty wave packet corresponds to a pure sinusoid of exactly defined amplitude and phase accompanied by the effective half photon of zero-point noise.

VII. SUMMARY AND DISCUSSION

Starting from the Heisenberg equations of motion for the field operators in a linear parametric amplifier, we have investigated the statistical characteristics of the output fields for a variety of input conditions. In all cases the amplification proceeds in a perfectly classical manner if we (1) consider that a minimum-uncertainty wave packet corresponds to an ideally monochromatic sinusoid with an exactly defined phase *and* (2) add to whatever real input noise energy (i.e., noise energy measurable by a power-sensitive device such as a photocell) enters each mode an effective $\frac{1}{2}$ photon to take into

[7] I. R. Senitzky, Phys. Rev. **95**, 904 (1954).
[8] E. Fermi, *Quantum Mechanics* (University of Chicago Press, Chicago, Illinois, 1961), p. 31.

account the response of the amplifier to the zero-point noise field. Also, the fields corresponding to real input noise may be assumed to maintain their *classical* statistical properties no matter how small the real input noise energy becomes. To say the same thing in another way, a classical description of the input fields and of the amplification process is completely correct if we always take care to add zero-point fields corresponding to an energy of $\frac{1}{2}$ photon to each input of the amplifier. The zero-point fields have the statistical properties of additive Gaussian noise.

Our results apply strictly only to the amplification of a single mode of the radiation field for each frequency, since our model involves a cavity resonator with one mode at each frequency. However, it is quite desirable to extend the theory to the more usual case of continuous amplification of varying signals received from a transmission line of some sort.[9] To do this we note that in a transmission line, in a bandwidth B, the radiation field received by the amplifier in each second may be resolved into just B orthogonal modes, and our results should apply for each such mode. Thus the zero-point energy of $\frac{1}{2}$ photon per mode translates into a zero-point noise power of $\frac{1}{2}h\nu B$ in the transmission line. Again, a classical description of the input fields and of the amplification process is valid if we take care to include this zero-point noise power.

APPENDIX

Let us expand Q_1 in the representation in which the number operator $a_1{}^\dagger a_1$ is diagonal. This gives

$$Q_1=(1-\tau_1{}^{-1})\sum_{n_1,m_1}\tau_1{}^{-n_1}\langle n_1|\exp(i\xi A^*a_1{}^\dagger)|m_1\rangle \times\langle m_1|\exp(i\xi Aa)|n_1\rangle.$$

Utilizing the well-known property that

$$(a_1{}^\dagger)^k|m_1\rangle=\left(\frac{(m_1+k)!}{m_1!}\right)^{1/2}|m_1+k\rangle,$$

and remembering that $\langle n_1|m_1+k\rangle=\delta_{n_1,m_1+k}$, we find

$$\langle n_1|\exp(i\xi A^*a_1{}^\dagger)|m_1\rangle=\frac{(i\xi A^*)^{n_1-m_1}}{(n_1-m_1)!}\left(\frac{n_1!}{m_1!}\right)^{1/2};$$

similarly

$$\langle m_1|\exp(i\xi Aa_1)|n_1\rangle=\frac{(i\xi A)^{n_1-m_1}}{(n_1-m_1)!}\left(\frac{n_1!}{m_1!}\right)^{1/2};$$

and so

$$Q_1=(1-\tau_1{}^{-1})\sum_{n_1,m_1}\tau_1{}^{-n_1}\frac{(-\xi^2|A|^2)^{n_1-m_1}n_1!}{(n_1-m_1)!^2m_1!}.$$

The sums may be performed easily if we change variables. Let $q_1=n_1-m_1$; doing this we have

$$Q_1=(1-\tau_1{}^{-1})\sum_{q_1}\tau_1{}^{-q_1}\frac{(-\xi^2|A|^2)^{q_1}}{q_1!}\sum_{m_1}\tau_1{}^{-m_1}\frac{(m_1+q_1)!}{q_1!m_1!}.$$

The sum over m_1 yields $(1-\tau_1{}^{-1})^{-(q_1+1)}$ and so

$$Q_1=\sum_{q_1}\frac{1}{q_1!}\left[-\frac{\xi^2|A|^2}{\tau_1-1}\right]^{q_1}=\exp\left[-\frac{\xi^2|A|^2}{\tau_1-1}\right]=\exp[-\xi^2|A|^2\bar{n}_{10}],$$

where we have made use of (22).

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[9] J. P. Gordon, in *Advances in Quantum Electronics*, edited by J. R. Singer (Columbia University Press, New York, 1961), p. 509.