leptonic isobaric spin, leptonic strangeness, and leptonic number through the operators $J_3'^+$, $J_3'^-$, and $S_3=J_3^++J_3^-$, the fundamental leptons corresponding to $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$, namely $e^-$, $\nu_e$, $\mu^-$, and $\nu_\mu$.

These proposals of Yukawa present the following advantages:

(a) They explain directly the separate conservation of baryon and lepton numbers.

(b) They establish a simple and beautiful correspondence between the four "fundamental" baryon states of $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$, namely $n$, $p$, $\Lambda^0$, and $V^+$, and the four fundamental lepton states of $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$, namely $e^-$, $\nu_e$, $\mu^-$, and $\nu_\mu$. This symmetry can be utilized as the starting point of a modified version of the Sakata model in which one utilizes four basic particles instead of three. In our case, as seen in II, all higher baryon states of $D(\frac{1}{2},1)$ and $D(1,\frac{1}{2})$ can be obtained as products of the eigenfunctions of $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$.

(c) This correspondence is strengthened by the recent discovery of a second neutrino (Brookhaven), and the existence shown in Berkeley, of a 1480-MeV backward-scattering resonance in $K^-+p=K^0+n$; since, as Yukawa and one of us (J.-P. V.) have remarked, the graph of Fig. 7 evidently implies backward scattering as a result of $V^+$ or $Y^+$ exchange.

(d) They lead, following step by step (with the new group $G$) the work of Ohnuki,[24] Ne'eman,[25] and Gell-Mann,[26] to an "$n$-fold way" which also introduces the $\omega$, $\rho$, $K^*$ vector mesons. Such bosons could also have been predicted directly from the fusion scheme of Sec. II, since, with every representation $D(l^+,l^-)$ one can associate spin 0 or spin 1.

The corresponding strong- and weak-interaction theories will be discussed in subsequent papers.[22]

[24] M. Ikeda, S. Ogawa, and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **22**, 715 (1959).
[25] Y. Ne'eman, Nucl. Phys. **26**, 222 and 230 (1961).
[26] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

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# Translational Inertial Spin Effect

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The following results are shown: (a) Contrary to widespread belief, two energy-momentum tensors $T^{ij}$ and $\Theta^{ij}$ with a divergenceless difference are not necessarily physically equivalent; in fact, they will *not* be equivalent if the flux $\iiint(T^{i\alpha}-\Theta^{i\alpha})ds_\alpha dt$ through the external surface of some test body between an initial and a final state is nonzero. (b) It follows necessarily from basic postulates of the Dirac one-electron theory that Tetrode's asymmetrical energy-momentum tensor is physically the good one, and that, in the circumstances mentioned above, use of the symmetrized $\Theta^{ij}=(T^{ij}+T^{ji})/2$ tensor would yield a wrong result for the variation of the energy-momentum between states 1 and 2. (c) This being so, a macroscopic experiment based on ferromagnetism or ferrimagnetism can be devised, which demonstrates these facts as a measurable "translational inertial spin effect." (d) It is highly plausible that the above predictions, based on the one-particle electron theory, would be valid in the framework of the many-particle electron theory obeying Fermi statistics (the argument is based on the so-called bound-interaction hyperquantized formalism). The last point can be verified experimentally.

## I. INTRODUCTION

Two energy-momentum tensors $T^{ij}$ and $\Theta^{ij}$ ($i$, $j$, $k$, $l=1, 2, 3, 4$; $x^1=x$, $x^2=y$, $x^3=z$, $x^4=ict$) are said to be *equivalent* if their difference is divergenceless:

$$\partial_j(T^{ij}-\Theta^{ij})=0. \tag{1}$$

This entails that the three-fold integral[1]

$$\iiint(T^{ij}-\Theta^{ij})du_j \tag{2}$$

is zero when taken over any closed domain, but not zero when taken over an open domain ($ic\epsilon^{ijkl}du_l=[dx^idx^jdx^k]$, 3-dimensional volume element; $\epsilon^{ijkl}$ is Levi-Civita's indicator).

One principal purpose of this note is to show how this remark yields the principle of physical experiments where mathematically equivalent energy-momentum tensors will not have physically equivalent behavior, so that (in the case we will consider) one of them may be selected as being, physically, "the good one."

The reason why such a fact has often been overlooked is that in a fairly large class of physical situations the values of the $T^{ij}$ tensors drop down at spatial infinity at a rate such that the integral (2), taken over any time-like domain at spatial infinity, is zero. When this is the case, the value of the integral (2) taken over any space-like domain[1] $S$ extending to infinity will be independent of

[1] To avoid confusion with the spin density $\sigma^i$, Schwinger's notations $d\sigma_j$ and $\sigma$ are discarded in favor of $du_j$ and $S$.

$S$, or conservative; thus, considering any two such integrals taken over nonintersecting $S_1$ and $S_2$ domains, the relation

$$\Delta P^i = \iiint_{S_2-S_1} T^{ij} du_j = \iiint_{S_2-S_1} \Theta^{ij} du_j \qquad (3)$$

will be true. In other words, when the integral (2) taken over any time-like domain at spatial infinity is zero, and when the total energy momentum is calculated over a space-like domain extending to infinity, then the two mathematically equivalent energy-momentum tensors are also physically equivalent, as yielding the same value for the variation of the total energy momentum.

But this will no longer be true if (I) the significant space-like domain is finite, as is the case with, say, a finite piece of matter; and (II) the integral (2) taken over the time-like wall generated in space-time by the contour of the piece of matter differs from zero:

$$Q^i = \iiint_{S_3} (T^{ij} - \Theta^{ij}) du_j \neq 0. \qquad (4)$$

Then (see Fig. 1), according to the general property of equivalent energy-momentum tensors, Eq. (3) is replaced by

$$-Q^i = \iiint_{S_2-S_1} (T^{ij} - \Theta^{ij}) du_j \neq 0, \qquad (5)$$

or equivalently

$$\Delta P^i = \iiint_{S_2-S_1} T^{ij} du_j = \iiint_{S_2-S_1} \Theta^{ij} du_j - Q^i. \qquad (6)$$

As the physical variation in energy momentum between states 1 and 2 is measurable, this amounts to say that in the case under consideration the two "mathematically equivalent" $T^{ij}$ and $\Theta^{ij}$ tensors are not physically equivalent; eventually, one of them may be selected as "the good one." If, in Eq. (6), $\Delta P^i$ represents the true, measurable, variation in energy momentum between states 1 and 2, $T^{ij}$ is a "good" and $\Theta^{ij}$ a "wrong" energy-momentum tensor.

It is clear that the cause of the difference of behavior between $T^{ij}$ and $\Theta^{ij}$ is the energy-momentum flux through the time-like $S_3$ domain, i.e., the integrated flux on a time interval through the external surface of the body. We will come back to this point later.

The case of interest for us is that of the family of energy-momentum tensors defined, in the Dirac electron theory, from Tetrode's[2] asymmetrical tensor

$$T^{ij} = -\tfrac{1}{2} c\hbar \bar{\psi} [\partial^i] \gamma^j \psi + ieA^i \bar{\psi} \gamma^j \psi \qquad (7)$$

[2] H. Tetrode, Z. Physik **48**, 852 (1928).

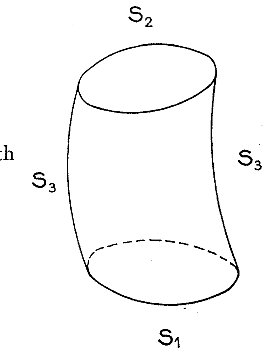


Fig. 1. Time-like world tube with two space-like cross sections.

by the linear combination

$$\Theta^{ij} = aT^{ij} + bT^{ji}, \quad a+b=1; \qquad (8)$$

where $h=2\pi\hbar$ is Planck's constant; $-e$ is the electron charge in emu; the $\gamma^i$ are spin matrices; $\bar{\psi}=\psi^\dagger\gamma^4$;

$$[\partial^i] = \partial^i - \rho^i, \qquad (9)$$

in the Jauch-Rohrlich[3] notation; and $A^i$ is the electromagnetic potential.

Among the $\Theta^{ij}$'s, the symmetrized tensor

$$\Theta_0^{ij} = \tfrac{1}{2}(T^{ij} + T^{ji}) \qquad (10)$$

has often been recommended as physically "the good one" (by Tetrode[2] and by Pauli[4]). On the other hand, the author[5] and Weyssenhof[6] have produced physical arguments in favor of using the asymmetrical Tetrode tensor (7).

The idea of an experimental test was presented earlier.[7]

According to Tetrode's well-known formula,

$$T^{ji} - T^{ij} = \partial_k \sigma^{ijk} = \tfrac{1}{2} ic \epsilon^{ijkl} (\partial_l \sigma_k - \partial_k \sigma_l), \qquad (11)$$

which is a consequence of the Dirac equation, the $\Theta^{ij}$ family is, of course, a class of mathematically equivalent energy-momentum tensors; $\sigma^{ijk} = \epsilon^{ijkl}\sigma_l$ denotes the Dirac spin density,

$$\sigma^{ijk} = c\hbar \bar{\psi} \gamma^{ijk} \psi; \qquad (12)$$

$\gamma^{ij\cdots} = \gamma^i\gamma^j\cdots$, if $i \neq j \neq \cdots$, and is zero if two or more indexes are equal.

It will be shown in Sec. II that, $T^{ij}$ being Tetrode's tensor defined by (7) and $S_3$ the time-like wall generated by the external surface of a ferromagnetic or ferrimagnetic solid, physical circumstances can be defined

[3] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), p. 53.

[4] W. Pauli, in *Handbuch der Physik*, edited by S. Flügge (Verlag Julius Springer, Berlin, 1933), Vol. 24, pp. 3, 235.

[5] O. Costa de Beauregard, Compt. Rend. **214**, 904 (1942); J. math. pures et appl. **22**, 131 (1943).

[6] J. Weyssenhof and A. Raabe, Acta Phys. Polon. **9**, 7 (1947).

[7] O. Costa de Beauregard, Cahiers Phys. **99**, 407 (1958) and **105**, 200 (1959).

such that

$$\iiint_{S_3}(T^{ji}-T^{ij})du_j=\iiint_{S_3}\partial_k\sigma^{ijk}du_j\neq 0; \quad (13)$$

thus the various $\Theta^{ij}$ tensors, though mathematically equivalent, are not physically equivalent in such circumstances.[8]

More specifically, (a) we shall deduce from the basic postulates of the Dirac electron theory that Tetrode's tensor is the real physical energy-momentum tensor, and that the finite energy momentum must be calculated according to Eqs. (6) and (7), with a dummy index for the $\gamma$'s; (b) this being so, we shall show that under the circumstances we shall define, the real physical energy-momentum 4-vector $P^i$ is not collinear with the kinematical 4-velocity; (c) we shall propose a macroscopic experiment based on ferromagnetism (or ferrimagnetism) where a measurable translational inertial spin effect should appear as a consequence of (b).

Of course, if the true, physical, energy-momentum tensor $T^{ij}$ is asymmetrical, then the left-hand side of Einstein's equation in general relativity,

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=\chi T_{ij},$$

must be generalized so as to become asymmetrical also.[9,10] Thus, a skew-symmetric part, yet unexplored, should exist in the gravitational potential, as generated by the very existence of spin. Our idea, which we intend to develop in a subsequent paper, is to explain the energy-momentum flux (13) as radiated in the skew-symmetrical gravitational field.

## II. DEDUCTION OF THE EFFECT IN THE FRAME OF THE DIRAC ONE-ELECTRON THEORY

In solid-state physics, where, rigorously speaking, one is dealing with a many-electron problem subject to Fermi statistics, the one-particle theory often yields experimentally sound results, together with a clear (though simplified) insight in the phenomena. For this reason, we shall also use the one-particle electron theory in the basic deduction of the translational inertial spin effect, postponing to a later section (IV) what can be said from the more rigorous point of view of hyperquantized theory.[11]

[8] A quite analogous situation occurs with the Dirac and the Gordon 4-currents, $j^i$ and $k^i$, the difference of which is divergenceless: the two integrals $\iiint j^i du_i$ and $\iiint k^i du_i$ are not necessarily equal when taken over on open domain. In Sec. II a situation shall be considered where the physical nonequivalence of the two "mathematically equivalent" vectors $j^i$ and $k^i$ will be quite obvious—and also closely connected with the energy momentum problem.

If, following Dirac, we take $j^i$ as the true or physical probability density current, then $k^i$ will be a "wrong" one, i.e., only part of the "good" one.

[9] A. Papapetrou, Phil. Mag. **40**, 937 (1949).

[10] D. W. Sciama, Proc. Cambridge Phil. Soc. **54**, 72 (1958).

[11] As we are using Dirac's covariant wave equation, a covariant treatment of the many-particle theory is also needed.

The four postulates we shall use are:

(A) The probability density of the electron's spatial location is $\psi^\dagger\psi$; an equivalent statement is that the space-time electron flux is $-1/e$ times the Dirac current

$$j^l=-ie\bar{\psi}\gamma^l\psi. \quad (14)$$

(B) The mean value $R$ at time $t$ of any Hermitian operator $\mathfrak{R}$ is

$$R=\iiint_t \psi^\dagger\mathfrak{R}\psi dxdydz. \quad (15)$$

(C) The energy-momentum operator of the electron is

$$P^i=i\hbar\partial^i+eA^i=(i\hbar/2)[\partial^i]+eA^i. \quad (16)$$

(D) Standard interpretation of various densities in the Dirac theory.

From these four postulates, and through the eight following theorems, we shall deduce the existence of the new translational inertial spin effect. Boldface characters will denote 3-vectors in the ordinary $x$, $y$, $z$, space.

(1) The Dirac 4-current associated with the electron cloud inside a solid runs in space-time tangent to the tube generated by the external surface of the body.

This follows from postulate A and the conservation of the electric charge associated to the electron cloud inside the body.

Thus, the direction of the space average of the Dirac 4-current must be close to that of the body's 4-velocity. A case (among others) where the relation

$$\iiint \mathbf{j}dxdydz=0 \quad (17)$$

holds exactly is that of a body at rest in some Lorentzian frame where the whole physical situation has an axis of symmetry in the ordinary $x$, $y$, $z$, space; this will be true in the experiment discussed below.

(2) If we introduce the Gordon current

$$k^l=\frac{ie}{2\kappa}\bar{\psi}[\partial^l]\psi-\frac{e^2}{\hbar\kappa}A^l\bar{\psi}\psi \quad (18)$$

($\kappa$, electron's mass term) and the electromagnetic polarization density $(\mathbf{M},\mathbf{P})$

$$m^{ij}=(ie/2\kappa)\bar{\psi}\gamma^{ij}\psi, \quad (19)$$

then, from Gordon's formula

$$j^l=k^l+\partial_k m^{lk}, \quad (20)$$

$$\mathbf{j}=\mathbf{k}+\text{curl}\mathbf{M}, \quad (21)$$

there follows, in the symmetric case considered above, the (noncovariant) formula

$$\int\int\int \mathbf{k}du = \int\int \mathbf{M}\times d\mathbf{s}, \tag{22}$$

where $du$ and $d\mathbf{s}$ denote the ordinary volume and surface elements.

(3) In the local rest frame of the Dirac current the magnetic polarization vector $\mathbf{M}$ and the spin density $\boldsymbol{\sigma}$ [defined by (12)] are collinear.[12]

This follows from one of the identities[13] arising from the algebra of $\gamma$ matrices,[14] that is,

$$\sum_{ijk}(2\bar{\psi}\gamma^{ij}\psi\ \bar{\psi}\gamma^k\psi - \bar{\psi}\gamma^{ijk}\psi\ \bar{\psi}\psi) = 0. \tag{23}$$

(4) The physical density associated by postulates $B$ and $C$ with the energy-momentum operator (16) is necessarily Tetrode's[2] asymmetrical tensor (7).

This is seen by substituting in (15) $\bar{\psi}\gamma^4$ for $\psi^\dagger$, replacing the constant-time integration by an integration over a space-like surface $S$,[1] and writing down the covariant product $\gamma^i du_i$ instead of the degenerate form $\gamma^4 du = (-i/c)\gamma^4 du_4$ ($du^i$ is the volume element 4-vector on $S$).

(5) In the case of a static solution $\psi(x) = \varphi(\mathbf{x}) \times \exp(iWt/\hbar)$, the "wrong" energy-momentum 4-vector

$$L^i = \int\int\int T^{ji}du_j \tag{24}$$

is, in each volume element, collinear with the Dirac 4-current.

This is evident in a constant-time integration: $[\partial^4]$ will be the only differential operator present, and it will yield a quantity proportional to $\bar{\psi}\gamma^i\psi$.

Thus, by virtue of theorem 1, $L^i$ may well be called the longitudinal energy-momentum.

(6) In the low-velocity limit, the true or physical energy-momentum 4-vector

$$P^i = \int\int\int T^{ij}du_j \tag{25}$$

is, in each volume element, nearly collinear with the Gordon 4-current (the angle between the two being of order $v^2/c^2$).

This is easily seen by using the so-called low-velocity representation of the $\gamma$'s, where $\gamma^4$ is diagonal with the signature $(+1, +1, -1, -1)$, and the $\psi$ has two large ($\psi_1$ and $\psi_2$) and two small ($\psi_3$ and $\psi_4$) components. The additional statement is seen, as usual, by inspection of the expressions of the components of the Dirac current (and, in our case, the use of theorem 5).

(7) According to Tetrode's formula (11), an expression for the transverse energy-momentum 4-vector

$$T^i = P^i - L^i \tag{26}$$

is

$$T^i = -\int\int\int \partial_k\sigma^{ijk}du_j = \tfrac{1}{2}\int\int \sigma^{ijk}ds_{jk}, \tag{27}$$

where the surface integral is over the contour of the volume integral domain.

Thus, in the symmetrical case considered above, the physical momentum 3-vector of a body endowed with a spin density $\boldsymbol{\sigma}$ has, in the rest frame, the nonzero value

$$\mathbf{P} = \mathbf{T} = -\int\int\int \mathrm{curl}\boldsymbol{\sigma}du = \int\int \boldsymbol{\sigma}\times d\mathbf{s}. \tag{28}$$

The compatibility of (28) with (22) through theorems (1) and (3) should be noted.

In the cases where the surface integral differs from zero, the translational inertial spin effect will follow from (28).

(8) Our last theorem will establish the connection between the contents of this section and the introductory one.

Consider (Fig. 1) an initial and a final state of the test body represented by finite nonintersecting space-like surfaces $S_1$ and $S_2$, and also the time-like wall $S_3$ generated by the contour surface of the body.

The variation, between states 1 and 2, of the true or physical energy momentum is, according to (25),

$$\Delta P^i = \int\int\int_{S_2-S_1} T^{ij}du_j, \tag{29}$$

while that of the longitudinal energy momentum is, according to (24),

$$\Delta L^i = \int\int\int_{S_2-S_1} T^{ji}du_j. \tag{30}$$

The difference between the two is, according to (26), the variation of the transverse energy momentum and it reads, according to Tetrode's formula (11),

$$\Delta T^i = \int\int\int_{S_2-S_1} (T^{ij} - T^{ji})du_j = -\int\int\int_{S_2-S_1} \partial_k\sigma^{ijk}du_j. \tag{31}$$

But, due to the identity

$$\partial_j\partial_k\sigma^{ijk} \equiv 0, \tag{32}$$

the last integral can be successively transformed as

$$\Delta T^i = \int\int\int_{S_3} \partial_k\sigma^{ijk}du_j = \tfrac{1}{2}\Delta\int\int \sigma^{ijk}ds_{jk}, \tag{33}$$

or, in a pre-relativistic fashion,

$$\Delta\mathbf{T} = \Delta\int\int \boldsymbol{\sigma}\times d\mathbf{s}. \tag{34}$$

[12] The same is true of $\boldsymbol{\sigma}$ and the electric polarization vector $\mathbf{P}$.
[13] W. Kofink, Ann. Physik **30**, 57 (1937).
[14] W. Pauli, Ann. Inst. Henri Poincaré **6**, 109 (1936).

The conclusion is that, if the expression $\int\int \boldsymbol{\sigma} \times d\mathbf{s}$ has varied between states 1 and 2, then the expressions (29) for $\Delta P^i$ and (30) for $\Delta L^i$ are not equivalent for calculating the variation of the experimentally measurable energy momentum.

It follows from the basic principles of quantum theory that the true, operational, energy momentum is the "oblique" (comprising a "transverse" component) energy momentum $P^i$ defined by (25), and not the "longitudinal" energy momentum $L^i$ defined by (24).

It should be noted that if one accepted the symmettized tensor $\Theta^{ij}$ defined by (10) as the real physical energy-momentum tensor, there would follow then not a zero effect as is usually tacitly assumed, but half the effect predicted here.

### III. THE PROPOSED EXPERIMENT

The idea is to use ferromagnetism or ferrimagnetism, as due to the electron spin, to display our translational inertial spin effect.

The essential point is to find a body shape and a magnetization procedure such that in the final, magnetized, state, the surface integrals in (22), (28), and (34) will be nonzero; the initial state is taken as unmagnetized.

We shall take (Fig. 2) as a macroscopic test body a small ferrite or iron-cobalt ring of cylindrical shape, and magnetize it to quasi-saturation[15] by means of a short current pulse in a rectilinear wire along its geometrical axis. According to classical electromagnetism, the switching on of the current will not entail the application of any force or torque on the test body.[16]

Now, the appearance, in the final state 2, of a macroscopic magnetic polarization $\mathbf{M}_2$ will imply that of a macroscopic spin density $\boldsymbol{\sigma}_2$ and thus, by virtue of formula (28), that of a "transverse momentum" $\mathbf{T}_2$. In the laboratory frame of reference the equations

$$\mathbf{P}_1 = \mathbf{P}_2 = 0, \quad (35)$$

$$\mathbf{L}_1 = 0, \quad \mathbf{L}_2 + \mathbf{T}_2 = 0, \quad (36)$$

will hold. These refer to space projections of space-time vectors; the kinematical situation in the final state is schematized in Fig. 3, from which it is clear that the words longitudinal and transverse refer to space-time, not to space vectors.

As, between states 1 and 2, the longitudinal energy momentum $L^i$ undergoes a transition of value $-T^i$ from its initial value $P^i$, the test body will recoil with the momentum $\mathbf{T}$. $\mathbf{T}$ is easily calculated as parallel to $z'z$ with the value (see Fig. 2)

$$T = 2\pi ab\sigma.$$

The mass $\mathfrak{M}$ of the ring of mean radius $r$ and specific mass $\rho$ is

$$\mathfrak{M} = 2\pi rab\rho,$$

so that the recoil velocity is

$$v = \sigma/\rho r. \quad (37)$$

Writing

$$\sigma/\rho = (\sigma/M)(M/\rho) \quad (38)$$

will exhibit a universal constant, the ratio of the electron mass to the electron charge:

$$\sigma/M = \text{electron spin/electron magnetic moment} = m/e, \quad (39)$$

and the specific magnetic polarization strength[17]

$$\sigma^* = M/\rho. \quad (40)$$

Equation (37) can thus be rewritten as

$$v = (m/e)(\sigma^*/r). \quad (41)$$

As $m/e = 5.7 \times 10^{-8}$ emu, one finds, for $r = 0.1$ cm and $\sigma^* = 70$ (manganese ferrite case) or 210 (iron-cobalt case),

$$v \simeq 3.9 \times 10^{-5} \quad \text{or} \quad 1.17 \times 10^{-4} \text{ cm/sec},$$

respectively. The sign of the effect is predicted to be opposite to that of the Einstein-de Haas effect, in the sense that while the Einstein-de Haas rotation follows the (conventional) exciting helix current, our translation should be opposite to the direction of the (conventional) exciting linear current.

Reversing the direction of the excitation current will double the effect, and mechanical resonance should amplify it (as in the Einstein-de Haas experiment). With an amplification coefficient of the order of 10 and a resonance frequency of the order of 10 sec$^{-1}$, the effect should be detectable by interferometric methods.

Ch. Goillot is constructing an apparatus following the scheme outlined in this section.

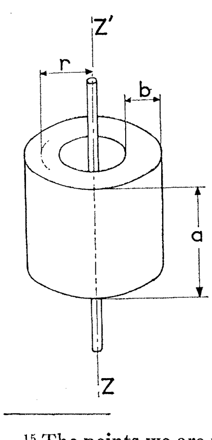


FIG. 2. Ferrite ring test body and magnetizing wire.

[15] The points we are working with in a "rectangular" hysteresis cycle are the points $\pm P_0$ corresponding to $H = 0$; our experimental parameters are such that $P_0$ is practically constant inside all the ring-shaped test body.

[16] Even if the ferrite test body were electrically charged, a short current pulse would cause no artifact; the electric field $\mathbf{E} = -\partial \mathbf{A}/\partial t$ is parallel to the wire, and the momentum imparted to the body of total charge $Q$ during a time interval $\Delta t$ comprising the pulse would be $Q \int \mathbf{E} dt = -\Delta \mathbf{A} = 0$.

[17] Usually denoted by $\sigma$ in magnetism literature.

### IV. HYPERQUANTIZATION

Physically, when going from the atomic scale up to the macroscopic scale of the test body, different averaging processes will occur in succession: atoms inside magnetic domains, domains inside individual crystals, crystals inside the macroscopic body. A detailed analysis of any one of these averaging processes would be a very formidable task. Fortunately, different kinds of general arguments, of a plausible rather than rigorous character, may be given in favor of maintaining the validity of the above conclusions. The main one is drawn from the hyperquantization technique.

If we ignore all particular details, our problem can be schematized as that of distributing, according to Fermi statistics, a large number of electrons among a set of orthogonal states. Moreover, it is sufficient for our purpose to consider the initial and the final state of our test body, which, macroscopically speaking, are stationary in the center-of-mass frame.

In Sec. II we have used the Dirac equation with an external potential. The obvious hyperquantized transposition is the so-called bound-interaction picture,[18] where the electron's $\psi$ is an operator, and the state function $\Phi$ varies according to the electron's interaction with the radiation field.

As the macroscopic states we are considering are stationary, $\Phi$ will fluctuate in time around some mean

[18] See for instance J. M. Jauch and F. Rohrlich, reference 3, p. 306.

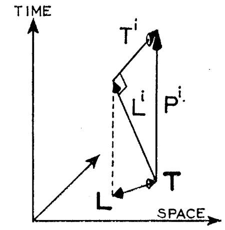


FIG. 3. Longitudinal, transverse, and total-energy momentum, with their space projections.

value $\Phi_0$. To find the observable mean value of such quantities as Tetrode's energy-momentum tensor, Dirac's spin density, etc., we will have to use expressions of the form

$$R = \langle \Phi_0 | \bar{\psi} \mathcal{R} \psi | \Phi_0 \rangle, \tag{42}$$

where the operators $\psi$ obey the same equations as in Sec. II and where $\Phi_0$ is constant. Thus, all the differential relations which are consequences of the Dirac theory, and among them Tetrode's and Gordon's formulas, which we have explicitly used, will still be valid in terms of the mean values (42).

In other words, the deduction of Sec. II may be extended to the more physical picture of hyperquantized theory.

Another consequence of formula (42) is

$$\langle \mathrm{curl}\,\boldsymbol{\sigma} \rangle_{\mathrm{av}} = \mathrm{curl} \langle \boldsymbol{\sigma} \rangle_{\mathrm{av}},$$

where $\langle\,\rangle_{\mathrm{av}}$ denote mean values; this relation has been tacitly assumed in Sec. III.