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# Space-Time Model of Relativistic Extended Particles in Minkowski Space. II. Free Particle and Interaction Theory

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In this paper, we examine new consequences of the idea that particles are extended structures in real space-time. Starting from the general quantum equations for stable states established in paper I, we discuss the general form of solutions satisfying simultaneously the internal state equations and the external wave equations. One sees in particular that the external part depending on the particle's position $x_\mu$ necessarily corresponds to state vectors belonging to irreducible finite-dimensional representations of the Lorentz group $\mathfrak{L}_4$. Assuming then that the general interaction Hamiltonians are invariant under $\mathfrak{L}_4$ and our new internal isobaric spin group $G$, one justifies the usual semiempirical scheme of strong interactions (invariant under $SO_3$) and introduce weak interactions in isobaric spin space. The theory also implies couplings between external and internal motions and breaks automatically the symmetries of strong interactions, in a natural development of Sakurai's ideas.

## INTRODUCTION

As indicated in paper I, the introduction of a unified space-time model of elementary particles should yield an interpretation of all their quantum properties. We shall thus attempt now the study of two problems left aside in paper I (which mainly concentrated on the understanding of the nature of the new quantum numbers such as isobaric spin, strangeness, and baryon number), namely, the interaction theory and spin states. To do this, we shall first resolve a purely mathematical problem: the construction of irreducible vectors belonging to any given irreducible representation $D(l^+,l^-)$ of our new group $SO_3^*$. This is important in our scheme for two reasons. First, as seen in paper I, $SO_3^*$ is isomorphic to the Lorentz group $\mathfrak{L}_4$, so that the construction of irreducible state vectors in $D(l^+,l^-)$ yields immediately the form of possible external waves (irreducible under $\mathfrak{L}_4$) which can be associated to various possible types of elementary particles with different spins and couplings. The second reason is that if we assume an interaction model which implies the existence of a highly excited intermediate state invariant under $SO_3^*$, we see that the irreducible vectors of $D(l^+,l^-)$ yield a possible way of grouping our internal "levels" in order to build strong invariant interaction Hamiltonians. For this reason we call them "interaction vectors" and will indicate here the corresponding interaction theory.

This discussion determines the plan of our paper. In Sec. I we shall establish the mathematical form of the irreducible vectors in $D(l^+,l^-)$ and build the strong and weak internal interaction theory. In Sec. II, we shall discuss the correspondence between internal and external states satisfying our general wave equations. Finally, in Sec. III, we shall take advantage of the "internal" invariance in order to introduce new "vector mesons" which break the symmetries in a suitable way and allow us to go more deeply into the analysis of strong interaction processes.

## SECTION I

According to our program we shall first discuss the possible interaction terms invariant under $G$ which can be built from suitable combinations of our fundamental state functions.

The deduction of such terms results simply from our model. Let us first deal with the "elementary" strong interactions, or Yukawa processes, represented by three-pronged graphs of the type shown in Fig. 1, the baryon states $B$ and $\bar{B}'$ being structures endowed with internal quantified motions corresponding to irreducible representations $D(l^+,l^-)$ and $D(k^+,k^-)$ of $SO_3^*$. We introduce the idea suggested to one of us (J. -P. V.) by Professor Yukawa, that such interactions consist of the "fusion" of the two quantized structures which overlap in space; then build a resulting quantized state (also expressed by an irreducible representation of $SO_3^*$) which may in turn decay into two various other quantized states. If we then make the fundamental physical assumption that the interaction process itself is invariant under our

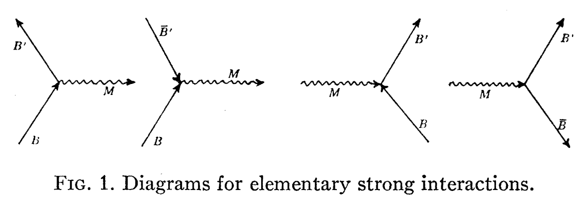



FIG. 1. Diagrams for elementary strong interactions.

group $G$, we get a strong selection rule for the resulting boson, namely: *The irreducible representation obtained must be one of the terms the Clebsch-Gordan splitting of the product of the interacting representations.* Thus the Clebsch-Gordan formula yields

$$D(\tfrac{1}{2},1)\otimes D(\tfrac{1}{2},1)\simeq D(1,0) \quad \text{for the interactions } \bar{N}N\pi,$$

$$D(1,\tfrac{1}{2})\otimes D(1,\tfrac{1}{2})\simeq D(1,0) \text{ for the interactions } \bar{\Sigma}\Sigma\pi,$$

$$D(\tfrac{1}{2},1)\otimes D(1,\tfrac{1}{2})\simeq D(\tfrac{1}{2},\tfrac{1}{2}) \text{ for the interactions } \bar{N}\Sigma K,$$

the sign $\simeq$ meaning that we keep among the decomposition terms the one which lies in our boson table.

We can express this invariance principle in a more precise mathematical form, namely: *The strong interaction Hamiltonians must be invariant under the group G.* We are thus faced here with a mathematical problem very similar to the abstract procedure applied in the usual isobaric spin space theory.

Now in recent years, great progress has been made in interaction theory. On the basis of semi-empirical considerations, one has been led to construct interaction Hamiltonians which conserve charge and baryon number (all interactions) and also isobaric spin (strong interactions) by considering particles as components of spinors (doublets) or vectors (triplets) in an abstract new "interaction space"; isobaric spin conservation being associated with invariance under $SO_3$ in that space. With the help of these considerations very important results have been obtained: selecting possible from forbidden interactions, studying parity conservation, etc. These results are very suggestive and valuable but evidently need theoretical justification. In particular, any complete theory must explain the nature of interaction space and the reason for the assimilation of any given particle to certain multiplets in that space. Until we are able to do that, we will not be in a position to predict the nature and form of all possible particle interaction Hamiltonians or, in fact, to say that we have bulit a satisfactory theory of elementary particles.

The advantage of any given model, such as the one we have tried to develop here, is that it leads to specific answers as to the nature of this "interaction space" so that its consequences can be compared with experiment. As we shall now see, our rotator model leads to promising definitions and answers to the preceding problems if we accept the proposal made by one of us (J.-P. V.) to identify the interaction space with the Hilbert space spanned by the irreducible representations $D(l^+,l^-)$ of $SO_3^*$; the particles of a given level being grouped in irreducible interaction vectors transforming under the considered representation.

This is a commonly accepted idea; and we now simply follow the usual procedure of quantum field theory, applied to our wider group $G$. We start from the evident remark that the eigenfunctions $Z(l^+,l^-,s';m^+,m^-,m')$ which span the Hilbert space $H(l^+,l^-)$ corresponding to $G$ can be split into families which span subspaces which remain irreducible under the transformations of some subgroups of our general isobaric group $G$. This is a well-known result of group theory and one sees immediately that one can form:

(1) subspaces transforming under irreducible representations $D(l^+)$ of $SO_3^+$ by fixing the values of $l^+$, $l^-$, $m^-$, $s'$ and $m'$ in $Z(l^+,l^-,s';m^+m^-,m')$;

(2) subspaces transforming under irreducible representations $D(l^-)$ of $SO_3^-$ by fixing the values of $l^+$, $m^+$, $l^-$, $s'$ and $m'$ in $Z(l^+,l^-,s';m^+,m^-,m')$;

(3) subspaces transforming under irreducible representations $D(l^+,l^-)$ of $SO_3^*$ by fixing the values of $l^+$, $l^-$, $s'$, and $m'$ in $Z(l^+,l^-,m';m^+,m^-,m')$;

(4) subspaces transforming under irreducible representations $D(s')$ of $SO_3'$ by fixing the values of $l^+$, $l^-$, $m^+$, $m^-$ and $s'$ in $Z(l^+,l^-,s';m^+,m^-,m')$.

(5) Moreover, as proposed by one of us (P. H.), we can utilize for our group $G$ an idea of Prentki and d'Espagnat, and group our functions $Z$ into linear combinations,

$$W(l^+,l^-,s,s';m,m') = \sum_{m^+,m^-} (l^+,l^-,s,m\,|\,m^+,m^-)Z(l^+,l^-,s';m^+,m^-,m'),$$

which are common eigenfunctions of the operators

$$J^{2+},\ J^{2-},\ S'^2,\ S_3',\ S^2,\ S_3,$$

with

$$S_k=J_3^+ + J_3^-,\quad S^2=S_kS_k,$$

the numbers $s$ and $m$ having the possible values

$$s=l^+ + l^-,\ l^+ + l^- -1,\ \cdots,\ |l^+ - l^-|,$$

$$m=-s,\ -s+1,\ \cdots,\ s-1,\ s.$$

These eigenfunctions provide us with another splitting of our functional space into new subspaces $M(l^+,l^-,s,s';m')$ which transform according to the irreducible representation $D(s)$ of the (real) rotation group $SO_3$.

Now as one knows, observed strong interactions can be described with various isobaric schemes, namely:

$$\text{(I)}\quad N=\begin{pmatrix} p \\ n \end{pmatrix},\quad \xi=\begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix},\quad k=\begin{pmatrix} K^+ \\ K^0 \end{pmatrix},\quad \tilde{k}=\begin{pmatrix} -\tilde{K}^0 \\ \tilde{K}^+ \end{pmatrix},$$

$$\Sigma=\begin{bmatrix} (\Sigma^+ + \Sigma^-)/\sqrt{2} \\ (\Sigma^- - \Sigma^+)/i\sqrt{2} \\ \Sigma^0 \end{bmatrix},\quad \pi=\begin{bmatrix} (\pi^+ + \pi^-)/\sqrt{2} \\ (\pi^- - \pi^+)/i\sqrt{2} \\ \pi^0 \end{bmatrix},\quad \Lambda,$$

or the "baryon doublet scheme":

$$\text{(II)}\quad N,\ \xi,\ k,\ \pi,\quad X=\begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix},\quad Z=\begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix},$$

with

$$Y^0=(\Lambda^0-\Sigma^0)/\sqrt{2},\quad Z^0=(\Lambda^0+\Sigma^0)/\sqrt{2}.$$

The so-called "triplet-singlet scheme" has also been tried:

$$\text{(III)}\quad N=\begin{Bmatrix}(p+\Xi^-)/\sqrt{2}\\(-p-\Xi^-)/i\sqrt{2}\\(-n-\Xi^0)/\sqrt{2}\end{Bmatrix},\quad N_0=(n+\Xi^0)/i\sqrt{2},\Sigma,\Lambda,\pi,$$

with

$$K=\begin{Bmatrix}(K^++\tilde{K}^+)/\sqrt{2}\\(-K^+-\tilde{K}^+)/i\sqrt{2}\\(-K^0+\tilde{K}^0)/\sqrt{2}\end{Bmatrix},\quad K^0=(K^0-\tilde{K}^0)/i\sqrt{2}.$$

To every scheme there belongs a well-defined isospace in which strong $\pi$ and $K$ interactions are described by scalar interaction Hamiltonians. The leptons have never been satisfactorily introduced in these schemes.

In these spaces a series of interaction Hamiltonians can be built which have to be invariant under the corresponding isobaric spin group. For instance according to the scheme I, where $N$ and $\xi$ are spinors, $\pi$ and $\Sigma$ isovectors, and $\Lambda$ an isoscalar, one can write the following set of invariant quantities:

$$H_{\text{int}}=g_{\pi_1}\bar{N}\sigma_i\pi_iN+g_{\pi_2}\bar{\Xi}\sigma_i\pi_i\Xi+g_{\pi_3}\epsilon_{ijk}\bar{\Sigma}_i\Xi_j\pi_k+g_{\pi_4}\bar{\Lambda}\Sigma_i\pi_i+g_{K_1}\bar{N}\sigma_iK\Sigma_i+g_{K_2}\bar{N}\Lambda K+g_{K_3}\bar{\Xi}\sigma_iK\Sigma_i+g_{K_4}\bar{\Xi}\Lambda K,$$

and one knows that most recent discussions deal with the assumption of higher symmetries deriving from the identification of some of the eight constants. Such symmetries are not implied by the basic isobaric spin symmetry, namely, the invariance under the isobaric spin group $SO_3$. Most authors seem to agree with the two assumptions of "global symmetry" (equality of the four $\pi$ coupling constants) and "cosmic symmetry" (equality of the four $K$ coupling constants).

Now the most simple way to generalize these conceptions to our new group $G$ is evidently to build up some irreducible vectors (such as the sets $N$, $\xi$, $k$, $\Sigma$, $\pi$, and $\Lambda$ of the preceding schemes) which are irreducible under suitable representations of the isobaric spin group $SO_3^*$. To construct such vectors we apply a general theorem of Wigner[1] stating that with suitable linear combinations of the elements $Z(l^+,l^-,s';m^+,m^-,m')$ belonging to a given subspace $H(l^+,l^-)$ it is possible to build entities which have a definite (spinor or tensor) variance and are irreducible under the group $G$ (that is, they remain under these transformations within the subspace under consideration). In order to remain as close as possible to usual isobaric spin theory, we shall thus consider the elements which belong to a given *level* $\mathcal{E}(l^+,l^-,s';m')$ [with given values for $l^+$, $l^-$, $s'$, and $m'$] and consider only the irreducible representations of the subgroup $SO_3^*$, which is in turn split into two groups $SO_3^+$ and $SO_3^-$ acting, respectively, in the complex space $E_3^+$ spanned by $A_k^{r+}$ and in the complex space $E_3^-$ spanned by $A_k^{r-}$.

We shall now deal with the construction of such combinations, which we shall call "interaction vectors." To do that, one can naturally, following Wigner[1]. introduce "spin tensors" built with the help of Pauli matrices. However, as we have defined in paper I two kinds of conjugation, namely the *charge conjugation* $Z^C(l^+,l^-,s';m^+,m^-,m')$ which transfers us from the level $\mathcal{E}(l^+,l^-,s';m')$ into the level $\mathcal{E}(l^+,l^-,s';-m')$ and the ordinary *complex conjugation* $Z^*(l^+,l^-,s';m^+,m^-,m')$ which transfers us into the level $\mathcal{E}(l^-,l^+,s';-m')$, we shall use quaternion matrices in order to avoid any mathematical difficulties.

As one knows, the Pauli matrices provide us in a two-dimensional complex Euclidean space, with a quaternion basis, namely $Q_k=i\sigma_k$, with the rules:

$$Q_iQ_j=-\delta_{ij}+\epsilon_{ijk}Q_k.\tag{1}$$

Now we define a *complex unimodular quaternion* by

$$S=a_0+a_kQ_k,\tag{2}$$

the four complex parameters $a_0$, $a_k$ being related by

$$a_0^2+a_ka_k=1.\tag{3}$$

Each quaternion is thus represented by a complex $2\times2$ matrix $S$. On these matrices one can define two kinds of conjugations:

the *quaternion conjugation*

$$\tilde{S}=S^{-1}=a_0-a_kQ_k,\tag{4}$$

and the *complex conjugation*

$$S^*=a_0^*+a_k^*Q_k.\tag{5}$$

These two operations are independent, commute, and play an essential role in what follows. Indeed, we can now define interaction vectors in the various spaces.

### A. Interaction Vectors (Spinors) in the Levels of the Representations $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$

Let us start with the eigenfunctions of $D(\frac{1}{2},0)$, that is, $Z(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,\frac{1}{2})$. and $Z(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,\frac{1}{2})$. One knows one can build with them a two-component spinor (lying in the level $m'=\frac{1}{2}$), namely,

$$\psi=\begin{pmatrix}Z(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,\frac{1}{2})\\Z(\frac{1}{2},0,\frac{1}{2};-\frac{1}{2},0,\frac{1}{2})\end{pmatrix},\tag{6}$$

which transforms by a definite unimodular quaternion matrix under each (complex) rotation of $SO_3^*$ acting on the complex triad $A_k^{r+}$. The quaternions $S$ build an irreducible representation $D(\frac{1}{2},0)$ of $SO_3^*$. Moreover, from $\psi$, we can build with the aid of the matrix $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

[1] E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

a contragredient spinor:

$$\tilde{\psi}=\{-Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,\tfrac{1}{2}),Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,\tfrac{1}{2})\}, \tag{7}$$

which remains in the same subspace and transforms under the quaternion conjugate matrix $\tilde{S}$:

$$\tilde{\psi}\rightarrow\tilde{\psi}\tilde{S}. \tag{8}$$

The product $\tilde{\psi}\psi$, evidently invariant under $SO_3{}^*$, vanishes in this case.

Now we have seen in paper I how to define (up to an arbitrary complex coefficient) the charge conjugate of each eigenfunction, namely,

$$Z^C(l^+,l^-,s';m^+,m^-,m') \\ =Z(l^+,l^-,s';-m^+,-m^-,-m'), \tag{9}$$

which we write (in order to construct spinors more easily)

$$Z^C(l^+,l^-,s';m^+,m^-,m') \\ =(-1)^{m^++m^-+m'}Z(l^+,l^-,s';-m^+,-m^-,-m'), \tag{9'}$$

the factor $\pm1$ in the right-hand side being picked out from the arbitrary coefficient of the $Z$ functions.

The correspondence $Z^*\rightarrow(Z)^C$ is well defined if we make a definite choice of arbitrary coefficients in associated pairs. It allows us to pass from $\psi$ to the charge conjugate spinor:

$$\tilde{\phi}=Z^C(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,\tfrac{1}{2}),Z^C(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,\tfrac{1}{2}) \tag{10}$$
$$=-Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,-\tfrac{1}{2}),Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,-\tfrac{1}{2}),$$

built with functions belonging to the level $m'=-\frac{1}{2}$. We see immediately that this spinor $\tilde{\phi}$ transforms also under the quaternion conjugate matrix

$$\tilde{\phi}\rightarrow\tilde{\phi}\tilde{S} \tag{11}$$

and we build the contragredient spinor

$$\phi=\begin{pmatrix} Z(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,-\frac{1}{2}) \\ Z(\frac{1}{2},0,\frac{1}{2};-\frac{1}{2},0,-\frac{1}{2})\end{pmatrix}, \tag{12}$$

with

$$\phi\rightarrow S\phi. \tag{13}$$

One checks also that the product $\tilde{\phi}\psi$ is invariant under $SO_3{}^*$ and takes the explicit form

$$Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,-\tfrac{1}{2}).\ Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,\tfrac{1}{2}) \\ -Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,\tfrac{1}{2}).\ Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,-\tfrac{1}{2}),$$

which is the trivial eigenfunction $Z(0,0,0;0,0,0)$ belonging to $D(0,0)$. The choice of the preceding arbitrary coefficients can be restricted by the relation

$$\tilde{\phi}\psi=1. \tag{14}$$

The same situation develops for the subspaces spanned by the eigenfunctions of $D(0,\frac{1}{2})$ which are provided by the complex conjugation of the preceding functions, so that the matrices acting on the spinors in these subspaces are the complex conjugate matrices $S^*$ and $\tilde{S}^*$.

We get:

$$\psi^*=\begin{pmatrix} Z(0,\frac{1}{2},\frac{1}{2};0,\frac{1}{2},\frac{1}{2}) \\ Z(0,\frac{1}{2},\frac{1}{2};0,-\frac{1}{2},\frac{1}{2})\end{pmatrix}\rightarrow S^*\psi^*, \tag{15}$$

$$\tilde{\psi}^*=\{Z(0,\tfrac{1}{2},\tfrac{1}{2};0,-\tfrac{1}{2},\tfrac{1}{2}),-Z(0,\tfrac{1}{2},\tfrac{1}{2};0,\tfrac{1}{2},\tfrac{1}{2})\}; \\ \tilde{\psi}^*\rightarrow\tilde{\psi}^*\tilde{S}^*, \tag{16}$$

$$\tilde{\phi}^*=\{Z(0,\tfrac{1}{2},\tfrac{1}{2};0,-\tfrac{1}{2},-\tfrac{1}{2}),-Z(0,\tfrac{1}{2},\tfrac{1}{2};0,\tfrac{1}{2},-\tfrac{1}{2})\}; \\ \tilde{\phi}^*\rightarrow\tilde{\phi}^*\tilde{S}^*, \tag{17}$$

$$\phi^*=\begin{pmatrix} -Z(0,\frac{1}{2},\frac{1}{2};0,\frac{1}{2},-\frac{1}{2}) \\ -Z(0,\frac{1}{2},\frac{1}{2};0,-\frac{1}{2},-\frac{1}{2})\end{pmatrix},\quad \phi^*\rightarrow S^*\phi^*. \tag{18}$$

We thus have to deal with two kinds of spinors with opposite "chiralities." As we shall see, we simply re-explain with their combinations many properties discovered long ago by various authors such as Cartan[2] who considered two kinds of spinors respectively related to self-dual and antidual tensors, or one of us[3] who introduced "right-handed" and "left-handed" spinors with different transformation laws.

Of course, we may express our spinors in terms of the elementary particles corresponding to the eigenfunctions under consideration:

$$\psi=\begin{pmatrix}\nu_e \\ e\end{pmatrix},\quad \tilde{\psi}=(-e,\nu_e),$$

$$\phi=\begin{pmatrix}\bar{e} \\ \bar{\nu}_e\end{pmatrix},\quad \tilde{\phi}=(\bar{\nu}_e,\bar{e}),$$

$$\psi^*=\begin{pmatrix}\nu_\mu \\ \mu\end{pmatrix},\quad \tilde{\psi}^*=(-\mu,\nu_r),$$

$$\phi^*=\begin{pmatrix}\bar{\mu} \\ \bar{\nu}_r\end{pmatrix},\quad \tilde{\phi}^*=(-\bar{\nu}_r,\bar{\mu}).$$

This is the form we shall retain for all expressions endowed with physical meaning.

### B. Interaction Vectors in the Subspace $D(1,0)$ and $D(0,1)$

We know from the case of the real group $SO_3$ that one can build from the four spinors $\psi$, $\tilde{\psi}$, $\phi$, and $\tilde{\phi}$ transforming under the matrices $S$ and $\tilde{S}$, three-dimensional vectors (more precisely skew self-dual tensors) with the help of the quaternion units $Q_k$, namely:

$$\tilde{\phi}Q_k\psi=\tilde{\psi}Q_k\phi,\quad \tilde{\psi}Q_k\psi,\quad \text{and}\quad \tilde{\phi}Q_k\phi. \tag{19}$$

As $\tilde{S}Q_kS=\Omega_{ki}Q_i$ amounts to a complex three-dimen-

[2] E. Cartan, *Leçons sur la théorie des spineurs* (Hermann et Cie, Paris, 1938).
[3] T. Takabayasi, Nucl. Phys. **7**, 237 (1958).

sional rotation ($\Omega_{k_i}\Omega_{k_j}=\delta_{ij}$) performed on the quaternion basis $Q_k$, these vectors transform under $S$ through the $3\times 3$ orthogonal matrix $\Omega_{ij}$. A simple calculation then shows that the bilinear combinations of the eigenfunctions of $D(\frac{1}{2},0)$ which appear in the vectors are just the eigenfunctions of $D(1,0)$ and belong, in each vector, to a definite value of $m'$, namely,

$$\begin{array}{ll} \tilde{\phi}Q_k\psi=\tilde{\psi}Q_k\phi & \text{to} \quad m'=0, \\ \tilde{\psi}Q_k\psi & \text{to} \quad m'=1, \\ \tilde{\phi}Q_k\phi & \text{to} \quad m'=-1; \end{array}$$

so that these three vectors are irreducible interaction vectors which rotate inside each of the three levels of $D(1,0)$ as the triad $A_k{}^{r+}$ undergoes any given rotation of $SO_3{}^*$. In particular, as we have physically to limit ourselves to the observed particles of this level, namely the pions, we have to consider the vector

$$\pi_k{}^{(0)}=\tilde{\phi}Q_k\psi=\tilde{\psi}Q_k\phi,$$

which can be written in terms of the state functions of the pions:

$$\pi_k{}^{(0)}=\begin{bmatrix} (\pi^+-\pi^-)/\sqrt{2} \\ (\pi^++\pi^-)/i\sqrt{2} \\ -\pi^0 \end{bmatrix},$$

which is equivalent to the above-recalled expression in the usual theory. Let us recall nevertheless that here $\pi_k{}^{(0)}$ is a three-vector under $SO_3{}^*$ acting in $E_3{}^+$ which is isomorphic to $\mathcal{E}(1,0,1;0)$ and a scalar under $SO_3{}^*$ acting in $E_3{}^-$.

Thus the above considerations yield two vectors, mathematically identical, in the level $\mathcal{E}(1,0,1;0)$, namely, the vector $\tilde{\phi}Q_k\psi$ built with the state functions of the leptons[3a] and the vector $\pi_k{}^{(0)}$ built with those of the pions. The scalar multiplication of both vectors provides us a combination of these two kinds of state functions, which is invariant under $G$, and which leads us (at least theoretically) to an invariant representation of interactions. The same vector combinations can be built up with the spinors of $E_3{}^-$ containing $\mu$ and $\nu_\mu$, namely $\tilde{\phi}^*Q_{k'}\psi^*$ (the relation to $E_3{}^-$ is denoted by primed indices) which would correspond to a set of three particles $\bar{\omega}$ (with $i_3=0$ and $S=2,\,0,\,-2$) associated with the $D(0,1)$ representation.

### C. Interaction Vectors in the Subspaces $D(\frac{1}{2},\frac{1}{2})$

We can follow a similar procedure in the case of $D(\frac{1}{2},\frac{1}{2})$ and combine the spinors of $D(\frac{1}{2},0)$ with those of $D(0,\frac{1}{2})$. In that case, however, the components of the form $\tilde{\phi}^*Q_k\psi$ mix, under a $SO_3{}^*$ transformation, with the noninvariant quantity $\tilde{\phi}^*\psi$. We thus have to add to the vector quaternion units $Q_k=i\sigma_k$ the scalar unit $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, denoted $Q_0$: $Q_\mu=Q_0$, $Q_k$ ($\mu=1,\,2,\,3,\,0$). Now it can be shown that:

$$\tilde{S}^*Q_\mu S=\Lambda_{\mu\nu}Q_\nu \tag{20}$$

[3a] Or $p$, $n$, $v$, and $\Lambda$ in the Yukawa classification.

(where $\mu=1,\,2,\,3,\,0$) with $\Lambda_{\mu\nu}\Lambda_{\lambda\nu}=\delta_{\mu\lambda}$ so that the quantity $\tilde{\phi}^*Q_\mu\psi$ transforms as a four-dimensional vector in a hyperbolic space with metric $1,\,-1,\,-1,\,-1$ since $S^*$ acts on the left-hand side and $S$ on the right-hand side. Four-vectors of the type can be built, namely:

$$\tilde{\phi}^*Q_\mu\psi,\quad \tilde{\psi}^*Q_\mu\phi,\quad \tilde{\psi}^*Q_\mu\psi,\quad \text{and}\quad \tilde{\phi}^*Q_\mu\phi.$$

They are not satisfactory since their components are not eigenfunctions of $S'^2$. We therefore introduce in their place suitable linear combinations such as

$$(\tilde{\psi}^*Q_\mu\psi-\tilde{\phi}^*Q_\mu\phi),$$

which belong to $s'=0$, $m'=0$. One discovers easily three other such combinations associated, respectively, with $m'=1$, $m'=0$, $m'=-1$, ($s'=1$).

In this way, we have discovered four irreducible interaction four-vectors which undergo four-dimensional rotations inside each of the four levels $\mathcal{E}(\frac{1}{2},\frac{1}{2},0;0)$, $\mathcal{E}(\frac{1}{2},\frac{1}{2},1;0)$, $\mathcal{E}(\frac{1}{2},\frac{1}{2},1;1)$, $\mathcal{E}(\frac{1}{2},\frac{1}{2},1;-1)$ as the triads $A_k{}^{r\pm}$ perform any rotation of $SO_3{}^*$.

In the same way as above, we can express the combination belonging to $\mathcal{E}(\frac{1}{2},\frac{1}{2},0;0)$ in terms of the state functions of the $K$ mesons belonging to this level:

$$K_\mu{}^{(0)}=\begin{bmatrix} (K^++K^-)/\sqrt{2} \\ (K^+-K^-)/i\sqrt{2} \\ (K^0+\bar{K}^0)/\sqrt{2} \\ (K^0-\bar{K}^0)/i\sqrt{2} \end{bmatrix}.$$

This is a four-vector with respect to $SO_3{}^*$, and in the level $\mathcal{E}(\frac{1}{2},\frac{1}{2},0;0)$ which builds a composed four-dimensional space $E_4$. This four-vector can combine into an invariant scalar product with the four-vector (20) which contains both kinds of leptons or baryons (Yukawa).

The $K$-meson functions can also be built in a square table which is spinor both in $E_3{}^+$ and in $E_3{}^-$.

$$\begin{pmatrix} K^+ & -\bar{K}^0 \\ K^0 & K^- \end{pmatrix}.$$

### D. Interaction Vectors in the Spaces $D(1,\frac{1}{2})$ and $D(\frac{1}{2},1)$

In these spaces, the construction of interaction vectors is analogous, but more complicated. It appears that we have two different ways to do this, namely, to use the product $D(1,0)\otimes D(0,\frac{1}{2})$ of the irreducible vectors built with $Z(0,\frac{1}{2},\frac{1}{2};0,m^-,m_1')$ and those built with $Z(1,0,s';m^+,0,m_2')$ and to use the product $D(\frac{1}{2},\frac{1}{2})\otimes D(\frac{1}{2},0)$ of the irreducible vectors built with $Z(\frac{1}{2},\frac{1}{2},s';m_1{}^+,m^-,m_1')$ and $Z(\frac{1}{2},0,\frac{1}{2};m_2{}^+,0,m_2')$. Of course, we obtain in both cases eigenfunctions of $D(1,\frac{1}{2})$ but we thus build two kinds of irreducible combinations (which, as we shall see later, account for the difference between the strong interaction Hamiltonians, yielding $\pi$ and $K$ mesons) which have different variance character under $SO_3{}^*$ transformations. First, if we take into account the preceding construction of the three-vectors in $D(1,0)$

and the well-known Clebsch-Gordan relation:

$$Z(\tfrac{1}{2},1,\tfrac{1}{2};\,m^+,m^-,\tfrac{1}{2})$$
$$=\frac{1}{\sqrt{3}}\{-Z(0,1,1;\,0,m^-,0)Z(\tfrac{1}{2},0,\tfrac{1}{2};\,m^+,0,\tfrac{1}{2})$$
$$+\sqrt{2}Z(0,1,1;\,0,m^-,1)Z(\tfrac{1}{2},\,0,\,\tfrac{1}{2};\,m^+,\,0,\,-\tfrac{1}{2})\},\quad (21)$$

we can check that the vector-spinor combination

$$A_k{}^{(\frac{1}{2})}=[(\tilde{\phi}^*Q_{k'}\psi^*)\otimes\psi+\sqrt{2}(\tilde{\psi}^*Q_{k'}\psi^*)\otimes\phi]\quad (22)$$

has the character of a three-vector, whose components are column spinors, all components belonging to $s'=\frac{1}{2}$, $m'=\frac{1}{2}$. The vector indices are primed because they are relative to the rotations of space $E_3{}^-$. More precisely, if we endow the spinors and quaternion matrices with (upper) primed or unprimed indices we have to write

$$A_{k'}{}^{(\frac{1}{2})t}=\tilde{\phi}^{*r'}Q_{k'}{}^{r's'}\psi^{*s'}\phi^t+\sqrt{2}(\tilde{\psi}^{*r'}Q_{k'}{}^{r's'}\psi^{*s'}\psi^{t'}).\quad (23)$$

In terms of the state functions of level $\mathcal{E}(\frac{1}{2},1,\frac{1}{2};\,\frac{1}{2})$, we have

$$A_{1'}{}^{(1/2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}\Xi^-+X^+\\ \Xi^0+X^{++}\end{pmatrix};\quad A_{2'}{}^{(1/2)}=\frac{1}{i\sqrt{2}}\begin{pmatrix}\Xi^--X^+\\ \Xi^0-X^{++}\end{pmatrix};\quad A_{3'}{}^{(1/2)}=\begin{pmatrix}n\\ p\end{pmatrix}.\quad (24)$$

The second procedure, however, takes into account the building of four-vectors in $D(\frac{1}{2},\frac{1}{2})$ and performs the direct product of suitable combinations $K_\mu{}'$ belonging to $s'=1$ ($K_\mu{}'^{(1)},K_\mu{}'^{(0)},K_\mu{}'^{(-1)}$ according to the values of $m'$), and the spinors of $E_3{}^-$. The values $s'=\frac{1}{2}$, $m'=-\frac{1}{2}$ particles) are provided by the general expression

$$\mathfrak{A}_\mu{}^{(\frac{1}{2})}=\sqrt{2}(\tilde{\psi}^*Q_\mu\psi+\tilde{\phi}^*Q_\mu\phi)\psi^*-2\tilde{\phi}^*Q_\mu\psi\phi^*.\quad (25)$$

This yields, in terms of the baryons:

$$\mathfrak{A}_1{}^{(1/2)}=\begin{pmatrix}p+X^+\\ \Xi^0+n\end{pmatrix},\quad \mathfrak{A}_2{}^{(1/2)}=i\begin{pmatrix}p-X^+\\ \Xi^0-n\end{pmatrix},\quad \mathfrak{A}_3{}^{(1/2)}=\begin{pmatrix}n-X^{++}\\ \Xi^--p\end{pmatrix},\quad \mathfrak{A}_4{}^{(1/2)}=i\begin{pmatrix}n+X^{++}\\ \Xi^-+p\end{pmatrix}.\quad (26)$$

These expressions build a four-vector in the composed space $E_3{}^-$.

The same procedure provides us for representation $D(1,\frac{1}{2})$:

(1) a three-vector in $E_3{}^+$, spinor in $E_3{}^-$, namely:

$$B_1{}^{(1/2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}Y^{++}+Y^0\\ \Sigma^++\Sigma^0\end{pmatrix},\quad B_2{}^{(1/2)}=\frac{1}{i\sqrt{2}}\begin{pmatrix}Y^{++}-Y^0\\ \Sigma^+-\Sigma^-\end{pmatrix},\quad B_3{}^{(1/2)}=\begin{pmatrix}Y^+\\ \Sigma^0\end{pmatrix};\quad (27)$$

(2) a four-vector in the composite space $E_4$, spinor in $E_3{}^+$, namely:

$$\mathfrak{B}_1{}^{(1/2)}=\begin{pmatrix}\Sigma^++Y^+\\ \Sigma^0+Y^0\end{pmatrix},\quad \mathfrak{B}_2{}^{(1/2)}=i\begin{pmatrix}\Sigma^+-Y^+\\ \Sigma^0-Y^0\end{pmatrix},\quad \mathfrak{B}_3{}^{(1/2)}=\begin{pmatrix}Y^{++}-\Sigma^0\\ Y^+-\Sigma^-\end{pmatrix},\quad \mathfrak{B}_4{}^{(1/2)}=i\begin{pmatrix}Y^{++}+\Sigma^0\\ Y^++\Sigma^-\end{pmatrix}.\quad (28)$$

With these irreducible vectors and their $C$ conjugates containing the antiparticles, we can build up invariant combinations with the aid of the quaternion operations $Q_k$ in three-dimensional spaces and $Q_\mu$ in four-dimensional subspaces. These invariant combinations will be the interaction Hamiltonians.

One remarks here that we succeed in this way to avoid an important difficulty emphasized by d'Espagnat and Prentki. It is, in fact, *not* possible to build within the frame of the Lorentz group a scalar with the irreducible vectors $\psi$ and their complex conjugates $\tilde{\psi}^*$. Here we use the *charge conjugate* $\tilde{\phi}$ of $\psi$, resulting from operation $C$, instead of the ordinary complex conjugate which results from operation $PC$, because the conjugation acts also on the complex variables and transforms $z_\mu{}^\pm$ into $z_\mu{}^\mp$, as we have seen in paper I. Thus, if the spinor $\psi$ undergoes transformation $S$, the complex conjugate $\tilde{\psi}^*$ undergoes transformation $\tilde{S}^*$, while the charge conjugate $\tilde{\phi}$ undergoes transformation $\tilde{S}$. Now $\tilde{S}S=1$, while $\tilde{S}^*S\neq 1$, so that $\tilde{\phi}\psi$ is invariant, unlike $\tilde{\psi}^*\psi$.

With this new scheme one can evidently obtain the so-called "elementary" Yukawa strong interactions (baryon-antibaryon boson interactions) represented by three-pronged graphs. These interactions result from scalars under $SO_3{}^*$ (multiplied by the usual external Hamiltonians) which determine all possible interactions. For example, the interaction between antibaryons and baryons of the $D(\frac{1}{2},1)$ representation to produce pions can be written

$$\begin{aligned}H_1=&\bar{A}_{k'}{}^{(1/2)}Q_j\pi_j{}^{(0)}A_{k'}{}^{(1/2)}+\text{H.c.}\\ =&\pi^0(\bar{X}^{++}X^{++}+\bar{\Xi}^0\Xi^0)-\pi^0(\bar{X}^+X^++\bar{\Xi}^-\Xi^-)+\text{H.c.}\\ &+\pi^-(\bar{X}^+X^{++}+\bar{\Xi}^-\Xi^0)+\pi^+(\bar{X}^{++}X^++\bar{\Xi}^0\Xi^-)+\text{H.c.}\\ &+\pi^0\bar{p}p+\pi^+\bar{p}n+\pi^-\bar{n}p-\pi^0\bar{n}n+\text{H.c.}\end{aligned}\quad (29)$$

These are the usual interactions as regards the known particles. One sees that this justifies immediately the absence of the unobserved strong interactions, and also that the unobserved $X$ particles can never arise from the observed ones.

Similar combinations may be built up with the baryons and antibaryons of $D(1,\frac{1}{2})$, and also with mixed combinations, with the aid of the four-vectors of $E_4$. Finally, we have, the $\Lambda^0$ interactions being left aside, the general interaction Hamiltonian:

$$H_{\text{int}}=g_1\bar{A}_{k'}{}^{(1/2)}Q_j\pi_j{}^{(0)}A_{k'}{}^{(1/2)}+g_2\epsilon_{ijk}\bar{B}_i{}^{(1/2)}B_j{}^{(1/2)}\pi_k{}^{(0)} +g_3(\bar{\mathfrak{A}}_\mu{}^{(1/2)}Q_\nu K_\nu{}^{(0)}\mathfrak{B}_\mu{}^{(1/2)}+\text{H.c.} \quad (30)$$

We see that we are left with only three independent coupling constants, the global symmetry corresponding to $g_1=g_2$. If the $\Lambda^0$ is assumed to belong, together with another (charged) particle $\Gamma^+$, to the representation $D(0,\frac{1}{2})$ obtained from the fusion of three spin units, we shall have a combination $\psi^{(1/2)}=\begin{pmatrix}\Lambda^0\\ V^+\end{pmatrix}$ which is scalar in $E_3{}^+$ and spinor in $E_3{}^-$. Now the corresponding interaction Hamiltonians are

$$g_2'(\bar{\psi}^{(1/2)}B_i{}^{(1/2)}\pi_i{}^{(0)}+\text{H.c.}) \text{ and } g_3'\bar{\mathfrak{A}}_\mu{}^{(1/2)}K_\mu{}^{(0)}\psi^{(1/2)}, \quad (31)$$

and the usual symmetries are obtained by

$$g_2'=g_1=g_2, \quad g_3'=g_3, \quad (32)$$

if they apply to $\Lambda^0$, which is still doubtful from the experimental data.[4]

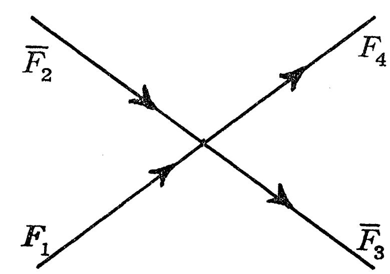
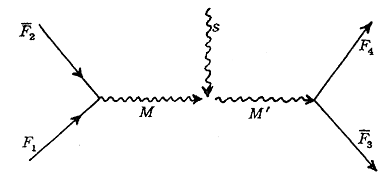



FIG. 2. Diagram for universal four-fermion interaction.

The application of this scheme to the weak interactions raises important difficulties, since a nonleptonic decay process, such as for instance $\Lambda^0\rightarrow p+\pi^-$, is evidently not invariant under $SO_3{}^*$. In our model, as in the usual theory weak interactions are not invariant under the total isobaric spin group. In order to make possible the use of scalar Hamiltonians under $G$ for these interactions, one thus introduces a pure $D(\frac{1}{2},\frac{1}{2})$ representation without spin, momentum, or mass, which is called a *spurion* as in the usual procedure. This can be represented in two equivalent forms. Let us consider the fundamental elementary process implied, at least virtually, in each weak inteaction, namely, the universal four-fermion process shown in Fig. 2. The four-pronged graph can be split into two three-pronged ones and we can apply to each of them our Clebsch-Gordan rule

[4] One can remark here that our invariance group provides us directly with strong interaction Lagrangians, implying the separate conservation of isobaric spin and strangeness. From a formal point of view one can check that the subspaces corresponding to the irreducible representations $D(l^+)$ of $SO_3{}^+$ regroup the $Z$ functions according to the scheme I of d'Espagnat and Prentki. Invariant Lagrangians under $SO_3{}^+$ thus correspond exactly to the classical isobaric formalism.

FIG. 3. Reduction of four-pronged diagram into two three-pronged ones.




(see Fig. 3). But (at least for the nonfermionic processes) we see that if $M$ belongs to $D(1,0)$, then $M'$ will belong to $D(\frac{1}{2},\frac{1}{2})$ and conversely, so that we are led to assume the mysterious intervention of a supplementary spurion representation, namely, $D(\frac{1}{2},\frac{1}{2})$ which transforms $M$ into $M'$ (see Fig. 4). This can be expressed in an alternative form, by considering the internal parity operator $P$ put into evidence in paper I, which transforms each function or vector of $D(l^+,l^-)$ into a function or vector belonging to $D(l^-,l^+)$. If this process, deprived as yet of any physical meaning, happens on one of the incident or created fermions during the interaction process, then the bosons $M$ and $M'$ come to fall in the same representation and can be considered as identical, according to the scheme shown in Fig. 5.

Now it is remarkable that the preceding analysis can be formulated in a simple "internal" scalar Hamiltonian scheme if one drops the idea that weak processes are invariant under $G$ and accept the assumption that they have weaker internal symmetries corresponding to subgroups of $G$. Indeed, following one of us (P. H.) one can check that these graphs correspond to processes invariant under the subgroup $SO_3\times SO_3'$ of our group $SO_3\times SO_3'$. If we note that the group $SO_3$ is built with the real rotations of $SO_3{}^*$, we see these rotations are identical to their product with the parity operation $P$ (which amounts to a complex conjugation). We thus construct the corresponding weak-interaction vectors irreducible under the representation $D(s)$ of $SO_3$ introduced before, and one checks easily they recover exactly the observed weak interaction Lagrangians. In other terms, the transposition in our scheme of the $M$ space of d'Espagnat and Prentki yields an explanation of weak-interaction decays. Detailed analysis of this question will be published by three of us (F. H., P. H., and J.-P. V.).

To conclude this study of "internal formalism," let us recall that each interaction vector component has to satisfy separately the second-order equation,

$$(J^{2+}+J^{2-})\psi=[l^+(l^++1)+l^-(l^-+1)]\hbar^2\psi. \quad (33)$$

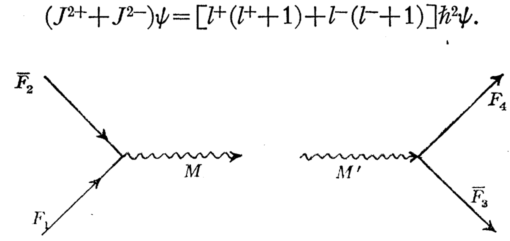



FIG. 4. Diagram for a four-fermion process with a supplementary spurion $s$.

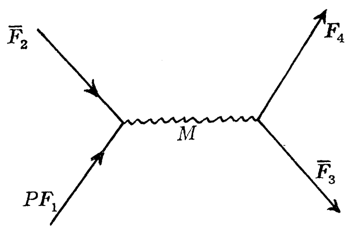



FIG. 5. Diagram for a four-fermion process with an intermediate boson.

However, in the case of fermions ($l^+ + l^-$ half-integer), as was shown by two of us (P. H. and J.-P. V.),[5] one can extend to the internal state vectors Dirac's linearization ideas. One checks indeed that in that case one can find linear equations whose solutions obey condition (33) to second order.

For instance, in the case of lepton interaction vectors of $D(\frac{1}{2},0)$, we can write

$$\sigma_k J_k{}^+ \psi = \chi_1 \psi, \quad J_k{}^+ \tilde{\phi} \sigma_k = \chi_2 \tilde{\phi}. \tag{34}$$

Multiplying each side of both equations with $\sigma_i J_i$ and taking into account the commutation relations of the $\sigma_i$'s and the $J_k$'s, we get easily

$$J^{2+}\psi = \chi_1(\chi_1 - \tfrac{1}{2}\hbar)\psi, \tag{35}$$

and

$$J^{2+}\tilde{\phi} = \chi_2(\chi_2 + \tfrac{1}{2}\hbar)\tilde{\phi}, \tag{36}$$

so that the constants $\chi_1$ and $\chi_2$ can be determined by identification with the general condition (33) which becomes

$$J^{2+}\psi = \tfrac{3}{4}\hbar^2\psi, \quad J^{2+}\tilde{\phi} = \tfrac{3}{4}\hbar^2\tilde{\phi}, \tag{37}$$

namely,

$$\chi_1 = \tfrac{1}{4}\hbar[1 \pm (13)^{1/2}], \quad \chi_2 = -\tfrac{1}{4}\hbar[1 \pm (13)^{1/2}]. \tag{38}$$

The same constants are used for the interaction vectors $\phi$ and $\tilde{\psi}$ of $D(\frac{1}{2},0)$ and for the interaction vector $\psi^*$ and $\tilde{\phi}^*$, $\phi^*$, and $\tilde{\psi}^*$ of $D(\frac{1}{2},0)$. In the baryon case, we have simultaneous linear and second-order equations. In the $D(\frac{1}{2},1)$ case, for example, we get:

$$(\sigma_k J_k{}^+ - \chi)\phi = 0, \quad (J_k{}^- J_k{}^- - \kappa)\phi = 0, \tag{39}$$

$\phi$ being the interaction vector, that is, a skew symmetric self-dual tensor with spinor components. In the boson case we have only (33).

Two essential facts result from the preceding discussions:

(I) Every internal level can be built out of sums or differences of bilinear terms combining levels associated to other representations.

(II) Any internal interaction vector of a given representation $D(l^+,l^-)$ can be built out of a combination of interaction vectors belonging to two other representations connected by Pauli matrices.

Both are summarized in the classical Clebsch-Gordan formula:

$$\begin{aligned} D(l^+,l^-)\otimes D(k^+,k^-) &= D(l^+ + k^+, l^- + k^-) \oplus D(l^+ + k^+ - 1, l^- + k^-) \oplus \cdots \\ &\quad \oplus D(|l^+ - k^+|, |l^- + k^-|) \oplus \cdots \\ &\quad \oplus D(l^+ + k^+, |l^- - k^-|) \\ &\quad \oplus D(l^+ + k^+ - 1, |l^- - k^-|) \oplus \cdots \\ &\quad \oplus D(|l^+ - k^+|, |l^- - k^-|), \end{aligned} \tag{40}$$

which describes the result of the product of two irreducible representations of a given group. The second member of (40) containing all vectors (or levels) which can be built out of the combination of vectors (or levels) belonging to the two representations $D(l^+,l^-)$ and $D(k^+,k^-)$.

Relation (40) plays an essential role in our theory. As one of us has shown,[6] it expresses in mathematical form the starting point of fusion theory and, because of the isomorphism between $SO_3{}^*$ and $S\mathfrak{L}_4$, governs interaction theory in our new isobaric spin space.

## SECTION II

We are now in a position to discuss the structure of external waves associated with the elementary internal states.

First, as we have seen in Appendix of paper I, if we start from a Lagrangian formalism based on very plausible assumptions, the general state function $\phi(x_\mu, z_\mu{}^\pm, \tau)$, restricted to stable states, splits into a product $\exp(-iMc^2\tau/\hbar)\Psi(x_\mu, z_\mu{}^\pm)$ with

$$\Psi(x_\mu, z_\mu{}^\pm) = \varphi_e(x_\mu)\cdot F(z_\mu{}^+, z_\mu{}^-), \tag{41}$$

and the general Lagrange equation splits into an external and an internal equation, namely:

$$(\Box - M^2c^2/\hbar^2)\varphi_e(x_\mu) = 0, \tag{42a}$$

$$(J^{+2} + J^{-2} - W)F(z_\mu{}^+, z_\mu{}^-) = 0. \tag{42b}$$

This means the internal factor $F(z_\mu{}^+, z_\mu{}^-)$ is necessarily an eigenfunction of $J^{+2}$ and $J^{-2}$ belonging to the irreducible representation $D(l^+,l^-)$ of our internal group $G$. Further the external wave $\varphi_e(x_\mu)$ must be an eigenfunction of $\Box$ and thus belongs to an irreducible representation $\mathfrak{D}(j,k)\oplus\mathfrak{D}(k,j)$ of the full Lorentz group, according to a very classical result of ordinary quantum mechanics. As was emphasized by one of us (T. T.), this can be expressed otherwise, independently of any Lagrangian assumption. Indeed our stable-state functions $\Psi(x_\mu, z_\mu{}^\pm)$, which must have the form $P(x_\mu, z_\mu{}^+)\cdot P'(x_\mu, z_\mu{}^-)$ ($P$ and $P'$ being polynomials in $z_\mu{}^\pm$) as we have said in paper I, can be developed in terms of our basic functions $Z(l^+,l^-,s';m^+,m^-,m')$ which build a complete set for such polynomials, so that we can write

$$\Psi(x_\mu, Z_\mu{}^\pm) = \sum_{l^+,l^-,s',m^+,m^-,m'} C(l^+,l^-,s';m^+,m^-,m')(x_\mu) \times Z(l^+,l^-,s';m^+,m^-,m'). \tag{43}$$

[5] P. Hillion and J. P. Vigier, Cahiers Phys. **121**, 345 (1960).

[6] L. de Broglie, *Introduction a la Nouvelle Théorie des Particules Elémentaires* (Gauthier-Villars, Paris, 1961).

Of course, according to our basic interpretation, which relates each internal function $Z(l^+,l^-,s';m^+,m^-,m')$ to a definite elementary particle, when we have to deal with an isolated particle, our state function reduces to the corresponding term of the development (43). Now the coefficient $C(l^+,l^-,s';m^+,m^-,m')\ (x_\mu)$ must express the whole behavior of the particle, taken as a block, with respect to the external world, in particular the properties bound to the *spin*, and this opens the problem of the relations between the internal state expressed by the quantum numbers $l^+$, $l^-$, $s'$, $m^+$, $m^-$, $m'$, and the external behavior—mainly the spin—expressed by the corresponding function $C(l^+,l^-,s';m^+,m^-,m')\ (x_\mu)$. The first approach to this problem is given by the requirement that the global field equation—whatever it shall be—which governs the function $\Psi$ must be invariant under any change of the laboratory frame, as a general condition of relativistic invariance. More precisely, as was pointed out in paper I, the assumed internal equation (33) is valid only in the $L$ frame, since the internal quantization must be performed in a well-defined kinematical frame, to acquire a specific meaning. Its solution $\Psi(x_\mu,z_\mu{}^\pm)$ represents, as we have seen, the stable state of motion of the frame $T$, referred to the $L$ frame, $x_\mu$ being their common origin.

On the contrary, there must exist also an "external" equation expressing the variations of the field at neighboring points $x_\mu$ and $x_\mu+dx_\mu$, in a form which can be expressed in any possible laboratory frame $\Sigma$, and it governs solutions $C(l^+,l^-,s';m^+,m^-,m')\ (x_\mu)$ expressed in this frame $\Sigma$. This external equation must have a form invariant under any change of $\Sigma$.

If we know a possible solution of (33) at a given point $x_\mu$ in the $L$ frame, we must express it in $\Sigma$ with the help of the parameters $\Lambda_{\alpha\beta}(x_\mu)$ of the Lorentz transform which carries $L$ into $\Sigma$, and of the transformation laws of such solutions $C(x_\mu)\rightarrow SC(x_\mu)$. This leads us, in the frame of the usual theory of Dirac, to the very important mathematical point, that such external wave functions $C(x_\mu)$ must belong to irreducible finite-dimensional representations of the external invariance group, which we shall choose, as usual, to be the full Lorentz group $\mathfrak{L}_4$. Consequently, we have to consider state functions $\Psi$ with several components, of the form

$$\Psi=Z(l^+,l^-,s';m^+,m^-,s')(z_\mu{}^\pm)\cdot\varphi_e(x_\mu), \qquad (44)$$

where $\varphi_e(x_\mu)$ is an irreducible vector of the representation $\mathfrak{D}(j,k)\oplus\mathfrak{D}(k,j)$ of the external group $\mathfrak{L}_4$, for instance, a four-component spinor for the lepton fields.

Now the question arises what is the relation between the "internal" quantum numbers $l^+$, $l^-$, $s'$; $m^+$, $m^-$, $m'$, expressing isobaric spin, strangeness, and baryon number, and the "external" quantum numbers $j$, $k$, related to the ordinary spin. A directing clue is given by the fundamental fact that the internal group $SO_3{}^*$ is isomorphic to the external group $\mathfrak{L}_4$ (or at least to the subgroup $S\mathfrak{L}_4$). This can relate a definite transformation of $SO_3{}^*$ considered in all representations $D(l^+,l^-)$, to a definite transformation of $S\mathfrak{L}_4$ considered also in all representations $\mathfrak{D}(j,k)$, but cannot give a relation between the representations $D(l^+,l^-)$ and $\mathfrak{D}(j,k)$ themselves. If we want that, we need a new principle.

We have seen in the preceding section that the functions $Z(l^+,l^-,s';m^+,m^-,m')(z_\mu{}^\pm)$ can be grouped in irreducible vectors belonging to definite internal levels $\mathcal{E}(l^+,l^-,s';m')$ and that the irreducible vectors belonging to higher values of $l^+$ and $l^-$ can be obtained as suitable combinations of those belonging to the lower values. We shall extend this conception to the external formalism by the fundamental assumption that in the global formalism, an interaction vector is represented by the direct product of an irreducible vector of the internal formalism, namely:

$$\Phi(x_\mu,z_\mu{}^\pm)=\psi(z_\mu{}^\pm)\times\varphi_e(x_\mu), \qquad (45)$$

each component of the internal vector $\psi(z_\mu{}^\pm)$ of the preceding section being multiplied by the *same* irreducible external vector $\varphi_e(x_\mu)$ of $\mathfrak{L}_4$, that is, each particle of the same internal level is assumed to belong to the same representation of $\mathfrak{L}_4$. This is evidently necessary, if the "interaction vectors" have a physical meaning at all (which is, in fact, needed by the interaction theory). Indeed, if it were not the case, any change of laboratory frame would "split" each particle into "subparticles," which is, of course, meaningless. Now if we want to pass from the lower representations to the higher ones, we must combine the interaction vectors *in the global formalism*, that is, perform the same combination on the external vectors and on the internal vectors. This limits strongly the possible associations between the external and internal representations.

This can be related to a basic idea of one of us[7] in his "fusion" theory. The statement is that in the frame of the "external" formalism, all the particles can be built up—at least in a formal point of view—from a certain number of elementary "spin units," according to a fusion process which follows the mathematical Clebsch-Gordan multiplication. For instance, two $\frac{1}{2}$ spin waves obeying the Dirac equation split after fusion into a 0-spin wave, scalar, obeying the Petiau-Kemmer (meson) equation and a 1-spin wave, relativistic vector, obeying the (generalized) Maxwell equation. Now this very powerful idea can be generalized in the following way in the frame of the present conception. There must exist two kinds of elementary spin units, which are simultaneously internal spin units and external spin units, namely,

$$[\mathfrak{D}(\tfrac{1}{2},0)\oplus\mathfrak{D}(0,\tfrac{1}{2})]\times D(\tfrac{1}{2},0)$$

and

$$[\mathfrak{D}(\tfrac{1}{2},0)\oplus\mathfrak{D}(0,\tfrac{1}{2})]\times D(0,\tfrac{1}{2}).$$

More precisely we can build two kinds of elementary

[7] L. de Broglie, *Théorie Générale des Particules à Spin* (Gauthier-Villars, Paris, 1954).

external-internal spinors, which are indeed our irreducible interaction vectors of the lepton levels $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$ considered as Dirac spinors in the external formalism, according to the usual assumption. Now all the other interaction vectors can be built up from several of these units, the Clebsch-Gordan multiplication acting simultaneously on the representations $\mathfrak{D}(d,k)\oplus\mathfrak{D}(k,j)$ of the external formalism, and on the representations $D(l^+,l^-)$ of the internal formalism.

The consequences of these conceptions are as follows:

(1) An odd number of *e*-spin–*i*-spin units yields irreducible vectors which are *fermions*, with half-integer spin, for the external formalism, and "*isofermions*", with half-integer values of $l^++l^-$, for the internal formalism.

(2) An even number of *e*-spin–*i*-spin units yields irreducible vectors which are *bosons* with integer spin, for the external formalism, and "*isobosons*" with integer values of $l^++l^-$ for the internal formalism. *Thus the integer or half-integer character is the same for the ordinary spin and for the sum $l^++l^-$ of isobaric spin and half-strangeness.*

Let us note that the relation between spin and isospin was a very cumbersome aspect of the original classification of Nishijima and Gell-Mann, who were puzzled by the fact that the fermions $\Lambda$ and $\Sigma$ have an integer isobaric spin, while the bosons $K$ have an half-integer isobaric spin. Here we give a basic argument which binds the character of the internal irreducible vectors to the sum $l^++l^-$ which happens, even in the Nishijima–Gell-Mann scheme, to have the same integer or half-integer character as the spin.

Of course, the preceding selection rule leaves open several possibilities. For instance, the boson-isoboson states $\pi$, belonging to $D(1,0)$, and $K$, associated with $D(\frac{1}{2},\frac{1}{2})$, may be represented in external formalism either as belonging to $\mathfrak{D}(0,0)$, that is as scalars (as is usually assumed), or as belonging to $\mathfrak{D}(1,0)\oplus\mathfrak{D}(0,1)$, that is, vector mesons, according to some recent proposals.[8] Similarly, the baryon states obtained from three *e*-spin–*i*-spin units can belong to $\mathfrak{D}(\frac{1}{2},0)\oplus\mathfrak{D}(0,\frac{1}{2})$ and obey Dirac equation (which is the assumption generally made in quantum field theory), but they can also belong to $\mathfrak{D}(0,\frac{3}{2})\oplus\mathfrak{D}(\frac{3}{2},0)$ as in Rarita and Schwinger's theory or also, as proposed in a recent paper from two of us[9], to $\mathfrak{D}(\frac{1}{2},1)\oplus\mathfrak{D}(1,\frac{1}{2})$.

This question evidently remains open and should only be solved by detailed examination of the consequences of each assumption in the domain of interactions. In particular, we can return to the problem of $\Lambda$. We have seen $\Lambda$ does not find any place in the $D(1,\frac{1}{2})$ level, as in this level the values $0, -\frac{1}{2}, -\frac{1}{2}$ for $m^+, m^-, m'$ already characterize $\Sigma^0$. It was therefore proposed to localize $\Lambda$ in the $D(0,\frac{3}{2})$ level; but this raises some difficulties in interaction theory. Now we can associate it to the level $D(0,\frac{1}{2})$ where we have also the values $0, -\frac{1}{2}, -\frac{1}{2}$. A possible objection[9a] is that these values also characterize already the particle $\nu_\mu$. However, if we recall that each particle is also characterized by an external part, we can propose that $\nu_\mu$ corresponds simply to one *e*-spin–*i*-spin unit, while $\Lambda$ corresponds, like the $\Sigma$ and $Y$ baryons, to the fusion of three *e*-spin–*i*-spin units, in such a way that two of the *i*-spin units are opposite, so that we get (in agreement with the usual isobaric spin theory) an isobaric spin singlet lying in the internal level $D(0,\frac{1}{2})$; but here the three *e*-spin units can be added, so that we get an *e*-spin state lying in $\mathfrak{D}(1,\frac{1}{2})\oplus\mathfrak{D}(\frac{1}{2},1)$, [which would differentiate this state from $\nu_\mu$ which lies in $\mathfrak{D}(0,\frac{1}{2})\oplus\mathfrak{D}(\frac{1}{2},0)$] or combined in the same way which would yield $\mathfrak{D}(\frac{1}{2},0)\oplus\mathfrak{D}(0,\frac{1}{2})$.

We are now in a position to discuss a very important point, namely, the possibility of writing Lagrangians of bare particles invariant under the whole external-internal group $\mathfrak{L}_4\times G$. Clearly we can limit our discussion to the rest mass term. If we assume that all particles associated with a certain representation $D(l^+,l^-)$ have the same external state vector $\varphi_e(x_\mu)$ [the antiparticles being associated with the corresponding bra vector $\tilde{\varphi}_e(x_\mu)$], the way to build a scalar expression invariant under $\mathfrak{L}_4\times G$, is to associate those particles into multiplets and form a scalar with the product of two internal interaction vectors belonging to two charge-conjugated representations. For example in the case of the $D(\frac{1}{2},0)$ representation where the state vectors of the bare particles can be written

$$\Psi_{\text{electron}}=\varphi_e(x_\mu)Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,\tfrac{1}{2})(z_\mu{}^\pm), \tag{46}$$

$$\Psi_{\text{neutrino}}=\varphi_e(x_\mu)Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,\tfrac{1}{2})(z_\mu{}^\pm), \tag{47}$$

$$\Psi_{\text{positon}}=\varphi_e(x_\mu)Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,-\tfrac{1}{2})(z_\mu{}^\pm), \tag{48}$$

$$\Psi_{\text{antineutrino}}=\varphi_e(x_\mu)Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,-\tfrac{1}{2})(z_\mu{}^\pm), \tag{49}$$

we get a scalar term of the form

$$L_m=m\varphi_e{}^+(x_\mu)\gamma_4\varphi_e(x_\mu)\cdot\{-Z(\tfrac{1}{2},0,\tfrac{1}{2};-\tfrac{1}{2},0,-\tfrac{1}{2}),Z(\tfrac{1}{2},0,\tfrac{1}{2};\tfrac{1}{2},0,-\tfrac{1}{2})\}\times\begin{pmatrix}Z(\frac{1}{2},0,\frac{1}{2};\frac{1}{2},0,\frac{1}{2})\\ Z(\frac{1}{2},0,\frac{1}{2};-\frac{1}{2},0,\frac{1}{2})\end{pmatrix}+\text{H.c.} \tag{50}$$

This treatment can clearly be generalized to all our representations. For example, in the case of $D(1,0)$ the rest mass term will contain the term $A_{k'}A_{k'}$ (with $A_{k'}=\bar{\psi}Q_{k'}\psi$) which splits into a sum of antiparticle-particle terms. More generally, all rest mass terms will contain a sum of antiparticle-particle terms which build together the internal scalar associated with the considered representation. The construction of such scalar Lagrangians evidently implies that all particles belong-

[8] The association of *e*-spin 1 *and* 0 with $i_3=1$, $S=0$, $B=0$, that is, scalar and vector mesons with $i_3=1, 0, -1$, seems now experimentally established.

[9] J. P. Vigier and P. Hillon, J. Phys. Radium (to be published).

[9a] This is not true in Yukawa's proposal.

ing to the same multiplet have the same external waves and bare rest masses. It also shows our general Lagrangians are invariant under the charge transformation $Q_{op}$, fermionic charge transformations $S_3'$, and the two three-dimensional complex subgroups corresponding to isobaric spin and strangeness.

We conclude this brief discussion on external waves by four remarks:

(a) In all cases one can linearize the external wave equations associated with fermions by using the antiparticle wave functions according to Dirac's ideas. One can also evidently stick to second-order equations within the frame of Feynman and Gell-Mann's conception.[10]

(b) The parity operation $P$ has not the same mathematical meaning for external and internal waves, since it is not an automorphism of $\mathfrak{L}_4$. This explains, as we shall see, the difference between external and internal interaction Hamiltonians.

(c) It is a general consequence of any unified theory of elementary particles in terms of a realistic model that it leads to certain connections between the external properties (spin, mass, parity) and the internal properties of particles. In the case of the original simple relativistic rotator model, we discover that spin and isospin for each particle become either both integer or both half-integer; while empirically all strange particles but $\Xi$ and $N$, i.e., $K$, $\Lambda$, and $\Sigma$, have integer (half-integer) spin and half-integer (integer) isospin (although in Tiomno's assignments this can be avoided). This point has until now always been regarded as an essential objection to rotator models.[11] Evidently, this type of difficulty does not appear in any theory which introduces internal space merely as an independent abstract space. However, this difficulty does not occur in the present theory where $e^+$ is identified with the magnitude of isospin ($l^+=I$, so $m^+=I_3$), while the magnitude of spin has the same integer–half-integer property as $l^+ + l^-$. Thus, when $l^-$ is half-integer, spin and isospin take different integer–half-integer properties, and we see that such cases just represent the $\Lambda$, $K$, $\Sigma$ particles of our table.

(d) The $e$-spin–$i$-spin fusion scheme provides a rather strong selection on possible physical terms of (43), but leaves open many as yet unobserved possibilities. These might naturally correspond to unstable states but further restriction can be obtained for two reasons. The first is that we have only discussed free-particle theory until now, and we know that such a thing never exists in nature; so that interaction theory will furnish a further selection on the external states $\varphi_e(x_\mu)$ attached to our internal levels. The second reason, if it turns out that fermionic leptons and baryons are correctly described by Dirac waves ($\pi$ and $K$ bosons corresponding to scalar external waves) would be that in the fusion scheme the most stable states correspond to the lowest states, meaning that the fusion of two spin $\frac{1}{2}$ units leads to $D(0,0)$, the fusion of three units going [preferably to $D(1,\frac{1}{2})$, for example], into $D(\frac{1}{2},0)$ or $D(0,\frac{1}{2})$, and this both for $i$ spin and $e$ spin. This is reasonable, since it fits with our general argument that only the lowest states (with $l^+, l^- \leq 1$) are easily observed in nature.

[10] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[11] H. Yukawa (private communication).

## SECTION III

Until now we have only discussed bare particle theory. Clearly this is not sufficient, since no such thing as bare particles exist in nature; and our scheme was essentially built to try to understand elementary particle interactions and decays.

Two questions thus appear immediately:

(A) What is in such a scheme the connection between internal and external motion?

(B) Is it possible to justify in this scheme the way in which the internal symmetries of the Lagrangian formalism should be broken in order to explain the known interaction coupling constants and rest mass differences in the known particle multiplets. In group-theoretical language: "Is it possible to find a wider group $G'$ which contains the necessary symmetry and asymmetry properties to account for experimental evidence?"[12]

Let us first discuss question (A). Following Nataf, we first note that such a connection must appear in any scheme which associates isobaric spin with internal motion (whether this motion happens in physical space or not). This results from the very existence of the Nishijima–Gell-Mann formula $Q=I_3+\frac{1}{2}S+\frac{1}{2}B$, which shows that electric charge, which has evident consequences in the particle's external behavior in the presence of electromagnetic field, is related to internal motions.

The answer to question (A) can be given in our opinion in the way opened by a remarkable paper of Utiyama[15] whose essential results we shall now recall.

Utiyama first considers the Lorentz-invariant Lagrangian $L(Q,\partial_\mu Q)$ of a free field $Q(x_\mu)$ and assumes it to be invariant under another group $G_I$ (for instance,

[12] The problem of the breaking of the symmetries is one of the most difficult of the theory of elementary particles, as was recently emphasized by Sakurai. Even for the usual isobaric spin group, we know that the basic assumption of charge independence is only a good approximation, since the various components of each charge multiplet have not exactly the same rest mass. Although this is usually related to the auxiliary influence of electromagnetic couplings, this is a rather arbitrary statement and the situation is not absolutely satisfactory. But if one wants to introduce higher symmetries, that is, wider invariance groups, such as $O^4$,[13] $SO^7$,[14] or our group $G$, one has to account for more important splitting, for instance between the nucleons and $\Xi$ particles. This means that we must immediately break the assumed symmetry and implies that the arbitrairness of the invariance statement is very unsatisfactory.

[13] J. Schwinger, Phys. Rev. **104**, 1164 (1956).

[14] J. M. Souriau, Compt. Rend. **250**, 2807 (1960); J. Tiomno, Nuovo Cimento **6**, 69 (1957).

[15] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

the Pauli gauge group). Let us then introduce the infinitesimal transformation $\epsilon^a$ of the group $G_I$ such that $Q(x_\mu)$ becomes $Q(x_\mu)+T_aQ(x_\mu)\epsilon^a$, where $T_a$ are the transformation matrices for the $Q$ fields. These matrices satisfy the Lie relations $[T_a,T_b]=f_{ab}{}^cT_c$ where $f_{ab}{}^c$ denote the structure constants ($f_{ab}{}^c=-f_{ba}{}^c$) of the group $G_I$. If we further "extend" $G_I$ and define the group $G_I'$ in which these transformations will depend on the coordinates $x_\mu$ [so that the terms $\epsilon^a$ become arbitrary functions $\epsilon^a(x_\mu)$ of $x_\mu$], we see this determines the form of a general Lagrangian invariant under the "extended" group $G_I'$. Such a Lagrangian can only be obtained:

(1) if one introduces *a supplementary field* described by relativistic vectors $A_\mu{}^a(x_\mu)$, each of them corresponding to one of the parameters $\epsilon^a$ of the group $G_I$;

(2) if the initial Lagrangian is supplemented by *an interaction Lagrangian* (expressing the interaction of the original field $Q(x)$ and the new field $A_\mu(x)$ of the form $j_\mu{}^aA_\mu{}^a$, where $j_\mu{}^a$ is the current derived from the initial Lagrangian $L_0$, by substituting to the operator $\partial_\mu=\partial/\partial x_\mu$ the operator $\nabla_\mu=\partial_\mu-T_aA_\mu{}^a$.

(3) if the fields $A_\mu{}^a$ transform under the extended group $G_I'$ according to the law

$$\delta A_\mu{}^a=f_{bc}{}^aA_\mu{}^b\epsilon^c+\partial_\mu\epsilon^a(x), \tag{51}$$

the field equations for the free field $A_\mu{}^a$ being deduced from a Lagrangian $L_0(F_{\mu\nu}{}^a)$ satisfying the supplementary condition

$$\frac{\partial L_0}{\partial F_{\mu\nu}{}^a}\cdot f_{cb}{}^aF_{\mu\nu}{}^b=0,$$

with

$$F_{\mu\nu}{}^a=\partial_\mu A_\nu{}^a-\partial_\nu A_\mu{}^a-\tfrac{1}{2}f_{\mu\nu}{}^a(A_\mu{}^bA_\nu{}^c-A_\nu{}^bA_\mu{}^c). \tag{52}$$

Though very interesting in itself, this formalism presents an evident difficulty. As an example, let us treat the case of the electromagnetic field. Every free-particle Lagrangian is invariant under the Pauli gauge transformation: $\delta Q=-i\epsilon Q$, so that the application of Utiyama's formalism implies that all gauge-invariant fields have the same interaction with the Maxwell field $A_\mu$. Naturally, this is not true, so that one gets out of trouble by introducing the *a priori* assumption that particles have various coupling constants (electric charge) with the field. This amounts in ordinary particle theory to an arbitrary breaking of the Pauli gauge symmetry of the Lagrangian.

In our scheme, the situation is different if we make the fundamental assumption that the $G_I$ groups considered by Utiyama result from Lagrangian invariance under subgroups of our internal group $G$. For instance, let us treat the case of electron-neutrino doublet. Clearly, the corresponding Lagrangian is invariant under the operator $Q=J_3{}^++J_3{}^--S_3'$ which gives, applied to the total wave field, the infinitesimal transform

$$\begin{aligned}\epsilon Q_{\mathrm{op}}[\varphi_e(x)\,|\,\psi(z^+)]&=\varphi_e(x)\epsilon\begin{pmatrix}0&0\\0&-1\end{pmatrix}\psi(z^+),\\ \epsilon Q_{\mathrm{op}}[\varphi_e{}^+(x)\,|\,\bar{\phi}(z^+)]&=\varphi_e{}^+(x)\epsilon\bar{\phi}(z^+)\begin{pmatrix}0&0\\0&1\end{pmatrix}.\end{aligned} \tag{53}$$

As a consequence, Utiyama's extension to our present scheme of the internal Pauli gauge group $\epsilon\rightarrow\epsilon(x)$ leads to the correct splitting between charged and neutral particle; the latter ones having no coupling with the electromagnetic fields given by the theory. We thus correctly break the gauge symmetry, explain the electron-neutrino mass difference, as a consequence of the electron's electromagnetic self-energy, and also answer question (B) with respect to electromagnetic interactions.

### A. Strong Interactions

Now it is clear that we can use our internal irreducible interaction vectors and write as a global Hamiltonian a combination of antibaryon-baryon state functions. This combination is invariant under our internal group $G$, which plays the role of a gauge group for our global Hamiltonian. Then, extending this group to a local gauge $G'$ according to Utiyama's method, we obtain the corresponding set of interaction vectors. As our group $G$ is the direct product $SO_3{}^+\times SO_3{}^-\times SO_3'$ of three rotation groups, it yields three triads of relativistic vectors, the vectors of each triad being bound together by Yang-Mills equations with intervention of the structure constants of the rotation group. These interaction vector mesons, coupled in different ways with the various baryon fields, break the symmetry between the baryons of the same level according to Sakurai's ideas.[16] One thus sees that question (A) and question (B) are closely related in our scheme, and we can make the fundamental statement that the complete particle theory (including the breaking of the symmetries) are not invariant under $\mathfrak{L}_4\times G$, but under the group $\mathfrak{L}_4\times G'$, where $G'$ is the local gauge group, that is, Utiyama's extension of our internal group $G$.

If one accepts the preceeding ideas, this implies that in our model the external motions of particles (and their behavior in interactions) is essentially determined by their coupling with the preceding vector mesons, according to Sakurai's point of view.

In our scheme, the theory of strong interactions of fermions with bosons can thus be represented, as Fujii[17] first suggested, by graphs of the form indicated in Fig. 6. If we assume, for example, that the $B_1$ and $B_2$ baryons belong to the $D(1,\frac{1}{2})$ representation, the graph expresses the basic idea of our interaction formalism, namely: *There is no physical difference between the observed boson ($\pi$ meson) and the bound baryon–antibaryon state.* This binding evidently results from the exchange of the inter-

[16] J. J. Sakurai, Ann. Phys. (New York) **11**, 1 (1960).
[17] Y. Fujii, Progr. Theoret. Phys. (Kyôtō) **21**, 232 (1959).

action vector mesons introduced through Utiyama's formalism.

With these assumptions, we see that Sakurai's theory of strong interactions results naturally from our theory, with slight differences (resulting from the utilization of different internal gauge groups) which will be discussed in a subsequent paper. More generally, we can say that all interactions can be constructed along his (or Utiyama's) line of thought, as a result of the extension of some local gauge groups.

Indeed we can summarize our conceptions as follows:

(A) To each internal quantized attribute (bound in our model to new "hidden" internal kinematical variables) there correspond external dynamical features.

(B) Each conservation law results from the invariance of the Lagrangians under a particular gauge group which must be a particular subgroup of our general isobaric group $G$.

(C) Since all strong Lagrangians must be invariant under $\mathfrak{L}_4$ (the external Lorentz group) and $G$ (our internal isobaric group), the strong and weak interactions should result from specific Lagrangian invariance under definite subgroups of $G$.

(D) In the case of strong interactions we have shown that, with the help of our preceding interaction vectors, we can build scalars under $G$ which are also:

(a) invariant under the internal complex rotation group $SO_3{}^*$, that is, as is well known, under two separate irreducible representations $D(l^+)$ for any rotation $\Omega$, and $D(l^-)$ for the complex conjugate $\Omega^*$;

(b) invariant under the internal three-dimensional real rotation group $SO_3'$;

(c) invariant under the Abelian one-dimensional gauge group $Q_{\mathrm{op}}=J_3{}^+ + J_3{}^- - S_3'$.

These invariances imply:

(a) the conservation of isobaric spin $I_3$ through the intervention of three vector mesons $B_\mu{}^{(T)}$ (namely, the classical Yang-Mills field) with their coupling constant $f_T$ and the interaction Lagrangian $\mathfrak{L}_T=-f_T B_\mu{}^{(T)}\cdot J_\mu{}^{(T)}$;

(b) the conservation of strangeness through the intervention of three vector mesons $B_\mu{}^{(S)}$ with their coupling constant $f_B$ snd the interaction Lagrangian $\mathfrak{L}_S=-f_S B_\mu{}^{(S)}\cdot J_\mu{}^{(S)}$.

(c) the conservation of baryon (fermion) number through the intervention of the three vector mesons $B_\mu{}^{(B)}$ with their coupling constant $f_B$ and the Lagrangian $\mathfrak{L}_B=-f_B B_\mu{}^{(B)}\cdot J_\mu{}^{(B)}$.

(d) the conservation of electromagnetic charge through the intervention of a single vector meson $A_\mu$ with the usual interaction Lagrangian $\mathfrak{L}_{em}=-A_\mu J_\mu$.

With Sakurai, we can assume that $f_B \geq f_S \geq f_T$.

(E) The question of the masses of these fields (a stumbling block in Sakurai's theory) can be solved in our scheme according to one of us[18] by assuming that the three fields $B_\mu{}^k$ which satisfy the usual Yang-Mills bare field eauations, are in reality built out of the sum of a strongly fluctuating unobserved vacuum part $B_\mu{}^{k0}$ and a slowly varying effective observable part $b_\mu{}^k$. If one then considers average values over small space-time cells, one sees that this amounts to the existence of an "effective" mass for each observable $b_\mu{}^k$ field which results from the $B_\mu{}^{k0} b_\mu{}^k$ interaction: the total field satisfying the usual Yang-Mills equation and the total Lagrangian satisfying strictly our new gauge principle.

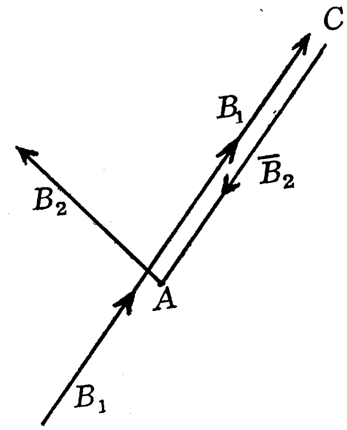



FIG. 6. Diagram for strong interaction of fermions with bosons.

(F) The above qualitative theory yields strong arguments for the elimination of some strong-reaction processes which correspond to mathematical possibilities (according to Sec. I) but are not observed in experiment. In accordance with the objection[18a] raised by Feynman at the Aix en Provence Conference, we can formally build invariant interaction Hamiltonians with lepton-antilepton (or lepton-antibaryon) pairs with creation of $\pi$ or $K$ mesons, a thing which never happens in nature. But in these processes we would have to deal at least with a $B_\mu{}^{(B)}$ exchange process in our scheme; that is, with a loss of mass of the order of 3000 electron mass units. This is not possible since we start in both cases from a much too small initial mass. Our external vector meson theory thus provides us with supplementary selection rules which complete in a very suitable way our internal interaction formalism. This is a very important consequence of our external formalism: since it forbids strong baryon-lepton transition and secures *the separate conservation of baryon and lepton number*, while our internal fomalism implies only the conservation of the quantum number $m'$.

One sees also that our scheme evidently leaves room for a similar treatment of weak interactions. Indeed if, recalling the results of Sec. I, we build [following one of us (P. H.)] Lagrangians invariant under the groups $SO_3$ and $SO_3'$ the invariance under $SO_3$ implies, according to Utiyama's formalism, the introduction of three new vector mesons $B_\mu{}^{(W)}$ which insure the conservation

[18] J. P. Vigier, Nuovo Cimento **23**, 1171 (1962).
[18a] Not applicable of course to Yukawa's proposal.

of $m^{+}+m^{-}$, that is, $I_3+\frac{1}{2}S$. These vector mesons, first introduced by Feynman and Gell-Mann,[10] will be responsible for weak interactions in our scheme. Their rest mass must be assumed to be of the order of 1200 electron masses. The corresponding interaction Lagrangians take the form $\mathfrak{L}_W=-f_W B_\mu^{(W)}\cdot J_\mu^{(W)}$; the currents $J_\mu^{(W)}$ contain the usual $\gamma_5$ matrices, according to Sakurai's[16] demonstration.

Let us emphasize that these conceptions provide us with a deeper, but qualitatively equivalent, interpretation of the strong interaction theory developed in Sec. I. The irreducible vectors built with the baryons allow us evidently to build up a collective Hamiltonian of the isolated baryon-antibaryon pairs, such as

$$\tilde{A}_{k'}\cdot A_{k'}=\bar{n}n+\bar{p}p+\bar{\Xi}^0\Xi^0+\bar{\Xi}^-\Xi^-+\bar{X}^+X^++\bar{X}^{++}X^{++},$$

which expresses simply the basic invariance of our whole free-particle formalism under the group $\mathfrak{L}_4\times G$ (the symbol $\bar{n}n$ recovers the internal-external Lagrangian, with derivative part, invariant under $\mathfrak{L}_4$).

As seen before, the assumption that the group $G$ must first be considered as a local gauge group, then extended to the corresponding enlarged Utiyama group $G'$, implies the introduction of the vector mesons and the corresponding interaction term to obtain extended Hamiltonians invariant under $\mathfrak{L}_4\times G'$. We are then led to the idea that, when a baryon and an antibaryon come near enough, they interact and build a stable edifice (a bound state) which appears as a boson. Of course the whole process is basically invariant under the group $\mathfrak{L}_4\times G$, so that we are compelled to adopt the statement which was primarily introduced as a principle in Sec. I: *Strong interactions are invariant under the internal group* $G$. The graph (Fig. 6) where vector mesons are assumed to act at the bound state branch, is indeed a more detailed, but equivalent, picture for the Yukawa graphs (Fig. 1) of Sec. I. Finally, the currents contained in the interaction terms to be added to the "bare" Hamiltonians introduce just the combinations of internal baryon functions which build the suitable internal boson functions; and this is just the mathematical relation between the present vector meson formalism and the internal formalism used in Sec. I.

As emphasized by Sakurai, this bound-state conception evidently implies a qualitative theory of the rest mass problem of elementary particles. If a vector meson is emitted by one of the baryons and absorbed by the other, it yields an attractive coupling which amounts to a mutual potential well. Such an attractive "exothermic" coupling entails a "loss of mass" which insures the stability of the created boson, and we can consider the corresponding mass difference as characterizing the coupling energy.

## CONCLUSION

In conclusion, we want to stress certain new aspects of our model and some of its consequences.

As stated in our introduction, the main difference between our ideas and the treatment of other authors is that we consider that the new quantum number of the Nishijima–Gell-Mann classification correspond to real physical periodic motions within the extended particles in Minkowski space. This is a new step since practically all former attempts are associated with abstract new spaces (four-dimensional with Euclidean metric, etc.) in order to preserve the point-like character of elementary particles.

Such a step is in line with the general idea of three of us[19] to re-interpret quantum mechanics with the help of new realistic (as yet "hidden") parameters. The rough idea is that there are no such things as points in space or instants in time but only space-like domains and time-like intervals. As a consequence, seemingly point-like structures at one level contain in reality an infinite number of field parameters out of which we can abstract a finite number of collective variables (which characterize a deeper level) governed by specific mechanical laws. For example, the new parameters utilized here correspond to motions happening within distances smaller than $10^{-13}$ cm so that it is not surprising we should discover for them new qualitative laws. This picture also implies that any seemingly stable structure at one level always recovers violent internal periodic space-time motions at a deeper level; so that the qualitatively different particles we observe, when we consider as points distances smaller than $10^{-13}$ cm, are really different quantized states of excitations of deeper field concentrations. In a crude sense we thus propose to make with respect to elementary particles the step made by quantum theory when it first attempted to explain the levels of the hydrogen atom.

The second point we want to make clear is that our model, like any other model, can only be compared with experiment if it yields a correct theory of experiments and accounts for the experimental mass spectrum. We purposely put the problems in that order, for a simple discussion shows that the mass problem should be attacked last in our scheme. The reason is that the observed masses are probably built of different elements. Clearly part of the masses result from "vacuum polarization" and cannot be calculated without a complete knowledge of interactions. Besides, one sees immediately that our model offers various new specific possibilities to account for the mass spectrum which require further investigation. For the moment, we shall leave aside this question until we achieve the complete analysis of interactions in our scheme.

Finally, we want to say a few words on possible future implications of this theory. If our starting point is correct, it clearly implies a research to explain the laws of quantization themselves in terms of topological and physical properties of deeper subquantum material

[19] L. de Broglie, D. Bohm, F. Halbwachs, P. Hillion, T. Takabayasi, and J. P. Vigier, preceding paper [Phys. Rev. 129, 438 (1963)].

behavior, possibly along the lines developed by two of us (D. B. and J.-P. V.). Moreover, subquantum properties which have already been used to justify the statistical laws of quantum mechanics[19] may also provide a justification of the "fusion" procedure as suggested by one of us.[6]

Anyway, it is clear that the substitution of relativistic extended rotators by point-like elements as the starting point of quantum theory implies deep modifications never considered before. For example, in all that precedes, it has not been necessary to introduce the dimensions of the particles themselves; but a more detailed analysis will have to do so in our scheme. If, as results from preliminary considerations, it turned out that the de Broglie wavelength were to be greater than the basic particle radius $r_0$, then $r_0$ would play the part of the fundamental length introduced in many modern attempts to eliminate the divergences of quantum field theory and introduce a natural "cutoff" in quantum mechanics.

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The authors would first like to express their gratitude to Professor Hideki Yukawa for his support of their investigations. Many of the preceding results have been inspired or had been anticipated by his research and that of his colleagues. As stressed before, our basic rotator model can be considered as a natural extension of Professor Yukawa's bilocal model,[20] the reasoning of Sec. II presenting evident analogies with Professor Sakata's[21] compound model of elementary particles.

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### APPENDIX. YUKAWA CLASSIFICATION

The model discussed in this paper and the preceding one can evidently be modified and improved in various ways. One of the most promising attempts to do so has been recently worked out by Yukawa.[22] We wish to indicate his results since they seem to fit nicely with very recent experimental data.

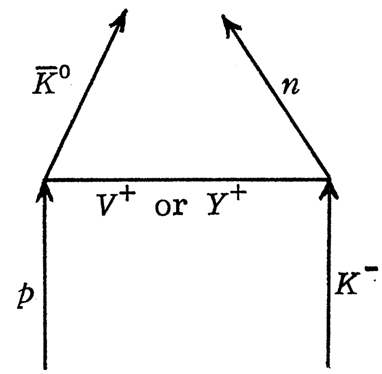


FIG. 7. Diagram for the process $K^- + p \to \bar{K}^0 + n$, with $V^+$ or $Y^+$ exchange.

Starting from our model, Yukawa remarks that our fundamental bilateral group $SO_3^+ \times SO_3^- \times SO_3'^+ \times SO_3'^-$ can be collapsed (as we have done) into

$$G = SO_3^+ \times SO_3^- \times SO_3',$$

but also into

$$G' = SO_3 \times SO_3'^+ \times SO_3'^-.$$

$G'$ is obtained by interchanging the primed and unprimed operators in $I$, an operation which amounts to interchange the role of $L$ and $T$.

Assuming then the same operators for $T_3$, $S$, and $B$, Yukawa proposes to associate baryons and bosons with the $D(l^+l^-)$ representations of $G$, the leptons corresponding to the representations $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$ of $G'$. In other terms, the Yukawa classification

(1) associates $D(\frac{1}{2},0)$ with the nucleons $N$ $(n,p)$;

(2) associates $\Lambda^0$ and a new particle $V^+$ with a strangeness doublet in $D(0,\frac{1}{2})$;

(3) associates higher baryons to the representations in Table I;

(4) defines the bosons as we have done by the representations $D(0,0)$, $D(1,0)$, $D(\frac{1}{2},\frac{1}{2})$, and $D(0,1)$, which correspond respectively to $\pi_0'$, pions, kaons, and possible strangeness 2 particles.[23]

In such a scheme, one can evidently introduce

TABLE I. Representations for various particles and resonances.

| Representation | $-m'=\frac{1}{2}B$ | $m^+=T_3$ | $m^-=\frac{1}{2}S$ | Particle |
|---|---|---|---|---|
| $D(\frac{1}{2},1)$, $S'=\frac{1}{2}$, | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $X^{++}$ |
| $m'=-\frac{1}{2}$ (particles) | $\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | $X^+$ |
| | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $p^*$ |
| | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $n^*$ |
| | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\Xi^0$ |
| | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\Xi^-$ |
| $D(1,\frac{1}{2})$, $s'=\frac{1}{2}$, | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $Y^{++}$ |
| $m'=-\frac{1}{2}$ (particles) | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $Y^+$ |
| | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $Y^0$ |
| | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\Sigma^+$ |
| | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\Sigma^0$ |
| | $\frac{1}{2}$ | $-1$ | $-\frac{1}{2}$ | $\Sigma^-$ |

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leptonic isobaric spin, leptonic strangeness, and leptonic number through the operators $J_3'^+$, $J_3'^-$, and $S_3=J_3^++J_3^-$, the fundamental leptons corresponding to $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$, namely $e^-$, $\nu_e$, $\mu^-$, and $\nu_\mu$.

These proposals of Yukawa present the following advantages:

(a) They explain directly the separate conservation of baryon and lepton numbers.

(b) They establish a simple and beautiful correspondence between the four "fundamental" baryon states of $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$, namely $n$, $p$, $\Lambda^0$, and $V^+$, and the four fundamental lepton states of $D'(\frac{1}{2},0)$ and $D'(0,\frac{1}{2})$, namely $e^-$, $\nu_e$, $\mu^-$, and $\nu_\mu$. This symmetry can be utilized as the starting point of a modified version of the Sakata model in which one utilizes four basic particles instead of three. In our case, as seen in II, all higher baryon states of $D(\frac{1}{2},1)$ and $D(1,\frac{1}{2})$ can be obtained as products of the eigenfunctions of $D(\frac{1}{2},0)$ and $D(0,\frac{1}{2})$.

(c) This correspondence is strengthened by the recent discovery of a second neutrino (Brookhaven), and the existence shown in Berkeley, of a 1480-MeV backward-scattering resonance in $K^-+p=K^0+n$; since, as Yukawa and one of us (J.-P. V.) have remarked, the graph of Fig. 7 evidently implies backward scattering as a result of $V^+$ or $Y^+$ exchange.

(d) They lead, following step by step (with the new group $G$) the work of Ohnuki,[24] Ne'eman,[25] and Gell-Mann,[26] to an "$n$-fold way" which also introduces the $\omega$, $\rho$, $K^*$ vector mesons. Such bosons could also have been predicted directly from the fusion scheme of Sec. II, since, with every representation $D(l^+,l^-)$ one can associate spin 0 or spin 1.

The corresponding strong- and weak-interaction theories will be discussed in subsequent papers.[22]

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# Translational Inertial Spin Effect

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The following results are shown: (a) Contrary to widespread belief, two energy-momentum tensors $T^{ij}$ and $\Theta^{ij}$ with a divergenceless difference are not necessarily physically equivalent; in fact, they will *not* be equivalent if the flux $\iiint(T^{i\alpha}-\Theta^{i\alpha})ds_\alpha dt$ through the external surface of some test body between an initial and a final state is nonzero. (b) It follows necessarily from basic postulates of the Dirac one-electron theory that Tetrode's asymmetrical energy-momentum tensor is physically the good one, and that, in the circumstances mentioned above, use of the symmetrized $\Theta^{ij}=(T^{ij}+T^{ji})/2$ tensor would yield a wrong result for the variation of the energy-momentum between states 1 and 2. (c) This being so, a macroscopic experiment based on ferromagnetism or ferrimagnetism can be devised, which demonstrates these facts as a measurable "translational inertial spin effect." (d) It is highly plausible that the above predictions, based on the one-particle electron theory, would be valid in the framework of the many-particle electron theory obeying Fermi statistics (the argument is based on the so-called bound-interaction hyperquantized formalism). The last point can be verified experimentally.

## I. INTRODUCTION

TWO energy-momentum tensors $T^{ij}$ and $\Theta^{ij}$ ($i, j, k, l=1, 2, 3, 4$; $x^1=x$, $x^2=y$, $x^3=z$, $x^4=ict$) are said to be *equivalent* if their difference is divergenceless:

$$\partial_j(T^{ij}-\Theta^{ij})=0. \tag{1}$$

This entails that the three-fold integral[1]

$$\iiint(T^{ij}-\Theta^{ij})du_j \tag{2}$$

is zero when taken over any closed domain, but not zero when taken over an open domain ($ic\epsilon^{ijkl}du_l=[dx^idx^jdx^k]$, 3-dimensional volume element; $\epsilon^{ijkl}$ is Levi-Civita's indicator).

One principal purpose of this note is to show how this remark yields the principle of physical experiments where mathematically equivalent energy-momentum tensors will not have physically equivalent behavior, so that (in the case we will consider) one of them may be selected as being, physically, "the good one."

The reason why such a fact has often been overlooked is that in a fairly large class of physical situations the values of the $T^{ij}$ tensors drop down at spatial infinity at a rate such that the integral (2), taken over any time-like domain at spatial infinity, is zero. When this is the case, the value of the integral (2) taken over any space-like domain[1] $S$ extending to infinity will be independent of

[1] To avoid confusion with the spin density $\sigma^i$, Schwinger's notations $d\sigma_j$ and $\sigma$ are discarded in favor of $du_j$ and $S$.