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Negative Pion Capture in Helium*†

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The capture rates of negative pions in He^4 into the three modes $t+n$, $d+2n$, and $p+3n$ are calculated phenomenologically, using a two-nucleon capture model. The amplitudes which appear in the phenomenological interaction are evaluated by a comparison with the pion production cross sections; and the capture rates are compared with experiment. The calculated ratio of the triton mode to all captures within the energy range observed by Schiff, Hildebrand, and Giese is 30%, which is in good agreement with the experimental ratio of 1/3.

1. INTRODUCTION

RECENTLY an experimental study of the capture of negative pions by helium was made by Schiff, Hildebrand, and Giese[1] which was in striking disagreement with an earlier work by Ammiraju and Lederman.[2] The reactions involved are

$$\begin{aligned} &\text{(a)} \quad \pi^- + \mathrm{He}^4 \to t+n, \\ &\text{(b)} \quad \pi^- + \mathrm{He}^4 \to d+2n, \qquad (1.1) \\ &\text{(c)} \quad \pi^- + \mathrm{He}^4 \to p+3n, \end{aligned}$$

where t, d, and p are tritons, deuterons, and protons Schiff, Hildebrand, and Giese, who studied the reactions in a hydrogen bubble chamber containing dissolved helium, found that the triton mode (1.1a) occurred in about 1/3 of all events whose prong ranges were between 5 and 110 mm (corresponding to proton energies of 7.2–38 MeV and deuteron energies of 9.9–53 MeV); whereas Ammiraju and Lederman, who reported a total of 60 events in a helium-filled diffusion cloud chamber, found that this mode occurred at most once among their events. Ammiraju and Lederman inferred that their result was in qualitative agreement with the two-nucleon capture model originally introduced by Brueckner, Serber, and Watson,[3] which assumes that the mechanism for pion capture is

$$\pi+N+N \to N+N. \qquad (1.2)$$

In a later theoretical paper, Ammiraju and Biswas[4] argue that the triton mode is very unlikely as a result of the two-nucleon capture model. In earlier quantitative calculations, based on a one-nucleon capture model, Petschek[5] predicted that it should occur in 22% of all events; and Clark and Ruddlesden[6] predicted a ratio of 3%. This difference in their results is mainly due to the fact that Petschek used a He^4 wave function of average kinetic energy 130 MeV, whereas the wave function used by Clark and Ruddlesden had an average kinetic energy of only 48 MeV; as a consequence, the He^4 wave function of Petschek is much larger at the kinetic energies of the triton mode. Indeed, substitution of the Clark and Ruddlesden wave function in Petschek's calculation resulted in a triton mode ratio of only 2%. However, these earlier calculations should perhaps not be taken too seriously, since the two-nucleon capture model has been quite well verified experimentally.[7]

Because of these experimental and theoretical inconsistencies, it was felt that a calculation of the capture rate of negative pions in He^4, using the two-nucleon capture model, would be of interest. In addition, the simplicity of the He^4 nucleus makes a quite complete analysis possible, and thus it affords an attractive opportunity to check the validity of the two-nucleon capture model.

In Sec. 2, a discussion of the phenomenological scattering matrix is given. The details of the calculations of the capture rates are found in Sec. 3; and in Sec. 4, results, comparison with experiment, and a discussion of the approximations used are given.

2. THE MATRIX ELEMENT FOR PION CAPTURE

The pion capture rates in helium were calculated phenomenologically.[8] The effective Hamiltonian has the

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[8] An identical approach was used to describe the inverse process of pion production by L. Wolfenstein, Phys. Rev. **98**, 766 (1955).

form

$$\mathcal{H}_{\text{eff}}=\int d^3x_1d^3x_2[\psi_N{}^\dagger(x_1)\psi_N{}^\dagger(x_2)\mathfrak{M}\psi_N(x_1)\psi_N(x_2)], \quad (2.1)$$

where $\psi_N(x)$ is the nucleon operator

$$\psi_N(x)=\begin{pmatrix}\psi_p(x)\\ \psi_n(x)\end{pmatrix}. \quad (2.2)$$

Assuming that the pions are captured from s states only,[9] and using a zero-range approximation, the most general form of $\mathfrak{M}$ will be

$$\begin{aligned}\mathfrak{M}=\textstyle\sum_\pm [g_0{}^\pm\tfrac{1}{2}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(1\pm P_{12}{}^\tau)\\ +g_1{}^\pm\tfrac{1}{2}(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\\ \times\tfrac{1}{2}(1\pm P_{12}{}^\sigma)]\delta(\mathbf{x}_1-\mathbf{x}_2), \quad (2.3)\end{aligned}$$

where $\boldsymbol{\phi}=\boldsymbol{\phi}(x_1)$ is the pion operator; $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ the isotopic spin operators of the two nucleons; $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ the spin operators of the two nucleons; $\mathbf{k}=-\frac{1}{2}i(\nabla_{N_1}-\nabla_{N_2})$ is the relative momentum of the two final-state nucleons; $P_{12}{}^\tau=\frac{1}{2}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ and $P_{12}{}^\sigma=\frac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ are operators which exchange the isospin and spin of the two nucleons; and $g_0{}^\pm$, $g_1{}^\pm$ depend only on the magnitude of $\mathbf{k}$; however, in the following, this $\mathbf{k}$ dependence will be assumed to be negligible, and $g_0{}^\pm$, $g_1{}^\pm$ will be treated as constants.

In the helium nucleus, which is the initial state under consideration, all of the nucleons are in s states, so that each pair of nucleons is in a spatially symmetric state. Therefore, the spin and isospin part of their wave function is antisymmetric, so that $\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\frac{1}{2}(1+P_{12}{}^\tau)$ and $\frac{1}{2}(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\frac{1}{2}(1+P_{12}{}^\sigma)$ contribute zero when acting on the initial state. Thus, the only terms which need be considered are (after a slight rearrangement)

$$\begin{aligned}\tfrac{1}{2}(1+P_{12}{}^\tau)\tfrac{1}{2}(1+P_{12}{}^\sigma)[g_0{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\\ +g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}](\mathbf{x}_1-\mathbf{x}_2). \quad (2.4)\end{aligned}$$

When the matrix multiplication of the isospin part of $\mathcal{H}_{\text{eff}}$ is carried out explicitly, and only the terms relevant to negative pion capture are retained, (2.1) becomes

$$\begin{aligned}\mathcal{H}_{\text{eff}}=-\sqrt{2}\int d^3\mathbf{x}_1d^3\mathbf{x}_2\,\delta(\mathbf{x}_1-\mathbf{x}_2)\phi^-\{[-\tfrac{1}{2}i(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{x}_2})\psi_n{}^\dagger(\mathbf{x}_1)\psi_n{}^\dagger(\mathbf{x}_2)]\tfrac{1}{2}(1+P_{12}{}^\sigma)\\ \cdot[g_0{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\psi_p(\mathbf{x}_1)\psi_n(\mathbf{x}_2)+[-\tfrac{1}{2}i(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{x}_2})\psi_p{}^\dagger(\mathbf{x}_1)\psi_n{}^\dagger(\mathbf{x}_2)]\tfrac{1}{2}(1+P_{12}{}^\sigma)\\ \cdot[g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\psi_p(\mathbf{x}_1)\psi_p(\mathbf{x}_2)\}. \quad (2.5)\end{aligned}$$

The first term in the curly brackets of (2.5) induces the reaction $\pi^-+p+n\rightarrow n+n$. If the initial nucleon state is ${}^1S_0$, the only available final state of correct parity is ${}^3P_0$, and if the initial state is ${}^3S_1$, the final state is ${}^3P_1$. It may be noted that $g_0{}^-$ is the amplitude for the transition $I=0\rightarrow I=1$, ${}^3S_1\rightarrow{}^3P_1$; and $g_1{}^-$ the amplitude for $I=1\rightarrow I=1$, ${}^1S_0\rightarrow{}^3P_0$. Similarly, the second term in (2.5) induces the reaction $\pi^-+p+p\rightarrow n+p$. Because of the exclusion principle, the only initial S state is ${}^1S_0$, and the final state available is ${}^3P_0$. As a result of charge independence, the amplitude for this transition is also $g_1{}^-$.

The evaluation of the matrix element $\langle f|\mathcal{H}_{\text{eff}}|i\rangle$ is carried out using a relationship given by Fock.[10] Consider the state vector $|\Phi\rangle$ of a system of N fermions

$$|\Phi\rangle=(N!)^{-1/2}\int\phi(\mathbf{x}_1\cdots\mathbf{x}_N)\psi^\dagger(\mathbf{x}_1)\cdots\times\psi^\dagger(x_N)|0\rangle d^3\mathbf{x}_1\cdots d^3\mathbf{x}_N. \quad (2.6)$$

It follows that

$$\psi(\mathbf{x})|\Phi\rangle=[N/(N-1)!]^{1/2}\int\phi(\mathbf{x},\mathbf{x}_1\cdots x_{N-1})\times\psi^\dagger(\mathbf{x}_1)\cdots\psi^\dagger(\mathbf{x}_{N-1})|0\rangle d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-1}. \quad (2.7)$$

Using (2.7) to evaluate the matrix element for pion capture in He^4, it is found that

$$\begin{aligned}\langle f|\mathcal{H}_{\text{eff}}|i\rangle=(6/m_\pi)^{1/2}\int d^3\mathbf{x}_1d^3\mathbf{x}_2d^3\boldsymbol{\xi}d^3\boldsymbol{\eta}\,\delta(\mathbf{x}_1-\mathbf{x}_2)\phi_\pi(\mathbf{x}_1)\{2[-\tfrac{1}{2}i(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{x}_2})\phi_f{}^*(\boldsymbol{\xi};\mathbf{x}_1,\mathbf{x}_2,\boldsymbol{\eta})]\tfrac{1}{2}(1+P_{12}{}^\sigma)\\ \cdot[g_0{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\phi_i(\boldsymbol{\xi},\mathbf{x}_1;\mathbf{x}_2,\boldsymbol{\eta})+[-\tfrac{1}{2}i(\nabla_{\mathbf{x}_1}-\nabla_{\mathbf{x}_2})\phi_f{}^*(\mathbf{x}_1;\mathbf{x}_2,\boldsymbol{\xi},\boldsymbol{\eta})]\tfrac{1}{2}(1+P_{12}{}^\sigma)\\ \cdot[g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\phi_i(\mathbf{x}_1,\mathbf{x}_2;\boldsymbol{\xi},\boldsymbol{\eta})\}, \quad (2.8)\end{aligned}$$

where $\phi_i(\xi_1,\xi_2;\xi_3,\xi_4)$ is the He^4 wave function of two protons with coordinates ξ_1, ξ_2, and two neutrons with coordinates ξ_3, ξ_4; $\phi_f(\xi_1;\xi_2,\xi_3,\xi_4)$ is the final-state wave function of one proton with coordinate ξ_1 and 3 neutrons; and $\phi_\pi(\mathbf{x}_1)$ is the pion wave function relative to

The function H, defined by Wolfenstein, is identical with $-(2)^{-1/2}g_0{}^-$.

[9] The rate of pion capture from p states is only a few percent of that from s states. See G. A. Snow, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 407; and also remarks by Bethe in same reference.

[10] V. Fock, Z. Physik **75**, 622 (1932); R. Becker and G. Liebfried, Phys. Rev. **69**, 34 (1946).

the center of mass of the He^4 nucleus. The spin indices have been suppressed throughout, but the notation is obvious: $\boldsymbol{\sigma}_j$ refers to the nucleon whose coordinate is $\mathbf{x}_j$.

The He^4 wave function may be written as a product of a spatial and spin wave function. Since both the neutron pair and the proton pair are in singlet states

$$\phi_i(1,2;\,3,4)=\phi_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\chi^0(1,2)\chi^0(3,4), \quad (2.9)$$

where χ^0 is a singlet spin wave function and $\phi_i(\mathbf{x}_1\cdots\mathbf{x}_4)$ is symmetric in all four variables.

The final-state wave function is antisymmetric in all three neutron variables. Therefore $\phi_f(1;\,2,3,4)$ may be written

$$\phi_f(1;\,2,3,4)=(6)^{-1/2}\sum_{\nu=1}^{6}(-)^{P_\nu}\times P_\nu F(x_1;\,x_2,x_3,x_4)\chi(1;\,2,3,4), \quad (2.10)$$

where P_ν is one of the six permutation operators of (234) and $(-)^{P_\nu}$ is the sign of the permutation.

After substituting (2.9) and (2.10) in (2.8) and using the symmetry of $\phi_i(\mathbf{x}_1\cdots\mathbf{x}_4)$, the expression for the matrix element simplifies to

$$\langle f|\mathcal{H}_{\text{eff}}|i\rangle=2\chi^\dagger(1,2,3,4)[(\mathbf{J}_4+P_{34}{}^\sigma\mathbf{J}_3+P_{24}{}^\sigma\mathbf{J}_2)(1+P_{23}{}^\sigma)\cdot\{g_0{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)+g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3)\}$$
$$+(\mathbf{K}_2-P_{23}{}^\sigma\mathbf{K}_3-P_{24}{}^\sigma\mathbf{K}_4)\cdot g_1{}^-\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]\chi^0(1,2)\chi^0(3,4), \quad (2.11)$$

where

$$\mathbf{J}_j=(m_\pi)^{-1/2}\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_4\,\delta(\mathbf{x}_k-\mathbf{x}_l)\phi_\pi(\mathbf{x}_k)[-\tfrac{1}{2}i(\boldsymbol{\nabla}_{\mathbf{x}_k}-\boldsymbol{\nabla}_{\mathbf{x}_l})F^*(\mathbf{x}_1;\,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)]\phi_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4),$$
$$\mathbf{K}_j=(m_\pi)^{-1/2}\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_4\delta(\mathbf{x}_1-\mathbf{x}_j)\phi_\pi(\mathbf{x}_1)[-\tfrac{1}{2}i(\boldsymbol{\nabla}_{\mathbf{x}_1}-\boldsymbol{\nabla}_{\mathbf{x}_j})F^*(\mathbf{x}_1;\,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)]\phi_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4), \quad (2.12)$$

where (jkl) is a cyclic permutation of (234).

As a result of the exclusion principle, which was used in the antisymmetrization of the final-state neutrons, the matrix element (2.11) contains terms corresponding to the capture of the pion by *each* pair of nucleons in the He^4 nucleus. Therefore, even though $g_0{}^-$ and $g_1{}^-$ are amplitudes for transitions of different total angular momentum in the "free" reaction $\pi+N+N\rightarrow N+N$, the capture rates will contain cross terms in $g_0{}^{-*}g_1{}^-$ because of interference between capture by different sets of pairs. Part of these cross terms would appear even in the absence of the exclusion principle, since they are due to interference between capture by a $p-p$ pair and an $n-p$ pair. Additional cross terms will appear in the triton and deuteron modes, because in these modes the final spin state is partially correlated, so that not all possible states contribute in the summation over final spin states.

3. CALCULATIONS OF CAPTURE RATES

A. The Mode $t+n$

The wave functions of the He^4 and H^3 nuclei were chosen so as to lead to the nucleon distributions measured by the Stanford electron scattering experiments. It was found that the nucleon distribution in the He^4 nucleus is well fitted by a Gaussian shape of rms radius $R=R_4=1.44\pm0.07$ F.[11] Therefore, the He^4 wave function was chosen to be

$$\phi_i(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=N_\alpha\exp[-\tfrac{1}{2}\lambda\sum_{i<j}(\mathbf{x}_i-\mathbf{x}_j)^2], \quad (3.1)$$

where $\lambda=9/(32R_4{}^2)$ and N_α is a normalization constant: $N_\alpha{}^2=V^{-1}(16\lambda^3/\pi^2)^{3/2}$, and where V is the normalization volume.

Similarly, electron scattering data on He^3 show that the nucleon distribution of the nucleus is well fitted by a Gaussian shape of rms radius $R=R_3=1.65\pm0.12$ F.[12] Assuming that the nucleon distribution in H^3 is the same as that in He^3, the H^3 wave function was chosen to be

$$\phi_t(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=N_t\exp[-\tfrac{1}{2}\lambda'\sum_{i<j}(\mathbf{x}_i-\mathbf{x}_j)^2], \quad (3.2)$$

where $\lambda'=(3R_3{}^2)^{-1}$ and $N_t{}^2=V^{-1}(3\lambda'^2/\pi^2)^{3/2}$.

The wave function for the state $t+n$ is given by (2.10) where we use (3.2) for the triton wave function and a plane wave state for the neutron

$$F^t(\mathbf{x}_1;\,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=(2V)^{-1/2}\phi_t(\mathbf{x}_1,\mathbf{x}_3,\mathbf{x}_4)\times\exp[i\mathbf{q}\cdot\tfrac{1}{3}(\mathbf{x}_1+\mathbf{x}_3+\mathbf{x}_4)+i\mathbf{p}\cdot\mathbf{x}_2],$$
$$\chi^t(1;\,2,3,4)=\chi^0(3,4)\chi(1,2)=\tfrac{1}{2}(1-P_{34}{}^\sigma)\chi(1,2,3,4), \quad (3.3)$$

where $\mathbf{q}$ is the triton momentum and $\mathbf{p}$ the neutron momentum; $\chi^0(3,4)$ is a singlet spin function; and $\chi(1,2)$ and $\chi(1,2,3,4)$ are arbitrary spin functions.

The wave function used for the pion was that of an s-state Bohr orbit of He^4. Since the variation of this wave function over the dimensions of the He^4 nucleus is very small, it was replaced by its value at the origin.

The capture rate into the mode $t+n$ was then found by substitution of (3.3) into (2.11) and (2.12), and using the "golden rule,"

$$W=2\pi\sum|\langle f|\mathcal{H}_{\text{eff}}|i\rangle|^2\delta(E_i-E_f)\rho_f. \quad (3.4)$$

[11] R. Hofstadter, Revs. Mod. Phys. 28, 214 (1956).

[12] R. Hofstadter (private communication).

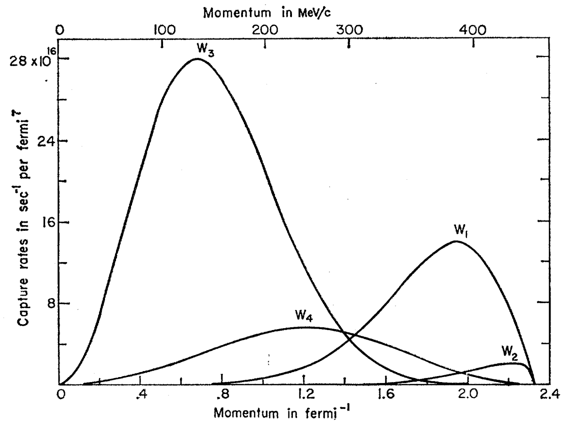

FIG. 1. The functions $W_i(q)$ which appear in the expression for the deuteron momentum spectrum (3.8).

A discussion of the integrals involved in this calculation, as well as the integrals involved in the calculation of the capture rates of the other modes will be found in Appendix A. The result is that the capture rate into the mode $t+n$ is

$$\begin{aligned} W_t = & |g_0^- + g_1^-|^2 N_t^2 N_\alpha^2 V^2 (n^3 m_\pi)^{-1} |\phi_\pi(0)|^2 2(6)^{1/2}\pi^5 \\ & \times m(m\Delta_t)^{3/2} \exp[-m\Delta_t/3(2\lambda+\lambda')] \\ & \times[(2\lambda+\lambda')(\lambda+\tfrac{3}{4}\lambda')]^{-3}[1+\tfrac{1}{2}\lambda'/(2\lambda+\lambda')]^2 \\ = & 9.68|g_0^- + g_1^-|^2 \times 10^{16} n^{-3}\ \mathrm{sec}^{-1}, \end{aligned} \quad (3.5)$$

where m is the nucleon mass, m_π the pion mass, $\Delta_t = 118.1$ MeV is the energy release in this reaction, n is effective radial quantum number of the Bohr orbit from which the pion is captured; and the g_i^- are given in F[4].

B. The Mode $d+2n$

The wave function for the state $d+2n$ was chosen as a product of plane waves for the free neutrons and a deuteron wave function. Thus, using (2.10),

$$\begin{aligned} F^d(\mathbf{x}_1;\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) & \\ = V^{-1}\phi_d(|\mathbf{x}_1-\mathbf{x}_3|) & \\ \times\exp[i\mathbf{p}_1\cdot\mathbf{x}_2+i\mathbf{p}_2\cdot\mathbf{x}_4+i\mathbf{q}\cdot\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_3)], & \quad (3.6) \\ \chi^d(1;2,3,4)=\chi^1(1,3)\chi(2,4)=\tfrac{1}{2}(1+P_{13}{}^\sigma)\chi(1,2,3,4), & \end{aligned}$$

where $\mathbf{q}$ is the deuteron momentum, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the neutron momenta; $\chi^1(1,3)$ is the triplet spin function of the deuteron, and $\chi(2,4)$ and $\chi(1,2,3,4)$ are arbitrary spin functions. The deuteron spatial wave function, $\phi_d(r)$ was chosen as[13]

$$\phi_d(r)=N_d[\exp(-0.232r)-\exp(-1.202r)]/r, \quad (3.7)$$

where N_d is a normalization constant, and r is given in fermis.

The momentum spectrum of the deuteron is found by integrating the capture rate of this mode over the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ only. The result is

$$\begin{aligned} dW_d(q)=dq\,[& (|g_0^-|^2+3|g_1^-|^2+2\,\mathrm{Re}g_0^{-*}g_1^-)W_1 \\ & +(|g_0^-|^2-|g_1^-|^2+2\,\mathrm{Re}g_0^{-*}g_1^-)W_2 \\ & +|g_0^-|^2W_3+2\,\mathrm{Re}g_0^{-*}g_1^-W_4], \end{aligned} \quad (3.8)$$

where the W_j are functions of q, and are shown graphically in Fig. 1. The function W_1 corresponds to captures in which one of the participating nucleons is bound in the deuteron; W_3 corresponds to those captures in which the free neutrons are the participants; W_2 is a cross term in the sense that it is due to interference between those cases in which one participant is bound in the deuteron, and the other participant is one or the other of the free neutrons—thus this term is entirely due to the antisymmetrization of the final state; similarly, W_4 is a cross term due to interference between the cases in which both free neutrons are participants and only one free neutron is a participant. It should be noted that the factor $2\,\mathrm{Re}g_0^{-*}g_1^-$ which multiplies W_1 is due to the fact that the final spin state is correlated—i.e., the deuteron has spin 1.

The total rate for capture into this mode is found by integrating over the deuteron spectrum. The result is that the total capture rate is

$$\begin{aligned} W_d=27.5[1.31|g_0^-|^2+|g_1^-|^2+1.19\,\mathrm{Re}g_0^{-*}g_1^-] & \\ \times10^{16}n^{-3}\ \mathrm{sec}^{-1}. & \quad (3.9) \end{aligned}$$

C. The Mode $p+3n$

Up to this point we have not taken final-state interactions into account because, in the modes $t+n$ and $d+2n$, the relative momenta of the final-state particles would seem to be sufficiently high so that the effect of final-state interactions would be negligible. However, the effect of final-state interactions is expected to be quite important in the mode $p+3n$, because in this case there are *two* bystanders of low relative momentum, which should interact quite strongly. Since the participants have high momenta, it would appear logical to choose a final-state wave function which is a product of plane waves for the participants and an interaction wave function for the bystanders. However, use of such a wave function would be inconsistent with the correct antisymmetrization, since the bystanders and participants exchange roles under antisymmetrization. The only consistent way of taking final-state interactions into account would seem to be to take a final-state wave function of four mutually interacting particles. Since this is rather difficult to carry out, the matrix element was first calculated using a product of plane waves as the wave function; then the relative wave function of the bystanders in the matrix element was replaced by the correct interaction wave function.

Thus, the matrix element was first calculated using the free-particle wave function

$$F^p(\mathbf{x}_1;\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=V^{-2}\exp(i\sum_1^4\mathbf{p}_j\cdot\mathbf{x}_j), \quad (3.10)$$

[13] M. J. Moravcsik, Nucl. Phys. 7, 113 (1958).

where the $\mathbf{p}_i$ are the momenta of the final-state nucleons. This wave function was substituted in the expression for the matrix element (2.11) and (2.12). Then, in the integral $\mathbf{J}_j$, for example, the bystanders are labeled by the indices $(1,j)$. The function $F^P(\mathbf{x}_1;\,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$ may be rewritten as

$$V^{-1}\exp(i\mathbf{p}_k\cdot\mathbf{x}_k+i\mathbf{p}_l\cdot\mathbf{x}_l)\times V^{-1}\exp[\tfrac{1}{2}i(\mathbf{p}_1+\mathbf{p}_j)\cdot(\mathbf{x}_1+\mathbf{x}_j)]\times\exp[\tfrac{1}{2}i(\mathbf{p}_1-\mathbf{p}_j)\cdot(\mathbf{x}_1-\mathbf{x}_j)]. \quad (3.11)$$

The s-wave part of the relative wave function of the bystanders (which is the only part which contributes to the integral) is $(pr)^{-1}\sin pr$, where $\mathbf{p}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_j)$, $\mathbf{r}=(\mathbf{x}_1-\mathbf{x}_j)$. This function is replaced by that interaction wave function $f_p(r)$ whose asymptotic form is $(pr)^{-1}\sin(pr+\delta)$. The correct interaction wave functions were found in the following way: The amplitude g_0^- describes capture from an $I=0$ state. Therefore the bystanders are also in an $I=0$ state, so that their configuration must be $^3S_1$; the corresponding $f_p(r)$ is the solution of the Schrödinger equation in a $^3S_1$ potential. Similarly, the amplitude g_1^- describes capture from an $I=1$ state, so that the bystanders will be in an $I=1$, $^1S_0$ configuration, with a corresponding $f_p(r)$. These wave functions were found by solving the Schrödinger equation with square well potentials, whose parameters were chosen to agree with the scattering lengths and effective ranges known from nucleon-nucleon scattering.[14]

The results of these calculations were as follows: The total capture rate for this mode in the absence of final state interactions was found to be

$$W_p(\text{free particles})=23.6[|g_0^-|^2+|g_1^-|^2+0.605\,\mathrm{Re}g_0^{-*}g_1]\times10^{16}n^{-3}\,\mathrm{sec}^{-1}. \quad (3.13)$$

After making the above corrections for final-state interactions, the total capture rate was found to be

$$W_p(\text{interacting bystanders})=23.6[0.19|g_0^-|^2+1.20|g_1^-|^2+0.29\,\mathrm{Re}g_0^{-*}g_1^-]\times10^{16}n^{-3}\,\mathrm{sec}^{-1}. \quad (3.14)$$

Comparison of (3.13) with (3.14) shows that the $^3S_1$ well causes an 81% reduction in the relevant term of the capture rate, whereas the $^1S_0$ well causes a 20% enhancement. The reason for the large reduction of the $^3S_1$ capture rate is that the state for which the bystanders are bound in a deuteron is implicitly included in the sum-over-states of the plane wave calculation, but does not, of course, appear in the final-state interaction calculation.

Because of the approximation method used, it is difficult to evaluate the effect of the cross terms. However, a plausible order-of-magnitude estimate is obtained by multiplying the coefficient of $\mathrm{Re}g_0^{-*}g_1^-$ in (3.13) by the square root of the product of the two correction factors for the direct terms. This leads to the cross term given in (3.14).

[14] M. J. Moravcsik, Ann. Rev. Nucl. Sci. **10**, 324 (1960). The corresponding square-well potential parameters are: $V_s=14.3$ MeV, $b_s=2.56$ F, $V_t=36.5$ MeV, $b_t=2.00$ F, where V is the potential depth and b the range of the potential.

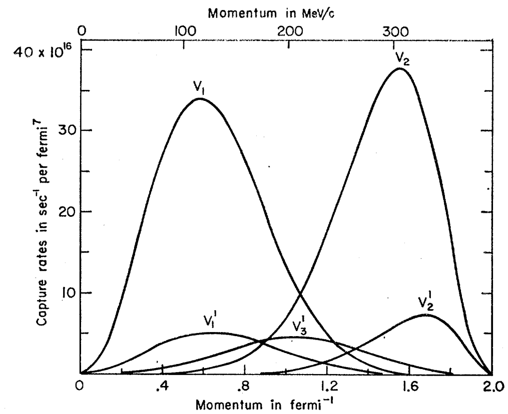

FIG. 2. The functions $V_i(p)$, $V_i'(p)$ which appear in the expression for the proton momentum spectrum (3.16). $V_3'(p)$ is the estimated function, as discussed in Appendix A.

The proton momentum spectrum is needed in order to be able to compare the ratios of the capture rates with experiment. However, since $f_p(r)$ is a function of the relative proton-neutron momentum, it is rather difficult to evaluate the absolute momentum spectrum of the proton accurately when final-state interactions are taken into account. Therefore, the spectrum was estimated simply by normalizing the various terms in the momentum spectrum calculated without interactions in the final state in such a way that the total rate agreed with (3.14).

When final-state interactions were neglected, the proton momentum spectrum was found to be

$$dW_p(p)=dp\,\{|g_0^-|^2V_1+\tfrac{1}{2}|g_1^-|^2\times[V_1+V_2-V_1'-V_2'+2V_3']+2\,\mathrm{Re}g_0^{-*}g_1^-[V_1'+V_3']\}, \quad (3.15)$$

where V_i and V_i' are functions of the momentum p, and are shown graphically in Fig. 2. The function V_1 corresponds to captures in which the participants are both neutrons; V_2 corresponds to captures in which the final-state proton is a participant; V_1' is the cross term due to interference between captures in which two different pairs of final-state neutrons are the participants; V_2' is the cross term due to interference between captures in which one of the participants is the final-state proton and the other participant is a different neutron in each case; and V_3' is the cross term due to interference between captures in which the final-state proton is a participant and those in which it is a bystander.

4. RESULTS AND DISCUSSIONS

In Appendix B the absolute squares of the amplitudes g_0^- and g_1^- are determined by a comparison of the experimental cross sections of the inverse reaction of pion production with the theoretical cross sections. It is found that

$$|g_0^-|^2=0.32\ \mathrm{F}^8;\quad |g_1^-|^2=0.29\ \mathrm{F}^8; \qquad (4.1)$$

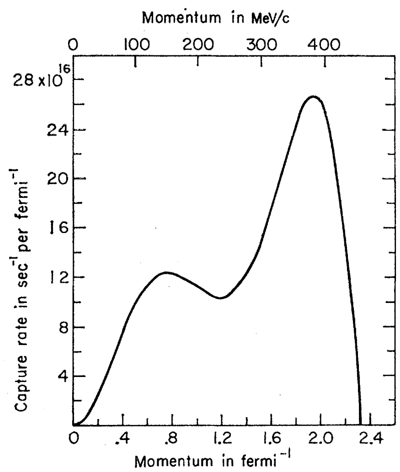

FIG. 3. Deuteron momentum spectrum.

and within experimental accuracy these amplitudes are equal in absolute value.

Under the assumption of time-reversal invariance, the phases of these amplitudes are identical with the nucleon-nucleon scattering phase shifts, i.e.,

$$\begin{aligned} g_0^- &= \pm|g_0^-|\exp[i\delta(^3P_1)]; \\ g_1^- &= \pm|g_1^-|\exp[i\delta(^3P_0)], \end{aligned} \qquad (4.2)$$

where $\delta(^3P_1)$ and $\delta(^3P_0)$ are the nucleon-nucleon phase shifts for c.m. energy 140 MeV. These phase shifts have not been unambiguously determined, but their difference is known to be small (about 20°),[14] so that g_0^- and g_1^- are, to a good approximation, relatively real. In order to determine the relative sign of the amplitudes, the total capture rates were calculated by substituting the values of $|g_0^-|^2$ and $|g_1^-|^2$ in (3.5), (3.9), and (3.15), and assuming that the difference in phase is either zero or 180°. For g_0^- and g_1^- relatively positive, the capture rates are

$$\begin{aligned} W_t &= 11.9\times10^{16}n^{-3}\ \mathrm{sec}^{-1}=22\%\ \text{of total}, \\ W_d &= 29.7\times10^{16}n^{-3}\ \mathrm{sec}^{-1}=56\%\ \text{of total}, \\ W_p &= 11.8\times10^{16}n^{-3}\ \mathrm{sec}^{-1}=22\%\ \text{of total}. \end{aligned} \qquad (4.3)$$

For g_0^- and g_1^- relatively negative, the results are

$$\begin{aligned} W_t &\approx 0, \\ W_d &= 9.4\times10^{16}n^{-3}\ \mathrm{sec}^{-1}=55\%\ \text{of total}, \\ W_p &= 7.6\times10^{16}n^{-3}\ \mathrm{sec}^{-1}=45\%\ \text{of total}. \end{aligned} \qquad (4.4)$$

The experimental result of Schiff *et al.* is that the $t+n$ mode occurs in about 1/3 of all captures in the energy range which they observed, so that it appears that (4.4) is inconsistent with experiment, and g_0^- and g_1^- are relatively positive; thus these amplitudes are equal within experimental accuracy. This is quite suggestive: Perhaps the nature of the pion-nucleon interaction is such that these amplitudes are required to be equal. In Appendix C it will be shown that, to the extent that the nucleon-nucleon interaction is due to the exchange of p-wave pions, this is, indeed, the case.

The deuteron and proton momenta spectra can now be found; they are given in Figs. 3 and 4. These spectra have double humps: the "slow" hump corresponds to captures in which the proton or deuteron is a bystander, and the "fast" hump corresponds to captures in which the proton or a nucleon bound in the deuteron is a participant.

In order to compare the results of this paper directly with the experimental results of Schiff *et al.*, the deuteron and proton spectra were integrated over the range of observed energies. Within these ranges, the calculated ratios of the three modes are as follows:

$$t+n:\ d+2n:\ p+3n=30\%:\ 54\%:\ 16\%, \qquad (4.5)$$

which is in good agreement with the experimental ratio of tritons to all captures of 1/3.

In addition, the deuteron and proton momenta spectra were transformed to range spectra by use of the range-energy relations in hydrogen; in Fig. 5, these range spectra are added and compared with the experimental range spectrum. The experimental and theoretical spectra appear to be in fair agreement.

The calculated ratio of captures in the triton mode to all captures does not depend very sensitively upon the assumed rms radius R_3 of the nucleon distribution in H^3. For example, an increase in R_3 from 1.65 F to 1.70 F causes a 10% decrease in the capture rate of the triton mode, and a decrease of R_3 to 1.60 F causes a 10% increase in the capture rate of this mode. However, the

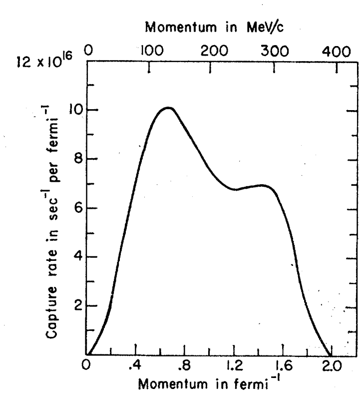

FIG. 4. Proton momentum spectrum, estimated as discussed in Sec. 3.

ratio of the capture rate of the triton mode to all captures changes by only 2%: for $R_3=1.70$ F, this ratio changes from 22% to 20%, and for $R_3=1.60$ F, the ratio becomes 24%.

This ratio is also very insensitive to the ratio of amplitudes $g_0^-/g_1^-=x$ (if it is assumed that the amplitudes are relatively positive). In fact,

$$\frac{W_t}{W_t+W_d+W_p}=\frac{9.68(1+x)^2}{50.2x^2+58.9x+65.5}. \quad (4.6)$$

The value of this ratio is 15% for $x=0$; 22% for $x=1$; and 19% for $x=\infty$. The ratio obtains its maximum value of 23% at $x=1.7$. All of these values agree fairly well with experiment. Thus, it is not possible at present to determine the ratio of the amplitudes from the experimental information of pion capture.

As it was pointed out in Sec. 2, cross terms in $g_0^{-*}g_1^-$ appear in the capture rates, even though g_0^- and g_1^- are amplitudes for transitions of different total angular momentum in the "free" capture $\pi+N+N\rightarrow N+N$. Although the cross term is most important in the triton mode, it is by no means negligible even for the sum of capture rates over all modes. Since these cross terms are due in large part to the exclusion principle, it appears that neglect of the exclusion principle is a rather poor approximation, at least in the case of very light nuclei, where the momentum of the bystander recoil nucleons is of the same order of magnitude as that of the participating nucleons. Brueckner, Serber, and Watson[3] calculated the total capture rate of pions by using a partial closure approximation, in which the final states of the bystander nucleons were summed, using the closure theorem. In order to use this method, one must neglect exchange effects between the participating nucleons and the bystanders, and so it would seem that the results of this approximation should be applied with some caution.[15]

It should be noted that although the ratios of the capture rates in the three modes are not very different from those given by Petschek, the results of this paper were obtained with the use of a He^4 wave function which did not differ significantly from that of Clark and Ruddlesden. The average kinetic energy of the He^4 wave function which was used is only 52 MeV, compared with 48 MeV for the wave function of Clark and Ruddlesden, and 130 MeV for the Petschek wave function.

At first sight the large ratio of the triton mode is rather puzzling in view of the low kinetic energy of the nucleons in the He^4 wave function which was used. It is, of course, a consequence of the requirement that the two nucleons responsible for the pion capture are correlated. This requirement, which was expressed as a δ function in configuration space—$\delta(\mathbf{x}_1-\mathbf{x}_2)$—becomes, in momentum space $\delta(\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')$ where $\mathbf{p}_1$, $\mathbf{p}_2$ ($\mathbf{p}_1'$,$\mathbf{p}_2'$) are the momenta of the participating initial (final) nucleons, and $\mathbf{q}$ is the pion momentum. This

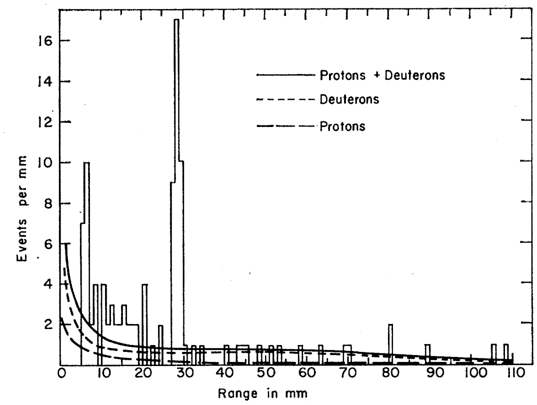

FIG. 5. Proton and deuteron range spectra; sum of these spectra; and histogram of events observed by Schiff, Hildebrand, and Giese (see reference 1). The total range spectrum is normalized to the number of events in the histogram. The experimental peak at 28 mm is due to the triton mode, which does not appear in the calculated curves.

condition is far less stringent than the condition which occurs in the absence of correlation, namely, $\delta(\mathbf{q}+\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')$. As a result of this latter condition, if $f(\mathbf{p}_1)$ is the probability that a nucleon in the He^4 nucleus have momentum $\mathbf{p}_1$, then the rate for the triton mode is proportional to $f(\mathbf{p}_1')$, where $\mathbf{p}_1'$ is the momentum of the free neutron (when the pion is absorbed from rest). Thus, if the He^4 wave function does not contain high momenta components, the rate for the triton mode is proportionately small. However the less stringent condition in the case of two-nucleon capture can be satisfied by low momenta components of the He^4 wave function, e.g., $|\mathbf{p}_1|=|\mathbf{p}_1'|-|\mathbf{p}_2|-|\mathbf{p}_2'|$.

The results of this paper are in complete disagreement with the qualitative arguments of Ammiraju and Biswas,[4] who concluded, on the basis of the two-nucleon capture model, that the deuteron and triton modes should be exceedingly rare. This difference between our results stems from their implicit assumption that the two-nucleon capture model implies that two fast nucleons must be ejected from the nucleus; the probability that one of these nucleons would then "stick" to the residual nucleus would be very small indeed, and furthermore, the fast nucleon would impart an extremely high excitation energy to the residual nucleus. However, the point of the two-nucleon capture model is not so much that the available momentum is shared by two nucleons, but that the capture of a pion proceeds primarily in the presence of two correlated nucleons. In fact, as was seen above, the pion, which is absorbed from rest, does not impart any momentum at all to the nucleons; the transition probability to a state of nucleons of given momenta is proportional to the

[15] The use of the closure approximation does not always necessitate the neglect of the Pauli principle. In the cases of muon capture and hypernuclear decay, the final state contains a neutrino or a pion in addition to nucleons; then the closure theorem may be used to sum over the entire nucleonic part of the final state, and exchange effects may be taken into account with respect to the nucleons. See, for example, R. H. Dalitz, Phys. Rev. 112, 605 (1958) for a discussion of the use of the closure approximation in hypernuclear decay.

probability that this distribution of momenta already exists within the nucleus. Thus, there is no reason to believe *a priori* that the two-nucleon capture model necessitates two fast nucleons in the final state.

ACKNOWLEDGMENT

It is a pleasure to thank Professor R. H. Dalitz for suggesting this topic, and for many helpful discussions.

APPENDIX A. DISCUSSION OF INTEGRALS USED IN CALCULATIONS

In this Appendix the integrals which appear in the evaluation of the matrix elements and in the integrations over phase space will be discussed.

First, in the mode $t+n$, if the wave function (3.3) is substituted in the expression for the matrix element (2.12), the result is

$$\mathbf{J}_2{}^t=\mathbf{K}_3{}^t=\mathbf{K}_4{}^t=0,$$

$$\begin{aligned}\mathbf{J}_3{}^t=-\mathbf{J}_4{}^t=\mathbf{K}_2{}^t &= (2Vm_\pi)^{-1/2}N_tN_\alpha\int d^3\mathbf{x}_1d^3\mathbf{x}_2d^3\mathbf{x}_3\,[\tfrac{1}{2}(\mathbf{p}-\tfrac{1}{3}\mathbf{q})+\tfrac{1}{2}i\lambda'(2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)]\phi_\pi(\mathbf{x}_2-\tfrac{1}{4}[2\mathbf{x}_2+\mathbf{x}_1+\mathbf{x}_3]) \\ &\quad\times\exp\{-\tfrac{1}{2}\lambda[2(\mathbf{x}_1-\mathbf{x}_2)^2+2(\mathbf{x}_2-\mathbf{x}_3)^2+(\mathbf{x}_1-\mathbf{x}_3)^2]\}\exp\{-\tfrac{1}{2}\lambda'[(\mathbf{x}_1-\mathbf{x}_2)^2+(\mathbf{x}_1-\mathbf{x}_3)^2+(\mathbf{x}_2-\mathbf{x}_3)^2]\} \\ &\quad\times\exp[-i\mathbf{q}\cdot\tfrac{1}{3}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3)-i\mathbf{p}\cdot\mathbf{x}_2].\end{aligned}\tag{A1}$$

Using the transformation:

$$\mathbf{r}=\tfrac{1}{2}\mathbf{x}_2-\tfrac{1}{4}(\mathbf{x}_1+\mathbf{x}_3),\quad \mathbf{s}=\mathbf{x}_1-\mathbf{x}_3,\quad \mathbf{t}=\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_3),\tag{A2}$$

and integrating over the variable $\mathbf{t}$, the expression for $\mathbf{J}_3{}^t$ simplifies to

$$\begin{aligned}\mathbf{J}_3{}^t &= (2Vm_\pi)^{-1/2}8N_tN_\alpha(2\pi)^3\delta(\mathbf{p}+\mathbf{q})\int d^3\mathbf{r}d^3\mathbf{s}\,\phi_\pi(r)[\tfrac{2}{3}\mathbf{p}+2i\lambda'\mathbf{r}]\exp[-4i\mathbf{p}\cdot\mathbf{r}/3]\exp[-8r^2(\lambda+\tfrac{1}{2}\lambda')-s^2(\lambda+\tfrac{3}{4}\lambda')] \\ &= \mathbf{p}(2Vm_\pi)^{-1/2}N_tN_\alpha\phi_\pi(0)(2\pi)^3\delta(\mathbf{p}+\mathbf{q})[\pi^2/\{(2\lambda+\lambda')(\lambda+\tfrac{3}{4}\lambda')\}]^{3/2}[1+\tfrac{1}{2}\lambda'/(2\lambda+\lambda')]\exp[-p^2/9(2\lambda+\lambda')],\end{aligned}\tag{A3}$$

where the pion wave function was replaced by its value at the origin, as discussed in the text. The phase space integrals for this mode are carried out completely by the use of the δ functions.

In the mode $d+2n$, when the wave function (3.6) is substituted in (2.12), the result is

$$\begin{aligned}\mathbf{J}_2{}^d=\mathbf{K}_4{}^d &= (m_\pi)^{-1/2}N_\alpha V^{-1}\int d^3\mathbf{x}_1d^3\mathbf{x}_2d^3\mathbf{x}_3\,\phi_\pi(\tfrac{1}{2}\mathbf{x}_2-\tfrac{1}{4}[\mathbf{x}_1+\mathbf{x}_3]) \\ &\quad\times[\tfrac{1}{2}(\mathbf{p}_2-\tfrac{1}{2}\mathbf{q})\phi_d(|\mathbf{x}_1-\mathbf{x}_2|)-\tfrac{1}{2}i(\mathbf{x}_2-\mathbf{x}_1)\phi_d'(|\mathbf{x}_1-\mathbf{x}_2|)/|\mathbf{x}_1-\mathbf{x}_2|] \\ &\quad\times\exp\{-\tfrac{1}{2}\lambda[2(\mathbf{x}_1-\mathbf{x}_2)^2+2(\mathbf{x}_2-\mathbf{x}_3)^2+(\mathbf{x}_1-\mathbf{x}_3)^2]\}\exp\{-i[\mathbf{p}_1\cdot\mathbf{x}_3+\mathbf{p}_2\cdot\mathbf{x}_2+\mathbf{q}\cdot\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)]\}, \\ \mathbf{J}_4{}^d(\mathbf{p}_1,\mathbf{p}_2) &= -\mathbf{J}_2{}^d(\mathbf{p}_2,\mathbf{p}_1)=-\mathbf{K}_2{}^d(\mathbf{p}_1,\mathbf{p}_2), \\ \mathbf{J}_3{}^d &= (m_\pi)^{-1/2}N_\alpha V^{-1}\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)\int d^3\mathbf{x}_1d^3\mathbf{x}_2d^3\mathbf{x}_3\phi_\pi(\tfrac{1}{2}\mathbf{x}_2-\tfrac{1}{4}[\mathbf{x}_1+\mathbf{x}_3])\phi_d(|\mathbf{x}_1-\mathbf{x}_3|) \\ &\quad\times\exp\{-\tfrac{1}{2}\lambda[2(\mathbf{x}_1-\mathbf{x}_2)^2+2(\mathbf{x}_2-\mathbf{x}_3)^2+(\mathbf{x}_1-\mathbf{x}_3)^2]\}\exp\{-i[(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{x}_2+\mathbf{q}\cdot\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_3)]\}, \\ \mathbf{K}_3{}^d &= 0.\end{aligned}\tag{A4}$$

Using the transformation (A2) and integrating over the variable $\mathbf{t}$ results in the simplified expression

$$\begin{aligned}\mathbf{J}_2{}^d &= (m_\pi)^{-1/2}4N_\alpha\phi_\pi(0)V^{-1}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\int d^3\mathbf{r}d^3\mathbf{s}\,[(\mathbf{p}_2-\tfrac{1}{2}\mathbf{q})\phi_d(|2\mathbf{r}-\tfrac{1}{2}\mathbf{s}|)-i(2\mathbf{r}-\tfrac{1}{2}\mathbf{s})\phi_d'(|2\mathbf{r}-\tfrac{1}{2}\mathbf{s}|)/|2\mathbf{r}-\tfrac{1}{2}\mathbf{s}|] \\ &\quad\times\exp\{-\lambda(8r^2+s^2)-i[2\mathbf{r}\cdot(\mathbf{p}_2+\tfrac{1}{2}\mathbf{q})-\tfrac{1}{2}\mathbf{s}\cdot(\mathbf{p}_1-\tfrac{1}{2}\mathbf{q})]\}.\end{aligned}\tag{A5}$$

Introducing the new variables

$$\mathbf{R}=2\mathbf{r}-\tfrac{1}{2}\mathbf{s},\quad \boldsymbol{\omega}=\tfrac{1}{3}(4\mathbf{r}+2\mathbf{s}),\tag{A6}$$

and integrating over the variable $\boldsymbol{\omega}$, (A5) becomes

$$\mathbf{J}_2{}^d=\mathbf{Q}\tfrac{1}{2}(m_\pi)^{-1/2}N_\alpha\phi_\pi(0)V^{-1}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})(2\pi/3\lambda)^{3/2}\exp[-p_1{}^2/6\lambda]J(Q),\tag{A7}$$

where $\mathbf{Q}=\frac{1}{3}(\mathbf{p}_2-\frac{1}{2}\mathbf{q})$ and

$$J(Q)=\int d^3\mathbf{R}\,[3\phi_d(R)-i\mathbf{Q}\cdot\mathbf{R}\phi_d'(R)/Q^2R]\exp[-\tfrac{4}{3}\lambda R^2-i\mathbf{Q}\cdot\mathbf{R}]. \tag{A8}$$

Integration of $J(Q)$ over angles, and integration of the $\phi_d'(R)$ term by parts, results in

$$J(Q)=16\pi Q^{-1}\int RdR\,\phi_d(R)\exp[-4\lambda R^2/3]\{\sin QR-(2\lambda/3Q^2)(\sin QR-QR\cos QR)\}. \tag{A9}$$

The function $J(Q)$ was then evaluated numerically.

The transformation of variables (A2) was also used to simplify $\mathbf{J}_3{}^d$. The result is

$$\mathbf{J}_3{}^d=(m_\pi)^{-1/2}N_\alpha\phi_\pi(0)V^{-1}\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})(\pi/2\lambda)^{3/2}\exp[-q^2/8\lambda]\int d^3\mathbf{R}\phi_d(R)\exp[-\lambda R^2]. \tag{A10}$$

The differential capture rate then becomes

$$\begin{aligned}dW_d=&[N_\alpha{}^2\phi_\pi{}^2(0)/m_\pi V](2\pi)^4\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\delta(\Delta_d-(p_1{}^2+p_2{}^2+\tfrac{1}{2}q^2)/2m)[V/(2\pi)^3]^3d^3\mathbf{p}_1d^3\mathbf{p}_2d^3\mathbf{q}\\ &\times\{(I_1{}^2\mathbf{K}^2+I_2{}^2\mathbf{Q}^2)(|g_0{}^-|^2+3|g_1{}^-|^2+2\,\mathrm{Re}g_0{}^{-*}g_1{}^-)+2(\mathbf{K}\cdot\mathbf{Q})I_1I_2(|g_0{}^-|^2-|g_1{}^-|^2+2\,\mathrm{Re}g_0{}^{-*}g_1{}^-)\\ &+\tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)^2I_3{}^2|g_0{}^-|^2+2\,\mathrm{Re}g_0{}^{-*}g_1{}^-[(\mathbf{p}_1-\mathbf{p}_2)\cdot(I_1\mathbf{K}-I_2\mathbf{Q})]I_3\},\end{aligned} \tag{A11}$$

where $\Delta_d=112.4$ MeV is the energy release of this reaction;

$$\mathbf{K}=\tfrac{1}{3}(\mathbf{p}_1-\tfrac{1}{2}\mathbf{q});\quad I_1=(2\pi/3\lambda)^{3/2}\exp(-p_2{}^2/6\lambda)J(K);\quad I_2=(2\pi/3\lambda)^{3/2}\exp(-p_1{}^2/6\lambda)J(Q);$$

and

$$I_3=2(\pi/2\lambda)^{3/2}\exp(-q^2/8\lambda)\int d^3\mathbf{R}\,\phi_d(R)\exp[-\lambda R^2].$$

The energy spectrum of the deuteron is found by integrating this expression over the neutron momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. The functions of the deuteron momentum W_i which appear in (3.8) are the following:

$$\begin{aligned}W_1(q)=&2q^2(2\pi/3\lambda)^3[N_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^4]\int d^3\mathbf{p}_1d^3\mathbf{p}_2\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\\ &\times\delta(\Delta_d-(\mathbf{p}_1{}^2+\mathbf{p}_2{}^2+\tfrac{1}{2}q^2)/2m)[KJ(K)]^2\exp(-p_2{}^2/3\lambda)\\ =&9q(2\pi/3\lambda)^3[mN_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^3]\exp(-m\Delta_d/2\lambda)\int_{L_-}^{L_+}K[KJ(K)\exp(3K^2/4\lambda)]^2dK,\end{aligned} \tag{A12}$$

where

$$L_\pm=\tfrac{1}{3}|q\pm(m\Delta_d-\tfrac{1}{2}q^2)^{1/2}|.$$

The right-hand side of (A12) was found by using the δ functions to integrate over $\mathbf{p}_2$ and over the angle between $\mathbf{K}$ and $\mathbf{q}$. The integral was evaluated numerically using the tabulated values of $J(K)$.

$$\begin{aligned}W_2(q)=&2q^2(2\pi/3\lambda)^3[N_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^4]\int d^3\mathbf{p}_1d^3\mathbf{p}_2\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\\ &\times\delta(\Delta_d-(p_1{}^2+p_2{}^2+\tfrac{1}{2}q^2)/2m)\mathbf{K}\cdot\mathbf{Q}J(K)J(Q)\exp[-(p_1{}^2+p_2{}^2)/6\lambda]\\ =&(2/9)(2\pi/3\lambda)^3[mN_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^3]\exp(-m\Delta_d/3\lambda)q^2(\tfrac{3}{2}q^2-m\Delta_d)(m\Delta_d-\tfrac{1}{2}q^2)^{1/2}\\ &\times\exp(q^2/12\lambda)\int_0^1 d(\cos\theta)J[\tfrac{1}{3}(A+B\cos\theta)^{1/2}]J[\tfrac{1}{3}(A-B\cos\theta)^{1/2}],\end{aligned} \tag{A13}$$

where

$$A=m\Delta_d+\tfrac{1}{2}q^2\quad\text{and}\quad B=2q(m\Delta_d-\tfrac{1}{2}q^2)^{1/2}.$$

The right-hand side of (A13) was found by transforming to the new variables $\mathbf{u}=\frac{1}{2}(\mathbf{K}+\mathbf{Q})$ and $\mathbf{v}=\frac{1}{2}(\mathbf{K}-\mathbf{Q})$; and then using the δ functions to integrate over $\mathbf{u}$ and the magnitude of $\mathbf{v}$. The values of the function J at the points $(A\pm B\cos\theta)$ were found by interpolation from the tabulated values of $J(K)$.

$$W_3(q)=2q^2(\pi/2\lambda)^3[N_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^4]\left[\int d^3\mathbf{R}\,\phi_d(R)\exp(-\lambda R^2)\right]^2$$
$$\times\int d^3\mathbf{p}_1d^3\mathbf{p}_2\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\delta(\Delta_d-(\mathbf{p}_1{}^2+\mathbf{p}_2{}^2+\tfrac{1}{2}\mathbf{q}^2)/2m)(\mathbf{p}_1-\mathbf{p}_2)^2\exp[-q^2/4\lambda]$$
$$=(\pi/\lambda)^3[mN_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^3]q^2(m\Delta_d-\tfrac{1}{2}q^2)^{3/2}\exp[-q^2/4\lambda]\left[\int d^3\mathbf{R}\,\phi_d(R)\exp(-\lambda R^2)\right]^2, \quad \text{(A14)}$$

$$W_4(q)=4q^2(\pi/\lambda)^3[N_\alpha{}^2\phi_\pi{}^2(0)V^2/3^{3/2}m_\pi(2\pi)^4]\left[\int d^3\mathbf{R}\,\phi_d(R)\exp(-\lambda R^2)\right]$$
$$\times\int d^3\mathbf{p}_1d^3\mathbf{p}_2\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{q})\delta(\Delta_d-(\mathbf{p}_1{}^2+\mathbf{p}_2{}^2+\tfrac{1}{2}\mathbf{q}^2)/2m)\mathbf{K}\cdot(\mathbf{p}_1-\mathbf{p}_2)J(K)\exp[-q^2/8\lambda-p_2{}^2/6\lambda]$$
$$=(12)^{1/2}(\pi/\lambda)^3[mN_\alpha{}^2\phi_\pi{}^2(0)V^2/m_\pi(2\pi)^3]\left[\int d^3\mathbf{R}\,\phi_d(R)\exp(-\lambda R^2)\right]q\exp[-(m\Delta_d+\tfrac{1}{2}q^2)/4\lambda]$$
$$\times\int_{L_-}^{L+}dK\,[KJ(K)\exp(3K^2/4\lambda)](\tfrac{1}{3}m\Delta_d-\tfrac{1}{2}q^2+3K^2). \quad \text{(A15)}$$

The right-hand side of (A15) was found [as in the case of (A12)] by using the δ functions to integrate over $\mathbf{p}_2$ and over the angle between $\mathbf{K}$ and $\mathbf{q}$.

In the mode $p+3n$, the calculation was first carried out for noninteracting final-state particles. In that case, when the plane wave function (3.10) was substituted in (2.12) the result was

$$\mathbf{J}_j{}^p=[N_\alpha\phi_\pi(0)/2m_\pi{}^{1/2}V^2](2\pi)^3\delta(\textstyle\sum\mathbf{p}_i)(\pi^2/2\lambda^2)^{3/2}\mathbf{J}_{lk},$$
$$\mathbf{K}_j{}^p=[N_\alpha\phi_\pi(0)/2m_\pi{}^{1/2}V^2](2\pi)^3\delta(\textstyle\sum\mathbf{p}_i)(\pi^2/2\lambda^2)^{3/2}\mathbf{J}_{j1}, \quad \text{(A16)}$$

where (jkl) is a cyclic permutation of (234) and

$$\mathbf{J}_{kl}=(\mathbf{p}_k-\mathbf{p}_l)\exp\{-[2(\mathbf{p}_k+\mathbf{p}_l)^2+(\mathbf{p}_m-\mathbf{p}_n)^2]/16\lambda\}, \quad \text{(A17)}$$

and $(klmn)$ is any permutation of (1234).

The differential capture rate for this mode is then found to be

dW_p(free particles)

$$=(2\pi)^4\delta(\textstyle\sum\mathbf{p}_i)\delta(\Delta-(\sum\mathbf{p}_i{}^2)/2m)[N_\alpha{}^2\phi_\pi{}^2(0)/m_\pi V^3](\pi^2/2\lambda^2)^3\prod_{j=1,\cdots,4}[Vd^3p_j/(2\pi)^3]$$
$$\times[(2|g_0{}^-|^2+|g_1{}^-|^2)(\mathbf{J}_{23}{}^2+\mathbf{J}_{24}{}^2+\mathbf{J}_{34}{}^2)+|g_1{}^-|^2(\mathbf{J}_{12}{}^2+\mathbf{J}_{13}{}^2+\mathbf{J}_{14}{}^2-\mathbf{J}_{12}\cdot\mathbf{J}_{13}-\mathbf{J}_{12}\cdot\mathbf{J}_{14}-\mathbf{J}_{13}\cdot\mathbf{J}_{14})$$
$$-(|g_1{}^-|^2-4\,\mathrm{Re}g_0{}^{-*}g_1{}^-)(\mathbf{J}_{23}\cdot\mathbf{J}_{24}+\mathbf{J}_{32}\cdot\mathbf{J}_{34}+\mathbf{J}_{42}\cdot\mathbf{J}_{43})+(|g_1{}^-|^2+2\,\mathrm{Re}g_0{}^{-*}g_1{}^-)$$
$$\times(\mathbf{J}_{21}\cdot\mathbf{J}_{23}+\mathbf{J}_{21}\cdot\mathbf{J}_{24}+\mathbf{J}_{31}\cdot\mathbf{J}_{32}+\mathbf{J}_{31}\cdot\mathbf{J}_{34}+\mathbf{J}_{41}\cdot\mathbf{J}_{42}+\mathbf{J}_{41}\cdot\mathbf{J}_{43})], \quad \text{(A18)}$$

where $\Delta=109.9$ MeV is the energy release in this reaction.

The proton momentum spectrum for free particles in the final state was found by integrating (A18) over the neutron momenta $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$. Because of the symmetric way in which the neutron momenta appear in the integrals, there are only five different terms appearing in the momentum spectrum. These are the functions V_i and V_i' discussed in Sec. 3. These functions are as follows:

$$V_1(p_1)=\alpha p_1{}^2\int\mathbf{J}_{kl}{}^2\delta(\textstyle\sum\mathbf{p}_j)\delta(\Delta-(\sum p_j{}^2/2m))d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4$$
$$=48(3)^{1/2}mp_1{}^2\alpha\exp(-p_1{}^2/3\lambda)(4\pi)^2R^6\int_0^1x^2(1-x^2)^{3/2}\exp(-3R^2x^2/2\lambda)dx,$$

where

$$R=R(p_1)=(\tfrac{1}{3}m\Delta-2p_1^2/9)^{1/2};\quad \alpha=[2N_\alpha^2\phi_\pi^2(0)V/m_\pi(2\pi)^7](\pi^2/2\lambda^2)^3 \quad \text{and} \quad k,\, l\neq 1,$$

$$V_2(p_1)=\alpha p_1^2\int \mathbf{J}_{1l}^2\delta(\textstyle\sum \mathbf{p}_j)\delta(\Delta-(\sum p_j^2/2m))d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4$$

$$=24(3)^{1/2}mp_1^2\alpha\exp[(2p_1^2/9-\tfrac{1}{2}m\Delta)/\lambda](4\pi)^2(\lambda/p_1)R^5\int_0^1 x(1-x^2)^{1/2}\exp(R^2x^2/2\lambda)$$

$$\times\{(4p_1x/3R)\cosh(2p_1Rx/3\lambda)+[x^2+(4p_1^2/9-2\lambda)/R^2]\sinh(2p_1Rx/3\lambda)\}dx,$$

$$V_1'(p_1)=\alpha p_1^2\int \mathbf{J}_{il}\cdot\mathbf{J}_{ik}\delta(\textstyle\sum \mathbf{p}_j)\delta(\Delta-(\sum p_j^2/2m))d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4,$$

where $i,\, l,\, k\neq 1$.

$$V_1'(p_1)=36(3)^{1/2}mp_1^2\alpha\exp[-(3R^2/8+p_1^2/3)/\lambda](4\pi)^2R^6\int_0^1 x^2(1-4x^2/3)(1-x^2)^{1/2}\exp(-3R^2x^2/4\lambda)dx,$$

$$V_2'(p_1)=\alpha p_1^2\int \mathbf{J}_{1l}\cdot\mathbf{J}_{1k}\delta(\textstyle\sum \mathbf{p}_j)\delta(\Delta-(\sum p_j^2/2m))d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4$$

$$=48(3)^{1/2}mp_1^2\alpha\exp[(5p_1^2/36-3m\Delta/8)/\lambda](4\pi)^2(\lambda/p_1)R^5\int_0^1 x(1-x^2)^{1/2}\exp(-R^2x^2/4\lambda) \tag{A19}$$

$$\times\{(2p_1x/3R)\cosh(p_1Rx/3\lambda)+[x^2-\tfrac{3}{4}+(4p_1^2/9-2\lambda)/R^2]\sinh(p_1Rx/3\lambda)\}dx,$$

$$V_3'(p_1)=\alpha p_1^2\int \mathbf{J}_{1l}\cdot\mathbf{J}_{kl}\delta(\textstyle\sum \mathbf{p}_j)\delta[\Delta-(\sum p_j^2/2m)]d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4.$$

It will be noticed that because of symmetry

$$V=\int V_1dp_1=\int V_2dp_1,$$
$$V'=\int V_1'dp_1=\int V_2'dp_1=\int V_3'dp_1. \tag{A20}$$

The simplified forms of V_1, V_1', V_2, and V_2' were obtained by using the 3-dimensional δ function to integrate over $\mathbf{p}_4$, say, and then using the transformation of variables:

$$\boldsymbol{\xi}=\tfrac{1}{2}(\mathbf{p}_2+\mathbf{p}_3+2\mathbf{p}_1/3);\quad \boldsymbol{\eta}=(\mathbf{p}_2-\mathbf{p}_3). \tag{A21}$$

The remaining δ function was then used to integrate over the magnitude of $\boldsymbol{\eta}$; and an added integration was carried out over the angle between $\mathbf{p}_1$ and $\boldsymbol{\xi}$. The remaining integration (which is essentially over the magnitude of $\boldsymbol{\xi}$) was then carried out numerically.

The integral V_3' could not, however, be simplified in such a manner. Although it was possible to reduce it to a single integral, the resulting integrand and its limits of integration were of a quite complicated nature, and it was felt that it could be estimated sufficiently well by an approximation procedure. Since V_3' is a cross term between capture by a (pp) pair and capture by a (pn) pair, it was approximated by

$$V_3'(p)\approx\tilde{V}_3'(p)=\frac{V'[V_1(p)V_2(p)]^{1/2}}{\int[V_1(p)V_2(p)]^{1/2}dp}. \tag{A22}$$

In Fig. 2, it is this approximated function, $\tilde{V}_3'$, which is shown.

In making the correction for interacting bystanders, as was discussed in Sec. 3, the relative wave function of the bystanders in the matrix element was replaced by an interaction wave function. This has the effect of replacing the factor $\exp[-(p_m-p_n)^2/16\lambda]$ in $\mathbf{J}_{kl}$ by

$$(\lambda/\pi)^{3/2}\int\exp(-\lambda r^2)f_p(r)d^3\mathbf{r}=I(p). \tag{A23}$$

The direct terms in the total capture rate for this case were then relatively simple to evaluate:

$$V\rightarrow\alpha m2^{1/2}(4\pi)^2\int\rho^6I^2(p)p^2dp\times\int_0^1 x^2(1-x^2)^{3/2}\exp(-\rho^2x^2/4\lambda)dx, \tag{A24}$$

where $\rho^2=2(m\Delta-p^2)$.

The simplified form of this integral was found by using the transformation of variables:

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{p}_m-\mathbf{p}_n),\quad \mathbf{p}'=\tfrac{1}{2}(\mathbf{p}_k-\mathbf{p}_l),$$
$$\mathbf{k}=(\mathbf{p}_m+\mathbf{p}_n),\quad \mathbf{k}'=(\mathbf{p}_k+\mathbf{p}_l), \tag{A25}$$

and using the δ functions to integrate over $\mathbf{k}'$ and the magnitude of $\mathbf{p}'$. The x integration which remains is essentially over the magnitude of $\mathbf{k}$.

APPENDIX B. EVALUATION OF g_0^- AND g_1^-

The amplitudes g_0^- and g_1^- were evaluated by calculating the cross sections of the inverse production reactions

$$\text{(a)}\quad p+p\rightarrow \pi^0+p+p,$$
$$\text{(b)}\quad p+p\rightarrow \pi^++p+n, \tag{B1}$$

and comparing the calculated values with the experimental cross sections. In order to do this, the effective Hamiltonian which describes the production reactions must be determined. Part of this Hamiltonian has already been given by (2.1), which, of course, describes s-state pion production with the nucleonic system undergoing $S\rightarrow P$ transitions. Since there are two identical protons in the initial state, the channel which is described by (2.1) is ${}^1S_0\rightarrow {}^3P_0$, so that only the g_1^- term contributes to this part of the cross section. Stallwood *et al.*[16] have fitted the experimental cross section to a power series in the maximum possible pion momentum η. The cross section of the ${}^1S_0\rightarrow {}^3P_0$ channel is proportional to η^6, but the calculated value cannot be compared directly with the η^6 term in the experimental cross section, because the ${}^1D_2\rightarrow {}^3P_2$ transition also contributes to this term. In addition, the experimental value of the η^6 term is ambiguous, even with respect to sign. Therefore the matrix (2.1) cannot be used at present to deduce the values of the amplitudes g_0^- and g_1^- directly from a knowledge of the production cross sections. Thus, it is necessary to find the terms which describe other production channels, in particular the $P\rightarrow S$ nucleonic transitions.

These terms in the effective Hamiltonian were found by noting that

$$\mathcal{H}_{\text{eff}}=T\mathcal{H}_{\text{eff}}^\dagger T^{-1}, \tag{B2}$$

where T is the time-reversal operator. This relationship may be proved by recalling that

$$\mathcal{H}_{\text{eff}}=V+V\frac{1}{E-H_0+i\epsilon}V +V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V+\cdots, \tag{B3}$$

from which (B2) follows immediately if V is invariant under time reversal.

Therefore, the effective Hamiltonian must contain a term which is the time-reversed Hermitian conjugate of (2.1). Since the field operators transform under time reversal as

$$\phi \xrightarrow{T} \eta_T\phi,\quad \psi\xrightarrow{T} i\sigma_y\eta_T'\psi, \tag{B4}$$

where η_T, η_T' are phase factors of amplitude 1, it is found that the time-reversed Hermitian conjugate of (2.1) is given by

$$\int d^3x_1d^3x_2\,\eta_T^*[\psi_N^\dagger(x_1)\psi_N^\dagger(x_2)\mathfrak{M}_T^\dagger\psi_N(x_1)\psi_N(x_2)], \tag{B5}$$

where

$$\mathfrak{M}_T^\dagger=\sum\nolimits_\pm [g_0^\pm\tfrac{1}{2}(1\pm P_{12}^\tau)\tfrac{1}{2}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}^\dagger\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k} +g_1^\pm\tfrac{1}{2}(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}^\dagger\tfrac{1}{2}(1\pm P_{12}^\sigma) \times\tfrac{1}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}]\delta(\mathbf{x}_1-\mathbf{x}_2),$$

where $\mathbf{k}$ is now the relative momentum of the two initial state nucleons.

It is clear that (B5) refers to $P\rightarrow S$ nucleonic transitions, with pion production in the s state; the g_0^- term describes $S=1\rightarrow S=1$ isospin-flip nucleonic transitions, and the g_1^- term describes $I=1\rightarrow I=1$ spin-flip transitions. It was therefore possible to determine g_1^- by using (B5) to calculate the cross section of reaction (B1a), $p+p\rightarrow\pi^0+p+p$ in the ${}^3P_0\rightarrow{}^1S_0$ channel. This part of the total cross section, which is proportional to η^2, was equated to the η^2 term in the experimental cross section.[17] In order to evaluate g_0^-, it is of course necessary to consider an isospin flip transition. Such a reaction is conveniently available, namely,

$$p+p\rightarrow \pi^++d. \tag{B1b$'$}$$

Crawford and Stevenson[17] have fitted the experimental cross section of this reaction to a power series in the pion momentum η_D. The part of the cross section due to the ${}^3P_1\rightarrow{}^3S_1$ transition is proportional to η_D. This part was calculated by using (B5), and g_0^- was found by equating the calculated expression to the η_D term in the experimental cross section.

The evaluation of the matrix elements of the production reactions was carried out in an analogous fashion to the calculations of Sec. 2. The wave function used for the initial state of two fast protons was an antisymmetrized plane wave function

$$\psi_i(1,2)=(2^{1/2}V)^{-1}[\exp(i\mathbf{k}_1\cdot\mathbf{x}_1+i\mathbf{k}_2\cdot\mathbf{x}_2)\chi_i(1,2) -\exp(i\mathbf{k}_2\cdot\mathbf{x}_1+i\mathbf{k}_1\cdot\mathbf{x}_2)\chi_i(2,1)], \tag{B6}$$

where $\chi_i(1,2)$ is an arbitrary spin wave function. In the final state, interactions between the nucleons cannot, of course, be neglected, because in order to obtain the values of the amplitudes corresponding to absorption of pions from rest, the production reactions must be considered at energies close to threshold. Therefore, even in the π^++p+p final state, the ${}^1S_0$ potential well must be taken into account because of the low energies of the

[16] R. A. Stallwood, R. B. Sutton, T. H. Fields, J. G. Fox, and J. A. Kane, Phys. Rev. **109**, 1716 (1958).

[17] F. S. Crawford and M. L. Stevenson, Phys. Rev. **97**, 1305 (1955). The results of this paper are consistent with the observations of R. Durbin, H. Loar, and J. Steinberger, *ibid.* **84**, 581 (1951); H. L. Stadler, *ibid.* **96**, 496 (1954); M. G. Mescerjakov, B. S. Neganov, N. P. Bogacev, and V. M. Siderov, Doklady Akad. Nauk S.S.S.R. **100**, 677 (1955); M. G. Mescerjakov, N. P. Bogacev, and B. S. Neganov, Suppl. Nuovo Cimento **3**, 120 (1956); and T. H. Fields, J. G. Fox, J. A. Kane, R. A. Stallwood, and R. B. Sutton, Phys. Rev. **109**, 1704 (1958).

protons. The final-state wave functions used were

$$\psi_f = V^{-1}\phi(|\mathbf{x}_1-\mathbf{x}_2|)\exp[i\mathbf{Q}\cdot\tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)+i\mathbf{q}\cdot\mathbf{x}_1] \times\tfrac{1}{2}(1\pm P_{12}{}^{\sigma})\chi_f(1,2), \quad \text{(B7)}$$

where $\phi(r)$ is the relative wave function of the two nucleons; $\mathbf{Q}$ is the c.m. momentum of the nucleons; $\mathbf{q}$ is the pion momentum; and the $\pm$ sign corresponds to the triplet or singlet spin state of the nucleons.

It was found that for the ${}^3P_0 \to {}^1S_0$ channel of reaction (B1a), $p+p\to\pi^0+p+p$,

$$\tfrac{1}{4}\sum|\mathcal{H}_{\text{eff}}|^2 = |g_1^-|^2(m_\pi V^3)^{-1}(2\pi)^3\delta(\mathbf{q}+\mathbf{Q}) \times\mathbf{k}^2|\phi(0)|^2. \quad \text{(B8a)}$$

The value of the relative wave function of the final-state protons $\phi(r)$ at the origin was found by the solution of the Schrödinger equation in a ${}^1S_0$ square-well potential, with the parameters given in footnote 14.[18] The result is

$$|\phi(0)|^2 dn_p = \frac{d^3p}{(2\pi)^3}\frac{p^2+mV_s}{p^2+mV_s\cos[b_s(p^2+mV_s)^{1/2}]} \approx mV_s dp/(2\pi^2), \quad \text{(B9a)}$$

where $\mathbf{p}$ is the internal momentum of the $p-p$ system, and dn_p is the number of states with momentum between p and $p+dp$.

For the ${}^3P_1\to{}^3S_1$ transition of reaction (B1b′) $p+p\to\pi^++d$, it was found that

$$\tfrac{1}{4}\sum|\mathcal{H}_{\text{eff}}|^2 = 2|g_0^-|^2(m_\pi V^3)^{-1}(2\pi)^3 \times\delta(\mathbf{q}+\mathbf{Q})\mathbf{k}^2|\phi_d(0)|^2. \quad \text{(B8b)}$$

The deuteron wave function at the origin was found in a manner analogous to the $p-p$ wave function, although this means that g_0^- may not be determined as accurately as it would have been had a better wave function been used. However, the value of the ratio $|g_0^-|^2/|g_1^-|^2$ will probably be more accurate because the same type of approximation is used in both cases. If B is the deuteron binding energy, then

$$|\phi_d(0)|^2 = (2\pi)^{-1}(mB)^{1/2}m(V_t-B)[1+B/(V_t-B)] \times[1+b_t(mB)^{1/2}-B/(V_t-B)]^{-1}. \quad \text{(B9b)}$$

The cross section for the ${}^3P_0\to{}^1S_0$ transition of $p+p\to\pi^0+p+p$ was found to be

$$|g_1^-|^2V_s m^3 m_\pi{}^2\eta^2/32(2)^{1/2}\pi^2, \quad \text{(B10)}$$

where η, the maximum available pion momentum, is given in units of the pion mass; and the cross section of the ${}^3P_1\to{}^3S_1$ transition of $p+p\to\pi^++d$ is

$$|g_0^-|^2(V_t-B)m^3m_\pi(m_\pi B)^{1/2}(2\pi^2)^{-1}[1+B/(V_t-B)] \times[1+b_t(mB)^{1/2}-B/(V_t-B)]^{-1}\eta_D, \quad \text{(B11)}$$

where η_D, the pion momentum, is again given in units of the pion mass.

The experimental value of the coefficient of the η^2 term in the $p+p\to\pi^0+p+p$ cross section is 25 μb to within about 50%[16]; and the coefficient of the η_D term in the $p+p\to\pi^++d$ experimental cross section is 138 μb.[17] Using these values it was found that

$$|g_0^-|^2=0.32\ \text{F}^8, \quad |g_1^-|^2=0.29\ \text{F}^8. \quad \text{(B12)}$$

It is clear that the amplitudes are equal in absolute value within experimental accuracy.

APPENDIX C

In Sec. 4, it was seen that the amplitudes g_0^- and g_1^- were equal within experimental accuracy. In this Appendix it will be shown that these amplitudes are predicted to be equal, within the approximation that only corrections due to the exchange of p-wave pions between the nucleons are included. The limitations of this approximation will be discussed briefly at the end of this Appendix.

The basic interaction between pions and nucleons is of the form $\boldsymbol{\sigma}\cdot\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi}$. In order to maintain Galilean invariance, an additional term of the form $\boldsymbol{\sigma}\cdot(\mathbf{p}_i+\mathbf{p}_f)\boldsymbol{\tau}\cdot\boldsymbol{\phi}$ (where $\mathbf{p}_i$ and $\mathbf{p}_f$ are the initial and final momenta of the nucleon) must be present in the interaction. It is this latter term which is responsible for s-wave pion absorption (emission).

Consider, first, the exchange of p-wave pions between the nucleons. In any order of perturbation theory there will be two closely related matrix elements corresponding to pion absorption by two nucleons. In Fig. 6, (a) will correspond to a matrix element of the form

$$M_1\propto[u^\dagger(p_1')\sigma_{i_1}\tau_{j_1}\cdots\sigma_{i_\nu}\tau_{j_\nu}\boldsymbol{\sigma}\cdot(\mathbf{p}_\nu+\mathbf{p}_{\nu+1})\boldsymbol{\tau}\cdot\boldsymbol{\phi} \times\sigma_{i_{\nu+1}}\tau_{j_{\nu+1}}\cdots\sigma_{i_n}\tau_{i_n}u(p_1)] \times[u^\dagger(p_2')\sigma_{k_1}\tau_{l_1}\cdots\sigma_{k_n}\tau_{l_n}u(p_2)], \quad \text{(C1)}$$

and (b) to a matrix element of the form

$$M_2\propto[u^\dagger(p_1')\sigma_{k_1}\tau_{l_1}\cdots\sigma_{k_n}\tau_{l_n}u(p_2)] \times[u^\dagger(p_2')\sigma_{i_1}\tau_{j_1}\cdots\sigma_{i_\nu}\tau_{j_\nu}\boldsymbol{\sigma} \cdot(\mathbf{p}_\nu'+\mathbf{p}_{\nu+1}')\boldsymbol{\tau}\cdot\boldsymbol{\phi}\sigma_{i_{\nu+1}}\tau_{j_{\nu+1}}\cdots\sigma_{i_n}\tau_{j_n}u(p_2)]. \quad \text{(C2)}$$

The set of indices $\{k_1\cdots k_n\}$ is a permutation of the set $\{i_1\cdots i_n\}$ and the set $\{l_1\cdots l_n\}$ is the *same* permutation

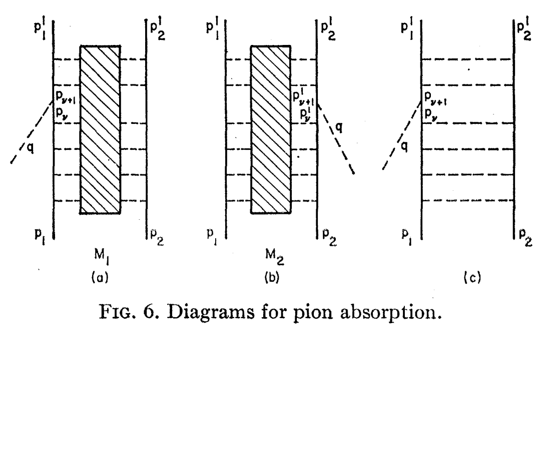

FIG. 6. Diagrams for pion absorption.

[18] Coulomb forces were neglected, since we are interested in the value of the wave function close to the origin, where the Coulomb interaction may be treated as a small perturbation.

of $\{j_1\cdots j_n\}$. As an example, for the simple ladder diagram (c) this permutation is the identity, and

$$M_1 \sim (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)^\nu(\boldsymbol{\sigma}_1\cdot\mathbf{p}_\nu+\mathbf{p}_{\nu+1})\times(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)^{n-\nu}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)^\nu(\boldsymbol{\tau}\cdot\boldsymbol{\phi})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)^{n-\nu}.$$

The sum of momenta $\mathbf{p}_\nu+\mathbf{p}_{\nu+1}$ is equal to $2\mathbf{p}_1'-\mathbf{q}-\sum\mathbf{q}_j$ where the $\mathbf{q}_j$'s are momenta of internal pion lines. The terms in the matrix element proportional to $\boldsymbol{\sigma}\cdot\mathbf{q}_j$ will vanish after integration over the internal pion momenta; and since by our assumption $\mathbf{q}=0$ (absorption from rest) M_1 will be proportional to $\mathbf{p}_1'$; and similarly M_2 will be proportional to $\mathbf{p}_2'$. In the c.m. system $\mathbf{p}_1'=-\mathbf{p}_2'=\frac{1}{2}(\mathbf{p}_1'-\mathbf{p}_2')=\mathbf{k}$. It can easily be seen that M_1 is the same function of $(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_1\cdot\mathbf{k})$ as of $(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2,\boldsymbol{\tau}_1\cdot\boldsymbol{\phi})$. Therefore, the most general form of the matrix element M_1 is

$$\begin{aligned}M_1=&[A(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}+B(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(1+P_{12}{}^\sigma)\\&+C(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(1-P_{12}{}^\sigma)]\\&\times[A(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}+B(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(1+P_{12}{}^\tau)\\&+C(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(1-P_{12}{}^\tau)].\end{aligned}\tag{C3}$$

The matrix element M_2 will have the same form, with $\boldsymbol{\sigma}_1\leftrightarrow\boldsymbol{\sigma}_2$, $\mathbf{k}\leftrightarrow-\mathbf{k}$ and $\boldsymbol{\tau}_1\leftrightarrow\boldsymbol{\tau}_2$. Thus

$$\begin{aligned}M_1+M_2=&2AB[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(1+P_{12}{}^\tau)\\&+(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(1+P_{12}{}^\sigma)]\\&+2AC[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\tfrac{1}{2}(1-P_{12}{}^\tau)\\&+(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}\tfrac{1}{2}(1-P_{12}{}^\sigma)].\end{aligned}\tag{C4}$$

A comparison with (2.3) shows that all diagrams in which only p-wave pions are exchanged between the nucleons contribute equally to g_0^- and g_1^- (and also contribute equally to g_0^+ and g_1^+).

In addition to the terms discussed above, the pion-nucleon interaction contains terms which are quadratic in the pion field, of the form

$$\lambda_0 4\pi(f/\mu)^2\phi^2+\lambda 4\pi(f/\mu)^2\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi},\tag{C5}$$

where $\boldsymbol{\pi}=\dot{\boldsymbol{\phi}}$ is the conjugate pion field. In lowest order these terms correspond to diagrams such as (b) of Fig. 7. In this case an s-wave pion is exchanged between the nucleons. It can easily be seen that the λ_0 term will give rise to a matrix element whose spin and isospin dependence is of the form

$$\begin{aligned}M_{\lambda_0}\propto&(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi})(\boldsymbol{\sigma}_1\cdot\mathbf{k})-(\boldsymbol{\tau}_2\cdot\boldsymbol{\phi})(\boldsymbol{\sigma}_2\cdot\mathbf{k})\\=&\tfrac{1}{2}[(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2\cdot\mathbf{k})\\&+(\boldsymbol{\tau}_1+\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{k}],\end{aligned}\tag{C6}$$

thus giving equal contributions to g_0^- and g_1^- (and also g_0^+ and g_1^+). The λ term will give rise to a matrix element whose spin and isospin dependence is of the form

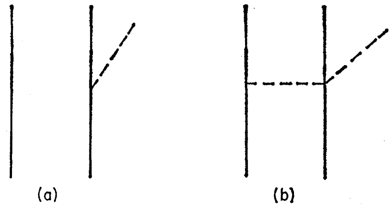

FIG. 7. Lowest order diagrams for pion absorption.

$$\begin{aligned}M_\lambda\propto&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}i(\boldsymbol{\tau}_1\times\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\\=&(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}(\boldsymbol{\tau}_1-\boldsymbol{\tau}_2)\cdot\boldsymbol{\phi}\\&\times[\tfrac{1}{2}(1-P_{12}{}^\tau)-\tfrac{1}{2}(1+P_{12}{}^\tau)].\end{aligned}\tag{C7}$$

This term contributes only to $g_0^\pm$ and not to $g_1^\pm$ at all. Thus g_0^- will not equal g_1^- to the extent that the λ term contributes to the interaction.

In a semiphenomenological analysis, Woodruff[19] calculated the production amplitudes for the process $p+p\to\pi^++d$ close to threshold, taking into account diagrams of the type (a) and (b) of Fig. 7. The parameters λ_0 and λ were fitted to the zero-energy s-wave pion-nucleon scattering data in Born approximation. Although his calculated results agreed with the p-wave pion production data to better than 10%, the calculated value of the s-wave production amplitude was 60% greater than the experimental value. (When the quadratic terms were neglected completely, the calculated value of the s-wave production amplitude was one-sixth of the experimental value.) If there is a large contribution from the quadratic s-wave scattering terms, the isospin dependent part would give rise to an appreciable difference in the values of g_0^- and g_1^-, contrary to the result we have obtained here from the analysis of the $\pi+N+N\to N+N$ process. However, Woodruff's calculation includes only rescattering effects and not the corrections arising from the exchange of p-wave pions between the nucleons.

Note added in proof. While this paper was in press, M. V. Bortolani, L. Lendinara, and L. Monari reported the results of an experiment on pion absorption by He^4 in a helium bubble chamber [Nuovo Cimento **25**, 603 (1962)]. According to this experiment, the ratio of captures in the triton mode (1.1a) to all modes is 0.22 ± 0.03. This is in excellent agreement with the ratio 22% calculated here [cf. Eq. (4.4)]. However, the inference drawn by Bortolani *et al.* that pion absorption takes place essentially on the proton-neutron pair is at variance with our result on the equality of g_0^- and g_1^-. This is due to certain unjustified assumptions which they made. For example, they assumed that the only deuterons which are emitted are bystanders; but as a result of our calculations, we have seen that there is a greater probability that a nucleon bound in the deuteron be a participant than that both nucleons be bystanders (see Fig. 3).

[19] A. E. Woodruff, Phys. Rev. **117**, 1113 (1960).