Nonlinear Interaction of Light in a Vacuum

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Semiclassical methods are used to study the nonlinear interaction of light in vacuum. The study was motivated by a desire to investigate the possibility of using recently developed light sources (lasers) to demonstrate the existence of these minute nonlinear effects. As is well known, Maxwell's equations can be modified by the addition of certain nonlinear terms so that they correctly describe the interaction of low-energy photons. Using these equations, an expression is derived for the counting rate for photons produced by two photons colliding inelastically in the presence of an external, static electric field. A derivation along these lines is also given for the well-known scattering cross section in the case of two photons colliding elastically to give two photons.

I. INTRODUCTION

It has been known for some time now that quantum electrodynamics predicts the existence of a nonlinear interaction between electromagnetic fields in vacuum.[1–7] The development within the past several years of optical lasers has provided very intense monochromatic light sources. One might think that such intense sources could be used to observe the extremely small scattering of light from light. However, if one estimates the counting rates in a typical experiment in which two laser beams are directed at one another and in which the scattered light intensity is measured, one finds (with available powers) extremely small counting rates. If one could cause three intense beams of light to intersect, there might be some hope of observing the small nonlinear interaction. The scattering of three radiative photons in an initial state to give a single photon in a final state is forbidden by phase-space considerations. The process could take place, however, if one of the initial fields were virtual ($\omega \neq k$).

The vacuum is, in fact, a polarizable continuum. The electrons filling the negative energy Dirac sea can be virtually excited by the absorption of radiation to form pairs. The pairs in turn annihilate themselves, giving rise to a scattering of the absorbed radiation. The cross section for this process can be calculated within the framework of quantum electrodynamics. The analytic expressions for the quantum-mechanical transition amplitudes are, in general, algebraically complicated since they involve fourth-order processes. Although the general expressions have been derived,[6,7] they have never been evaluated in their entirety. In fact, it is only in the case where all the participating photons are real ($\omega = k$) that the scattering cross section has been explicitly worked out.[7] Only in the high- and low-energy limits are the expressions simple.

The nonlinear interaction between electromagnetic fields has a structure which reflects the dynamics of the pair field. The range of the nonlinear interaction must be of the order of the electron's Compton wavelength. For sufficiently low energies, i.e., energies such that $\hbar\omega \ll mc^2$ (ω = frequency of the incident light), the radiation cannot "see" the structure of the interaction. The sole effect of the pair field in the low-energy limit is to produce a nonlinear point interaction of a certain strength between fields. The earliest papers in this field (see, for example, reference 5) showed that this interaction can be grafted onto the linear classical theory of light, (Maxwell's equations) by adding to the classical Maxwell Lagrangian density terms which are quartic in the fields.

We make this low-energy approximation throughout this paper. We use it to calculate the counting rate for the scattering of light by light in the presence of classical static fields. We also calculate the cross section for the scattering of light by light in the absence of external fields (a result which is well known). The calculations involved in this paper are simple compared with the corresponding quantum-mechanical calculations. The expressions derived are also simple, both mathematically and physically.

Numerical estimates are made of the kind of fields and field gradients needed in order to observe these extremely small nonlinear effects in the laboratory. The transition rates for the kinds of geometries and fields considered are still too small to be measured.

II. THE NONLINEAR MAXWELL EQUATIONS

The classical Lagrangian for slowly varying fields, which incorporates the effects of virtual pairs and is correct to terms of order e^4, is[6]

$$L=\int\left[\frac{1}{16\pi}f^{\mu\nu}f_{\mu\nu}+c_1(f_{\mu\nu}f^{\mu\nu})^2+c_2f_{\mu\nu}f^{\nu\rho}f_{\rho\sigma}f^{\sigma\mu}\right]d^4x, \quad (1)$$

where $c_1=(5/180)(\alpha^2/m^4)$, $c_2=-(14/180)(\alpha^2/m^4)$, α is the fine structure constant, and m is the mass of the electron. We use naturalized Gaussian cgs units (i.e., $\hbar=c=1$, so $\alpha=e^2$). The quantities $f_{\mu\nu}$ are the usual components of the Maxwell field tensor $f_{\mu\nu}=\partial A_\mu/\partial x^\nu-\partial A_\nu/\partial x^\mu$, where

[1] O. Halpern, Phys. Rev. **44**, 855 (1933).
[2] H. Euler, Ann. Physik **26**, 398 (1936).
[3] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).
[4] V. S. Weisskopf, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd **16**, No. 6 (1936).
[5] A. Achieser, Physik, Z. Sowjetunion **11**, 263 (1937).
[6] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950).
[7] R. Karplus and M. Neuman, Phys. Rev. **83**, 776 (1951).

A_μ is the four-potential. The tensor indices take on the values $\mu=0, 1, 2, 3$, and a repeated index denotes summation. Variation of the Lagrangian with respect to the potentials of the field yields the equations of motion of the field. In this case, an extended set of Maxwell's equations results:

$$\nabla\times\mathbf{E}=-(\partial\mathbf{B}/\partial t), \quad (2)$$

$$\nabla\times\mathbf{H}=\partial\mathbf{D}/\partial t, \quad (3)$$

$$\nabla\cdot\mathbf{D}=0, \quad (4)$$

$$\nabla\cdot\mathbf{B}=0, \quad (5)$$

where

$$D_i=\sum_{j=1}^{3}\epsilon_{ij}E_j\equiv E_i+\lambda\sum_{j=1}^{3}\epsilon_{ij}'E_j\equiv E_i+\lambda(\delta E_i), \quad (6)$$

$$H_i=\sum_{j=1}^{3}\mu_{ij}B_j\equiv B_i+\lambda\sum_{j=1}^{3}\mu_{ij}'B_j\equiv B_i+\lambda(\delta B_i), \quad (7)$$

and

$$\epsilon_{ij}=\delta_{ij}+\lambda[2(E^2-B^2)\delta_{ij}+7B_iB_j]\equiv\delta_{ij}+\lambda\epsilon_{ij}', \quad (8)$$

$$\mu_{ij}=\delta_{ij}+\lambda[2(E^2-B^2)\delta_{ij}-7E_iE_j]\equiv\delta_{ij}+\lambda\mu_{ij}'. \quad (9)$$

In Eqs. (8) and (9), δ_{ij} is the Kronecker delta, and λ is a constant:

$$\lambda=(1/45\pi)(e^4/m^4). \quad (10)$$

Several simple solutions of this system of equations can be determined by inspection. First, a single plane wave satisfying the classical linear Maxwell's equations is a solution. For this plane wave $|E|=|B|$, and $\mathbf{E}\cdot\mathbf{B}=0$, so from Eqs. (6)–(9), $\mathbf{D}=\mathbf{E}$, and $\mathbf{H}=\mathbf{B}$, and hence Eqs. (2)–(5) are satisfied. One expects this result, for quantum electrodynamics predicts that a single free photon can propagate undisturbed.

The next simplest solution is a superposition of two plane waves having different frequencies and an arbitrary relative phase, but propagating in the same direction. We can write the **E** and **B** vectors as

$$\mathbf{E}=\mathbf{E}_1\cos[\omega_1(t-\mathbf{v}\cdot\mathbf{r})]+\mathbf{E}_2\cos[\omega_2(t-\mathbf{v}\cdot\mathbf{r})+\varphi], \quad (11)$$

$$\mathbf{B}=\mathbf{v}\times\mathbf{E}_1\cos[\omega_1(t-\mathbf{v}\cdot\mathbf{r})]+\mathbf{v}\times\mathbf{E}_2\cos[\omega_2(t-\mathbf{v}\cdot\mathbf{r})+\varphi]=\mathbf{v}\times\mathbf{E}, \quad (12)$$

where $\mathbf{v}$ is the unit vector in the direction of propagation, φ is the arbitrary phase difference, and $\mathbf{v}\cdot\mathbf{E}_1=\mathbf{v}\cdot\mathbf{E}_2=0$. It is easy to show that $E^2-B^2\equiv0$ and $\mathbf{E}\cdot\mathbf{B}\equiv0$, so that $\mathbf{H}=\mathbf{B}$, $\mathbf{D}=\mathbf{E}$, and hence Eqs. (2)–(5) are satisfied. It is also simple to prove that no other combination of two plane waves is a solution of the system of Eqs. (2)–(5). In order that the superposition of two plane waves be a solution of these equations, it is necessary that their propagation vectors be parallel (*not* antiparallel). It is easy to extend these results and show that a superposition of n-plane waves having arbitrary frequencies and arbitrary relative phases is again a solution of the extended Maxwell equations, if their propagation vectors are all parallel. Physically, this means that no scattering takes place between photons travelling in the same direction.

In order to discuss the scattering of waves, it seems necessary to resort to an approximate analysis. Equations (2)–(5) can be rewritten in the form

$$\nabla\times\mathbf{E}=-(\partial\mathbf{B}/\partial t), \quad (13)$$

$$\nabla\times\mathbf{B}=\partial\mathbf{E}/\partial t+4\pi\lambda\mathbf{J}, \quad (14)$$

$$\nabla\cdot\mathbf{E}=4\pi\lambda\rho, \quad (15)$$

$$\nabla\cdot\mathbf{B}=0, \quad (16)$$

where

$$\rho=-(1/4\pi)\nabla\cdot\delta\mathbf{E}, \quad (17)$$

$$\mathbf{J}=(1/4\pi)[(\partial/\partial t)(\delta\mathbf{E})-\nabla\times\delta\mathbf{B}]. \quad (18)$$

In this way of writing the extended Maxwell's equations, the nonlinear terms have been lumped into two source terms which are proportional to the small parameter λ. We expand the assumed true solution of Eqs. (13)–(16) in a power series in λ,

$$\mathbf{E}=\mathbf{E}_0+\lambda\mathbf{E}_f+\cdots,\quad \mathbf{B}=\mathbf{B}_0+\lambda\mathbf{B}_f+\cdots, \quad (19)$$

and attempt to determine this true solution correctly to order λ.

It is readily seen that $\mathbf{E}_0$ and $\mathbf{B}_0$ are solutions of (13)–(16) with $\rho\equiv0$ and $\mathbf{J}\equiv0$, that is, they are solutions of the sourceless, linear Maxwell's equations. We discuss the nature of these solutions later. For the moment, we assume that they are known.

The fields $\mathbf{E}_f$ and $\mathbf{B}_f$ are then solutions of (13)–(16) when $\lambda=1$, $\rho=\rho_0\equiv\rho(\mathbf{E}_0\mathbf{B}_{,0})$, and $\mathbf{J}=\mathbf{J}_0\equiv\mathbf{J}(\mathbf{E}_0,\mathbf{B}_0)$. Since we are interested in solutions $\mathbf{E}_0$ and $\mathbf{B}_0$ which are periodic in time, we expand $\mathbf{E}_f$, $\mathbf{B}_f$, ρ_0, and $\mathbf{J}_0$ as Fourier series in the time and denote the respective Fourier coefficients by $\mathbf{E}_f(\mathbf{r},\omega)$, $\mathbf{B}_f(\mathbf{r},\omega)$, $\rho_0(\mathbf{r},\omega)$, and $\mathbf{J}_0(\mathbf{r},\omega)$ (the time factor corresponding to the ωth term is $e^{-i\omega t}$). The Fourier coefficients then satisfy the reduced wave equations

$$(\nabla^2+\omega^2)\mathbf{E}_f(\mathbf{r},\omega)=-4\pi[i\omega\mathbf{J}_0(\mathbf{r},\omega)-\nabla\rho_0(\mathbf{r},\omega)], \quad (20)$$

$$(\nabla^2+\omega^2)\mathbf{B}_f(\mathbf{r},\omega)=-4\pi\nabla\times\mathbf{J}_0(\mathbf{r},\omega). \quad (21)$$

The solutions of (20) and (21) which are defined over all space and correspond to outgoing waves at infinity are well known[8]:

$$\mathbf{E}_f(\mathbf{r},\omega)=\int\frac{[i\omega\mathbf{J}_0(\mathbf{r}',\omega)-\nabla\rho_0(\mathbf{r}',\omega)]}{R}e^{i\omega R}d^3r', \quad (22)$$

$$\mathbf{B}_f(\mathbf{r},\omega)=\int\frac{\nabla\times\mathbf{J}_0(\mathbf{r}',\omega)}{R}e^{i\omega R}d^3r', \quad (23)$$

where $R=|\mathbf{r}-\mathbf{r}'|$.

[8] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 213.

III. THE INITIAL FIELDS

In order to proceed further we must specify the initial fields, $\mathbf{E}_0$ and $\mathbf{B}_0$, which produce the source terms. We assume that the initial fields are a sum of fields, in one case consisting of two incoming fields plus a single outgoing field, and in another case consisting of two incoming fields plus a static electric field. Our approximate solution can be interpreted as follows. The initial fields polarize the vacuum and generate a current which in turn radiates the field $\mathbf{E}_f$. The question now is: Given the strengths, polarizations, spatial and time dispersion of the initial fields, how much power is radiated into the final field? In discussing the scattering of light by light in the absence of an external static field, two of the initial fields will be chosen to correspond to the incoming photons, and the third field will correspond to a photon in a specified final state. In discussing the scattering of light by light in the presence of an external static field, two of the initial fields correspond to incoming photons while the third field will be the prescribed static field. We show that in the first case the radiated field $\mathbf{E}_f$ corresponds to those photons in the final state which have not already been specified, while in the second case $\mathbf{E}_f$ corresponds to all the photons in the final state. The power radiated into a given solid angle determines the observable counting rate. This type of analysis is analogous to the calculation of the matrix elements for the scattering of light by light in quantum electrodynamics. There one considers a set of diagrams of the form shown in Fig. 1. Three of the photons (within the encircled area) create the current which radiates the final photon at the fourth vertex.

In writing down the initial fields, a certain amount of care must be taken. It might be assumed that the initial incoming fields should be plane waves in order that they correspond to single photons. This would lead to mathematical difficulties which reflect the fact that the assumption is not physical. In any actual scattering experiment, the incoming fields would be in the form of collimated beams which interact only over a finite volume.[9] While the cross sections of such collimated beams are finite, their linear dimensions are very large compared with the wavelength of the light, and so we assume that it is reasonable to approximate an incoming field by a plane wave which is finite over the volume of the beam and is zero elsewhere. Furthermore, in the case where a third wave is completely specified, it is physically reasonable that the wave should also be a collimated beam. This is equivalent to letting the initial fields be plane waves and taking the region of integration of the integrals in (22) and (23) to be a compact set, the interaction volume V_0.

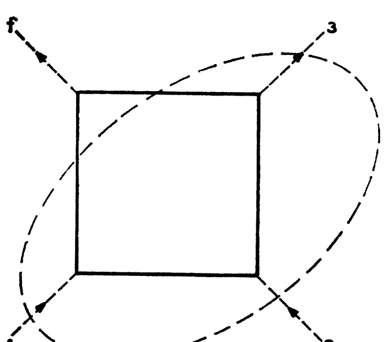

FIG. 1. Feynman diagram for the scattering of two free photons into two free photons.

We consider for a moment an initial field which is an arbitrary finite sum of linearly polarized plane waves. Since the initial fields appear nonlinearly in the expressions for the scattered field, we must use real expressions for the initial plane waves. We write

$$\mathbf{E}_0=\sum_{j=1}^{n} F_j\boldsymbol{\varepsilon}_j \cos[\omega_j t-\mathbf{k}_j\cdot\mathbf{r}], \tag{24}$$

$$\mathbf{B}_0=\sum_{j=1}^{n} F_j\boldsymbol{\nu}_j\times\boldsymbol{\varepsilon}_j \cos[\omega_j t-\mathbf{k}_j\cdot\mathbf{r}], \tag{25}$$

where $\boldsymbol{\varepsilon}_j$ is a unit polarization vector, $\boldsymbol{\nu}_j=\mathbf{k}_j/\omega$ is the unit propagation vector, $\boldsymbol{\nu}_j\cdot\boldsymbol{\varepsilon}_j=0$, $j=1, 2, \cdots, n$, and F_j is the real amplitude of the jth wave. If we substitute (24) and (25) into the expressions (17) and (18) to get ρ_0 and $\mathbf{J}_0$, and, in turn, substitute ρ_0 and $\mathbf{J}_0$ into (20) and (21), several different types of source terms appear. Schematically, $\mathbf{J}_0$ and ρ_0 are proportional to the cubes of $\mathbf{E}_0$ and $\mathbf{B}_0$. In the first place, no terms corresponding to the cube of a single initial field (F_j^3) is present due to the structure of $\delta\mathbf{E}$ and $\delta\mathbf{B}$. If $n\geq 2$, there are terms which are quadratic in one initial field and linear in another ($F_j^2F_k$). These source terms give a contribution to the scattered field only in the direction of one of the initial fields.[10] If $n\geq 3$, terms appear which are the product of three different fields ($F_jF_kF_l$). It is easy to show that no other types of terms appear.

In the following sections we examine in detail the scattered field arising from those source terms which are the product of three different fields. It should be noticed at this point, that had we chosen the initial field as the sum of two plane waves, no scattering other than forward scattering would have been described. This forward scattering can be interpreted as causing a change in the dielectric constant of the vacuum as seen by each beam. An extensive discussion of this point can be found in the paper by Schrödinger[10] in which he studied the Born-Infeld nonlinear theory of electromagnetism. In order to describe the scattering of two photons into two photons within the framework of this classical theory, *it is necessary* to prescribe three of the four fields.

[9] For a discussion of this point see L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., pp. 101–102. See also N. M. Kroll, Phys. Rev. **127**, 1207 (1962).

[10] E. Schrödinger, Proc. Roy. Soc. Irish Acad. **A47**, 77 (1942).

IV. THE SCATTERING OF LIGHT BY LIGHT IN THE ABSENCE OF AN EXTERNAL FIELD

In this section we calculate the low-energy limit of the cross section for the scattering of light by light in the absence of an external field. The result has been known for a long time,[3] but in this classical theory the calculations are simple and illustrate the techniques we use in the more interesting case of scattering in the presence of an external field.

For our initial fields we choose the sum of three plane waves; two of these waves correspond to the two incoming photons, and the third corresponds to an outgoing photon in a specified final state. As shown in Sec. III, we need consider only those source terms which are products of three different fields. If we write

$$\mathbf{E}_0=\sum_{j=1}^{3} F_j\boldsymbol{\varepsilon}_j \cos\varphi_j,$$

and

$$\mathbf{B}_0=\sum_{j=1}^{3} F_j\mathbf{v}_j\times\boldsymbol{\varepsilon}_j \cos\varphi_j, \tag{26}$$

where $\varphi_j=\mathbf{k}_j\cdot\mathbf{r}-\omega_j t$, it is easy to show that

$$\rho=\left(\frac{1}{4\pi}\right)\left(\frac{1}{8i}\right)(F_1F_2F_3)\sum_{\pm}\rho(\pm,\pm,\pm)\exp[i(\pm\varphi_1\pm\varphi_2\pm\varphi_3)], \tag{27}$$

$$\mathbf{J}=\left(\frac{1}{4\pi}\right)\left(\frac{1}{8i}\right)(F_1F_2F_3)\sum_{\pm}\mathbf{J}(\pm,\pm,\pm)\exp[i(\pm\varphi_1\pm\varphi_2\pm\varphi_3)], \tag{28}$$

where

$$\rho(\pm,\pm,\pm)=\mathbf{A}_1\cdot(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3), \tag{29}$$

$$\mathbf{J}(\pm,\pm,\pm)=(\pm\omega_1\pm\omega_2\pm\omega_3)\mathbf{A}_1+(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3)\times\mathbf{A}_2, \tag{30}$$

$$\mathbf{A}_1=\sum_{\substack{j,k,l=1\\ j\neq k\neq l}}^{3}[2(\boldsymbol{\varepsilon}_j\cdot\boldsymbol{\varepsilon}_k-\mathbf{v}_j\times\boldsymbol{\varepsilon}_j\cdot\mathbf{v}_k\times\boldsymbol{\varepsilon}_k)\boldsymbol{\varepsilon}_l+7\boldsymbol{\varepsilon}_j\cdot(\mathbf{v}_k\times\boldsymbol{\varepsilon}_k)\mathbf{v}_l\times\boldsymbol{\varepsilon}_l], \tag{31}$$

$$\mathbf{A}_2=\sum_{\substack{j,k,l=1\\ j\neq k\neq l}}^{3}[2(\boldsymbol{\varepsilon}_j\cdot\boldsymbol{\varepsilon}_k-\mathbf{v}_j\times\boldsymbol{\varepsilon}_j\cdot\mathbf{v}_k\times\boldsymbol{\varepsilon}_k)\mathbf{v}_l\times\boldsymbol{\varepsilon}_l-7\boldsymbol{\varepsilon}_j\cdot(\mathbf{v}_k\times\boldsymbol{\varepsilon}_k)\boldsymbol{\varepsilon}_l]. \tag{32}$$

The sums in (27) and (28) are over the eight possible combinations of + and − signs. In deriving (27)–(32), we have neglected all terms not involving the product of three different fields. We can now pick out the Fourier coefficients of ρ and $\mathbf{J}$ from (27) and (28) and substitute them into (22) and (23). Since we are interested in calculating the far field, and the region of integration in (22) and (23) is compact, we can make the approximation

$$(1/R)e^{i\omega_j R}\approx(1/r)\exp(i\omega_j r-i\omega_j\mathbf{v}\cdot\mathbf{r}'), \tag{33}$$

where $\mathbf{v}=\mathbf{r}/r$. Using (33), we get for the Fourier components of $\mathbf{E}_f$ and $\mathbf{B}_f$

$$\mathbf{E}_f(\mathbf{r},\pm\omega_1\pm\omega_2\pm\omega_3)=\frac{F_1F_2F_3}{32\pi}[(\pm\omega_1\pm\omega_2\pm\omega_3)\mathbf{J}(\pm,\pm,\pm)-(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3)\rho(\pm,\pm,\pm)]\times\frac{1}{r}e^{i(\pm\omega_1\pm\omega_2\pm\omega_3)r}I(\pm,\pm,\pm), \tag{34}$$

$$\mathbf{B}_f(\mathbf{r},\pm\omega_1\pm\omega_2\pm\omega_3)=\frac{F_1F_2F_3}{32\pi}[(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3)\times\mathbf{J}(\pm,\pm,\pm)]\times\frac{1}{r}e^{i(\pm\omega_1\pm\omega_2\pm\omega_3)r}I(\pm,\pm,\pm), \tag{35}$$

where

$$I(\pm,\pm,\pm)=\int_{V_0}\exp[i(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3)\cdot\mathbf{r}'-i(\pm\omega_1\pm\omega_2\pm\omega_3)\mathbf{v}\cdot\mathbf{r}']d^3r'. \tag{36}$$

Since the dimensions of V_0 are very much larger than $2\pi/\omega_j$, the integral (36) is essentially different from zero only when the total wave vector in the exponent is equal to zero. Therefore,

$$I(\pm,\pm,\pm)=V_0\delta(\pm\mathbf{k}_1\pm\mathbf{k}_2\pm\mathbf{k}_3,(\pm\omega_1\pm\omega_2\pm\omega_3)\mathbf{v}), \tag{37}$$

where the δ function in (37) is a Kronecker delta.

Now of the eight possible choices of + or − signs in the sums in (27) and (28), only two will be of interest to us. The eight possible combinations correspond to the different ways of choosing initial and final states. The use of real cosine expressions for the fields precludes the possibility of distinguishing between incoming and outgoing states. The two terms with all the signs the same, $\omega_1+\omega_2+\omega_3$ and $-\omega_1-\omega_2-\omega_3$, correspond to processes in which three photons produce one photon or one photon produces three photons. It is readily seen that this is possible only if $\mathbf{v}=\mathbf{v}_1=\mathbf{v}_2=\mathbf{v}_3$, that is, only in the case of forward scattering. However, we have seen that three plane waves traveling in the same direction do not scatter, so such processes are excluded.[11] The remaining six terms pair up into three groups, each group corresponding to two waves scattering to give the third wave plus the scattered field. We consider only the terms with $\omega_1+\omega_2-\omega_3$ and $-\omega_1-\omega_2+\omega_3$. These correspond to the processes: photons 1 and 2 in with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, and photons 3 and 4 out with momenta $\mathbf{k}_3$ and $\mathbf{k}_4=(\omega_1+\omega_2-\omega_3)\mathbf{v}$; and photons 1 and 2 out with momenta $-\mathbf{k}_1$ and $-\mathbf{k}_2$ and photons 3 and 4 in with momenta $-\mathbf{k}_3$ and $-\mathbf{k}_4$. These amplitudes are indistinguishable experimentally and the two Kronecker delta functions merely produce a factor of 2 in the amplitude. At this point it is clear that this classical theory has yielded the energy and momentum conservation laws for the processes under consideration.

We now calculate the time average power, dP, radiated into the solid angle $d\Omega$ by one of the fields (34)–(35). This is done by calculating the radial component of the time average Poynting vector and multiplying by $r^2d\Omega$ to get the average energy per unit time radiated into $d\Omega$. Since the two terms in which we are interested give identical results, we merely double the result for one of them. A standard calculation then gives[12]

$$\begin{aligned} dP&=2\mathbf{v}\cdot\frac{1}{4\pi}\frac{1}{2}\,\mathrm{Re}(\lambda\mathbf{E}_f\times\lambda\mathbf{B}_f^*)r^2d\Omega \\ &=\frac{\lambda^2}{4\pi}\left(\frac{F_1F_2F_3}{32\pi}\right)^2V_0^2d\Omega\omega_4^2|\mathbf{v}\times\mathbf{J}(++-)|^2. \end{aligned} \tag{38}$$

In (38), $\omega_4=\omega_1+\omega_2-\omega_3$, and we have used the Kronecker delta in (37) to set $\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3=\omega_4\mathbf{v}=\mathbf{k}_4$. In order to get the scattering cross section from (38), we must divide dP, the average energy per unit time radiated into $d\Omega$, by the average energy flux in beams 1 and 2. The latter quantity is just $(1/4\pi)(\frac{1}{2}F_1^2+\frac{1}{2}F_2^2)$. Therefore the differential scattering cross section, $d\sigma/d\Omega$, is

$$\frac{d\sigma}{d\Omega}=\frac{2\lambda^2}{F_1^2+F_2^2}\left(\frac{F_1F_2F_3}{32\pi}\right)^2V_0^2\omega_4^2|\mathbf{v}\times\mathbf{J}(++-)|^2. \tag{39}$$

In order to see that (39) gives the correct low-energy cross section, we transform to the center-of-mass system of beams 1 and 2. Then $\omega_1=\omega_2=\omega_3=\omega_4=\omega$, $\mathbf{v}_1=-\mathbf{v}_2$, and $\mathbf{v}=-\mathbf{v}_3$. We normalize each initial wave so that it corresponds to one photon per volume of interaction. That is, we set

$$\omega=\frac{1}{8\pi}\int_{V_0}2F_j^2\cos^2\varphi_jd^3r\sim\frac{V_0}{8\pi}F_j^2,\quad j=1,2,3, \tag{40}$$

or $F_j=(8\pi\omega/V_0)^{1/2}$. If these values are substituted into (39), and use is made of (10), we find

$$\frac{d\sigma}{d\Omega}=\left(\frac{1}{90}\right)^2\left(\frac{\alpha}{2\pi}\right)^2r_0^2\left(\frac{\omega}{m}\right)^6|\mathbf{v}_3\times(\mathbf{A}_1-\mathbf{v}_3\times\mathbf{A}_2)|^2, \tag{41}$$

where $\alpha=e^2$ is the fine structure constant and $r_0=e^2/m$ is the classical electron radius. By resolving the vectors $\boldsymbol{\varepsilon}_j$ into components parallel and perpendicular to the plane of scattering, it is easily shown that $|\mathbf{v}_3\times(\mathbf{A}_1-\mathbf{v}_3\times\mathbf{A}_2)|$ is equivalent to $(90/\omega^4)\ M_{\lambda_1\lambda_2\lambda_3\lambda_4}$, where $M_{\lambda_1\lambda_2\lambda_3\lambda_4}$ is the quantity defined in Eq. (6) in reference 7 of Karplus and Neuman. It should be noted that in our case only three polarizations can be specified, and the fourth one is determined by the other three. Thus, our formula has only eight possible polarization states, while in the Karplus and Neuman paper there are sixteen possible polarization states. However, some of these are equal to each other, and others are identically zero, and it can be shown that (41) yields all the possible polarization states. If we average over initial states and sum over final states, (41) yields the well-known formula

$$\frac{d\sigma}{d\Omega}=\frac{139}{(90)^2}\left(\frac{\alpha}{2\pi}\right)^2r_0^2\left(\frac{\omega}{m}\right)^6(3+\cos^2\theta)^2. \tag{42}$$

V. SCATTERING IN THE PRESENCE OF AN EXTERNAL FIELD

We now use the techniques of Sec. IV to calculate the counting rate for the production of photons by the inelastic scattering of two photons in the presence of a static, spatially inhomogeneous electric field. The basic process consists of two photons, 1 and 2, interacting once with the external field and producing a single photon (see Fig. 2). The energy of the outgoing photon is the sum of ω_1 and ω_2, the energies of the incoming photons. However, the momentum is not conserved; some momentum is imparted to the static field in the process.

We choose the initial fields as follows:

$$\mathbf{E}_0=\sum_{j=1}^{2}F_j\boldsymbol{\varepsilon}_j\cos\varphi_j-\nabla U(\mathbf{r}), \tag{43}$$

$$\mathbf{B}_0=\sum_{j=1}^{2}F_j\mathbf{v}_j\times\boldsymbol{\varepsilon}_j\cos\varphi_j. \tag{44}$$

In (43) and (44) the two incoming beams are represented by plane waves over the beam, $F_j\boldsymbol{\varepsilon}_j\cos\varphi_j$, where $\varphi_j=\mathbf{k}_j\cdot\mathbf{r}-\omega_jt$. The static field is represented by the negative gradient of its potential, and we write

$$U(\mathbf{r})=\sum_{\boldsymbol{\kappa}}U(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}),\quad U(\boldsymbol{\kappa})^*=U(-\boldsymbol{\kappa}). \tag{45}$$

We assume that the external static field is smoothly varying and is very small outside V_0. The approximation

[11] The authors are indebted to Dr. W. S. Brown for pointing out to them that this process is also excluded in quantum electrodynamics because of lack of phase space.

[12] See reference 8, pp. 216–218.

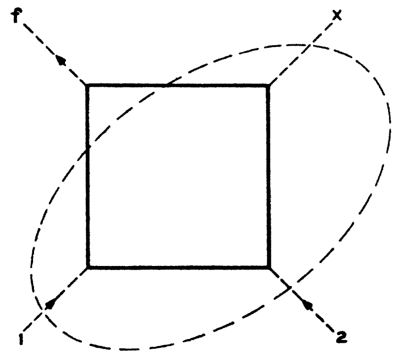

FIG. 2. Feynman diagram for the scattering of two free photons in the presence of an external field.

is made that the potential satisfies periodic boundary conditions at the walls of a cube of volume V_0. This introduces an error of the order $1/\kappa V_0^{1/3}$ in the Fourier transform $U(\boldsymbol{\kappa})$. Since we are interested in the case where $\kappa \gg 1/V_0^{1/3}$, the error is negligibly small.

Using (43) and (44) it is again a simple task to compute the source terms. However, in this case we must give additional arguments for neglecting certain new types of source terms. In the first place, there will be terms which are cubic in $U(\boldsymbol{\kappa})$. It is easy to see that these contribute a static term to $\mathbf{E}_f$ and make no contribution to $\mathbf{B}_f$. This part of the field can, therefore, radiate no energy, and we neglect it. In the second place, source terms appear which are quadratic in the static field and linear in one of the incoming fields. These terms cannot be argued away on grounds of energy-momentum conservation as before. In fact these terms correspond to an incoming photon interacting twice with the external field and being scattered by it (Delbrück scattering).[13] The cross section for this process could be calculated within this classical theory. The cross section so obtained would contain terms involving arbitrarily large momenta, and since this theory is valid only at low energies, the validity of such results would be doubtful. The elastic process (Delbrück scattering) does not physically interfere with the inelastic single scattering process.[14] They are experimentally distinct. Therefore, just as before, we consider only those source terms which are linear in each of the three initial fields.

It is now easy to show that

$$\rho=-\left(\frac{1}{4\pi}\right)\left(\frac{F_1F_2}{4}\right)\sum_{\kappa} U(\boldsymbol{\kappa})\sum_{\pm} \rho(\boldsymbol{\kappa}, \pm, \pm) \exp(i\boldsymbol{\kappa}\cdot\mathbf{r}\pm i\varphi_1\pm i\varphi_2), \tag{46}$$

$$\mathbf{J}=-\left(\frac{1}{4\pi}\right)\left(\frac{F_1F_2}{4}\right)\sum_{\kappa} U(\boldsymbol{\kappa})\sum_{\pm} \mathbf{J}(\boldsymbol{\kappa}, \pm, \pm) \exp(i\boldsymbol{\kappa}\cdot\mathbf{r}\pm i\varphi_1\pm i\varphi_2), \tag{47}$$

where

$$\rho(\boldsymbol{\kappa}, \pm, \pm)=(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2)\cdot\mathbf{c}_1(\boldsymbol{\kappa}), \tag{48}$$

$$\mathbf{J}(\boldsymbol{\kappa}, \pm, \pm)=(\pm\omega_1\pm\omega_2)\mathbf{c}_1(\boldsymbol{\kappa})+(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2)\times\mathbf{c}_2(\boldsymbol{\kappa}), \tag{49}$$

and

$$\mathbf{c}_1(\boldsymbol{\kappa})=4[(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\kappa})\boldsymbol{\varepsilon}_2+(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\kappa})\boldsymbol{\varepsilon}_1]+4[\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2-(\mathbf{v}_1\times\boldsymbol{\varepsilon}_1)\cdot(\mathbf{v}_2\times\boldsymbol{\varepsilon}_2)]\boldsymbol{\kappa}+7[\boldsymbol{\kappa}\cdot(\mathbf{v}_1\times\boldsymbol{\varepsilon}_1)\mathbf{v}_2\times\boldsymbol{\varepsilon}_2+\boldsymbol{\kappa}\cdot(\mathbf{v}_2\times\boldsymbol{\varepsilon}_2)\mathbf{v}_1\times\boldsymbol{\varepsilon}_1], \tag{50}$$

$$\mathbf{c}_2(\boldsymbol{\kappa})=4[(\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\kappa})\mathbf{v}_2\times\boldsymbol{\varepsilon}_2+(\boldsymbol{\varepsilon}_2\cdot\boldsymbol{\kappa})\mathbf{v}_1\times\boldsymbol{\varepsilon}_1]-7[\boldsymbol{\varepsilon}_1\cdot(\mathbf{v}_2\times\boldsymbol{\varepsilon}_2)+\boldsymbol{\varepsilon}_2\cdot(\mathbf{v}_1\times\boldsymbol{\varepsilon}_1)]\boldsymbol{\kappa}-7[\boldsymbol{\kappa}\cdot(\mathbf{v}_1\times\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_2+\boldsymbol{\kappa}\cdot(\mathbf{v}_2\times\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_1]. \tag{51}$$

The sums in (46) and (47) are over the four possible combinations of + and − signs. In deriving these equations we have neglected all terms not involving the product of three different fields. We can now pick out the Fourier coefficients of ρ and $\mathbf{J}$ from (46) and (47) and by calculations similar to those of Sec. IV, we get

$$\mathbf{E}_f(\mathbf{r}, \pm\omega_1\pm\omega_2)=-\frac{iF_1F_2}{16\pi}\sum_{\kappa} U(\boldsymbol{\kappa})[\pm\omega_1\pm\omega_2)\mathbf{J}(\boldsymbol{\kappa}, \pm, \pm)-(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2)\rho(\boldsymbol{\kappa}, \pm, \pm)] \times\frac{1}{r}\exp[i(\pm\omega_1\pm\omega_2)r]V_0\delta(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2,(\pm\omega_1\pm\omega_2)\mathbf{v}), \tag{52}$$

$$\mathbf{B}_f(\mathbf{r}, \pm\omega_1\pm\omega_2)=-\frac{iF_1F_2}{16\pi}\sum_{\kappa} U(\boldsymbol{\kappa})(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2)\times\mathbf{J}(\boldsymbol{\kappa}, \pm, \pm)\times\frac{1}{r}\exp[i(\pm\omega_1\pm\omega_2)r]V_0\delta(\boldsymbol{\kappa}\pm\mathbf{k}_1\pm\mathbf{k}_2,(\pm\omega_1\pm\omega_2)\mathbf{v}). \tag{53}$$

The Kronecker deltas in the sums (52) and (53) pick out that component of the static field which is needed to supply the momentum transfer.

There are four possible combinations of + or − signs in (52) and (53) and only two of these combinations are of interest to us: $\omega_1+\omega_2$ and $-\omega_1-\omega_2$. These combinations correspond to photons 1 and 2 going into photon 3, or photon 3 going into photons 1 and 2. These choices correspond to $\boldsymbol{\kappa}=\mp[\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3]$, where $\mathbf{k}_3=(\omega_1+\omega_2)\mathbf{v}$.

[13] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 379.

[14] The relevant amplitudes for Delbrück scattering in a Coulomb field cannot be computed in this approximation. For a discussion of this point see N. Kemmer, Helv. Phys. Acta **10**, 112 (1937); N. Kemmer and G. Ludwig, *ibid.* **10**, 182 (1937).

To calculate the number of photons radiated per unit time from the total interaction volume V_0 into the solid angle $d\Omega$, we compute the radial component of the time average Poynting vector, multiply by $r^2d\Omega$, and divide by $(\omega_1+\omega_2)$. We set $\boldsymbol{\kappa}=\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3$. It is easily seen that $\mathbf{J}(\boldsymbol{\kappa},-,-)=\mathbf{J}(-\boldsymbol{\kappa},+,+)$, and since $U(-\boldsymbol{\kappa})=U(\boldsymbol{\kappa})^*$, the two choices $++$ and $--$ make identical contributions to the counting rate. The differential counting rate, $d\Gamma$ (the number of photons scattered per unit time into the solid angle $d\Omega$), is

$$d\Gamma=\frac{\lambda^2}{4\pi}V_0^2\left(\frac{F_1F_2}{16\pi}\right)^2|U(\boldsymbol{\kappa})|^2(\omega_1+\omega_2)\times|\mathbf{v}\times\mathbf{J}(\boldsymbol{\kappa},-,-)|^2d\Omega. \quad (54)$$

The assumption of a static electric field is valid only for a particular reference frame, since after a general Lorentz transformation, a static electric field is transformed into a combined static electric and magnetic field. With this in mind, we consider the special experiment in which two beams of the same frequency ω collide head-on in the presence of a static electric field (that is, the center-of-mass frame and the laboratory frame are the same). Then we have $\omega_1=\omega_2=\omega$ and $\mathbf{v}_2=-\mathbf{v}_1$. We must still normalize the incident beams. We assume that each incident beam contains n_j photons in an interaction volume V_0. Then from Eq. (40), we see that

$$F_j=(8\pi n_j\omega/V_0)^{1/2}. \quad (55)$$

If these values are substituted into (54) and use is made of (10), we find that

$$\frac{d\Gamma}{d\Omega}=\frac{8}{(45)^2}\left(\frac{\alpha}{2\pi}\right)^2r_0^2\frac{1}{V_0^2}n_1n_2\left(\frac{\omega}{m}\right)^7\frac{m}{\pi}|V_0U(\boldsymbol{\kappa})|^2\times|\mathbf{v}\times[\mathbf{c}_1(\mathbf{v})+\mathbf{v}\times\mathbf{c}_2(\mathbf{v})]|^2. \quad (56)$$

If the total number of photons emitted in an experimental arrangement is n_j', then $n_j'=(V/V_0)n_j$, where V is the total volume of the beam. The quantity

$$V_0U(\boldsymbol{\kappa})=\int_{V_0}\exp(i\boldsymbol{\kappa}\cdot\mathbf{r})U(\mathbf{r})d^3r \quad (57)$$

is finite in the limit as $V_0\rightarrow\infty$, so that the transition rate is proportional to $1/V_0^2$. The ratio of the transition rate to the flux is proportional to $1/V_0$ and not independent of V_0 as in the case of the two-body processes discussed in Sec. IV. For a fixed number, n_3, of scatterers in the volume V_0, [the source of the potential $U(\mathbf{r})$], $d\Gamma/d\Omega\sim n_1n_2(n_3/V_0)(1/V_0)$. The ratio of $d\Gamma/d\Omega$ to the flux will then depend on n_1, n_2, and n_3/V_0, the density of scattering centers in V_0 (the collision of two photons depends on the existence of a nearby scattering center).

If the potential $U(\mathbf{r})$ is spherically symmetric, the ratio of the total counting rate for two photons scattering in the presence of an external electric field to the total counting rate for two photons scattering in a vacuum can be written in an interesting way:

$$\Gamma_{\text{ext}}/\Gamma_{\text{vac}}=\Lambda(V_0E(\boldsymbol{\kappa})^2/8\pi\hbar\omega), \quad (58)$$

where Λ is a dimensionless angular factor of order unity. In fact, if we average over initial polarizations and sum over final polarizations in calculating Γ_{ext} and Γ_{vac}, then $\Lambda=1.019$. Equation (58) then states that the vacuum counting rate is multiplied by a factor which is the ratio of the energy stored by the electric field in the interaction volume at the wave number $\boldsymbol{\kappa}$ to the energy of a single photon. If we assume some typical numbers, for example, $n_1=n_2=10^{20}$, $\omega=1$ eV, $A=10^{-5}$ cm^2, and $T=10^{-8}$ sec, where n is the number of photons in the scattering volume produced by a laser which has a pulse time T and whose beam is focused on an area A, then the flux is n/AT and we find that $\Gamma_{\text{vac}}\sim10^{-13}$ photon/sec. If we assume $V_0=10^{-2}$ cc, then in order that $\Gamma_{\text{ext}}/\Gamma_{\text{vac}}\geq1$, it is necessary that $E(\kappa)\geq2\times10^{-2}$ V/cm. It appears quite difficult, however, to produce fields having a Fourier component at this wavelength ($\omega=1$ eV$\Rightarrow\kappa\sim10^5$ cm^{-1}) which is much in excess of 2×10^{-2} V/cm. For example, consider a charged metal sphere of radius ρ such that the value of the field at the surface of the sphere is E_0 stat-V/cm. If $\rho\ll(V_0)^{1/3}$, then $E(\kappa)$ is given quite accurately by

$$E(\kappa)=(4\pi\rho^2E_0/V_0\kappa)(\sin\kappa\rho/\kappa\rho). \quad (59)$$

If $\rho\sim10^{-2}$ cm and $E_0=\frac{1}{3}\times10^6$ statV/cm, then $E(\kappa)\sim4\times10^{-4}$ statV/cm$=1.2\times10^{-1}$ V/cm. As ρ increases, formula (59) breaks down [$U(\mathbf{r})$ violates the condition that it be small outside V_0] and for order of magnitude calculations should be replaced by

$$\mathbf{E}(\boldsymbol{\kappa})=\frac{1}{V_0}\int_{V_0}\mathbf{E}(\mathbf{r})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r})d^3r. \quad (60)$$

It seems that the value 10^{-1} V/cm cannot be improved by much more than an order of magnitude. It might be thought that a high density of atomic nuclei would provide a more intense field. However, at these relatively long wavelengths, the light would not "see" the charged nuclei, but would "see" only neutral atoms. Thus, while ratios of the order of magnitude $\Gamma_{\text{ext}}/\Gamma_{\text{vac}}\sim10^4$ are probably obtainable, the nonlinear effects would still be unobservable.

If the calculation were performed for the static magnetic field case, the multiplication factor (just from dimensional considerations) would be $[V_0B^2(\boldsymbol{\kappa})/8\pi\hbar\omega]$, where $B(\boldsymbol{\kappa})$ is the magnitude of the κth Fourier component of the field. The angular factor and explicit numerical factors would, of course, be different. Experimentally, the question is how to generate the maximum energy storage for a given Fourier component of the field.

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