# Theoretical Values for Magnetic Moments of Mu-Mesonic Atoms*

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The magnetic moment of a negative muon bound in the field of a nucleus is slightly less than the moment of a free muon. The binding corrections to the moment have been calculated accurately for a number of nuclei, using realistic nuclear charge distributions. Polarization of the nucleus and of the atomic electrons by the muon also give rise to significant corrections to the muon moment. Precise determinations of the magnetic moment of heavy mu-mesonic atoms afford a way to test for possible structure of the muon which might be exhibited through the polarization of the muon in the strong electric field of the nucleus. The theoretical and experimental $g$ values are in reasonably good agreement in view of the uncertainties about chemical and solid state magnetic shielding effects.

## I. INTRODUCTION

IN 1958 Hughes and Telegdi[1] drew attention to the fact that the magnetic moment of a negative muon bound in an atomic orbit is altered from the moment of a free muon, and that the magnitude of this alteration is sensitive to the size of the nucleus. Measurements of magnetic moments of mu-mesonic atoms have recently been reported[2] which are precise enough to demonstrate the binding effect and the finite nuclear size effect. We present here the results of theoretical calculations of the magnetic moments for a number of elements.[3]

We limit attention at the outset to zero spin nuclei, since experiments have only been done for such nuclei and would be much more difficult for a nucleus with nonzero spin due to the depolarization of the muon associated with the muon-nucleus hyperfine structure interaction. Further, we consider that the muon is bound in a $1s$ orbit about the nucleus. We also assume an electronic structure with zero total angular momentum which usually requires that the mesonic atom in fact be a singly charged ion; this point is discussed in Sec. IV.

Corrections to the gyromagnetic ratio of a Dirac particle ($g_0=2$) bound in the field of a zero spin nucleus and surrounded by a zero spin electronic cloud may be listed as follows:

(1) radiative correction, $g_1$;
(2) binding correction to radiative correction, $g_2$;
(3) direct binding correction, $g_3$;
(4) nuclear polarization correction, $g_4$;
(5) electronic polarization correction, $g_5$;
(6) electronic diamagnetic shielding correction, $g_6$;
(7) center-of-mass correction, $g_7$;

The radiative correction, $g_1$, and the binding correction to the radiative correction, $g_2$, have already been calculated and are written down in this section. The remaining corrections are discussed one by one in Secs. II through V. The theoretical results together with their uncertainties are summarized in Sec. VI.

The familiar radiative correction is [4–6]

$$g_1/g_0=\alpha/2\pi+0.75(\alpha/\pi)^2=0.001165. \quad (1)$$

The binding correction to the radiative correction has been derived by Lieb,[7] and is given in nonrelativistic approximation by

$$g_2/g_0=(26/15\pi)\alpha\langle V\rangle/m_\mu c^2, \quad (2)$$

where $\langle V\rangle$ is the expectation value of the potential energy and $m_\mu$ is the muon rest mass. [There is a reduced mass correction to Eq. (2) of the order of $m_\mu/M$ where $M$ is the nuclear mass, but since $g_2/g_0$ is small the reduced mass correction is negligible.] This expression is adequate for mu-mesonic atoms, since relativistic effects are small even for large $Z$, and $g_2$ never exceeds 2% of the direct binding correction, $g_3$. The quantity $\langle V\rangle$ has been calculated numerically, as outlined in Sec. II.

The radiative corrections embodied in (1) and (2)

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[7] E. H. Lieb, Phil. Mag. **46**, 311 (1955).

are accurate to order $\alpha(\alpha Z)^n$ and $\alpha^2$. The first omitted terms are of order $\alpha^2(\alpha Z)^2$ and $\alpha^3$ and $\alpha(\alpha Z)^2(m_\mu/M)$. These omitted terms are negligible for the present application as, indeed, is the $\alpha^2$ term in $g_1$.

### I.1 The Hamiltonian

It is convenient to enumerate the interactions contributing to the magnetic moment schematically as illustrated in Fig. 1. The parts of the system are the muon field ($\mu$), the nucleus ($N$), the electron field ($e$), the radiation field ($\gamma$), and the static external magnetic field ($B$). Zero-order Hamiltonian terms may be designated by $H(\mu)$, $H(N)$, etc., and interaction terms by $H(\mu B)$, $H(e\gamma)$, etc. The muon-nucleus interaction consists of three parts: $H_0(\mu N)$, the interaction of the muon with the average central static Coulomb electric field seen by the muon; $H_1(\mu N)$, the residual nonstatic, noncentral Coulomb interaction between muon and protons; and $H_2(\mu N)$, the spin-dependent muon-nucleus interaction. Similarly, the muon-electron interaction may be broken into a Coulomb part, $H_1(\mu e)$, and a spin-dependent part, $H_2(\mu e)$, which is the Breit interaction.

The zero-order Hamiltonian is $H(\mu)+H_0(\mu N)+H(N)+H(e)+H(\gamma)$, the first two terms of which are used to obtain numerically the muon eigenfunctions. The radiative correction (1) includes third- and fifth-order corrections contributed by $H(\mu\gamma)$, $H(e\gamma)$, and $H(\mu B)$, with intermediate muons and electrons in free states. The term (2) corrects the third-order term for the action of the central nucleus Coulomb field on the virtual muons.

The direct binding correction arises from the expectation value of $H(\mu B)$ in the muon bound state. The interactions $H_1(\mu N)$ and $H_1(\mu e)$ contribute to the magnetic moment only through higher-order terms which are negligible. The important electronic diamagnetic shielding correction and significant nuclear and electronic polarization contributions due to $H_2(\mu N)$ and $H_2(\mu e)$ occur in second order [$H_2(\mu N)$ or $H_2(\mu e)$ acts once and $H(NB)$ or $H(eB)$ acts once]. The polarization contributions give rise to the largest uncertainties in the predicted magnetic moments. In Fig. 1 solid lines designate those interactions which prove to be significant. In the following we set $\hbar=c=1$.

## II. DIRECT BINDING CORRECTION

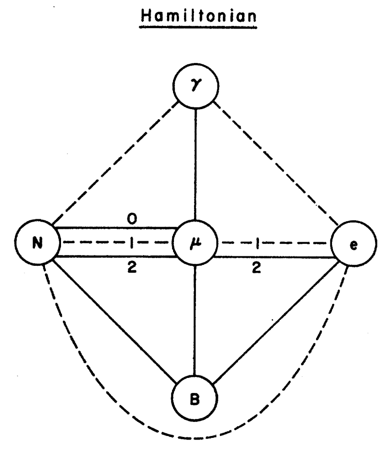
The direct binding correction was given in nonrelativistic approximation in reference 1. It is

$$g_3(T)/g_0=-2\langle T\rangle/3m, \tag{3}$$

where $\langle T\rangle$ is the expectation value of the muon kinetic energy and $m$ is the reduced mass. The $T$ on the left side of (3) identifies the method of calculating $g_3$. For a point nucleus, this expression becomes for the $1s$ state:

$$g_3(pt)/g_0=-\tfrac{1}{3}(\alpha Z)^2. \tag{4}$$

A relativistic treatment of the magnetic moment of a Dirac particle in a central field may be carried through




$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$$
$$\mathcal{H}^{(0)} = H(\mu)+H_0(\mu N)+H(N)+H(e)+H(\gamma)$$
$$\mathcal{H}^{(1)} = H_1(\mu N)+H_2(\mu N)+H_1(\mu e)+H_2(\mu e) + H(\mu\gamma)+H(\mu B)+H(e\gamma)+H(eB) + H(eN)+H(NB)+H(N\gamma)$$

FIG. 1. Diagram of interactions relevant to magnetic moment of mu-mesonic atom. The various symbols denote the nucleus, the muon, the electrons, the radiation field, and the static external magnetic field. Solid lines indicate those interactions which are significant, dashed lines those interactions whose effect is negligible (if nucleus and electrons have zero angular momentum in the lowest state). The lines 1 and 2 connecting muon and electrons refer to the Coulomb interaction and the Breit interaction, respectively. The three lines connecting muon and nucleus refer to (0) the average central potential, (1) the residual muon-proton Coulomb interaction, and (2) the spin-dependent muon-nucleus interaction.

readily, using the interaction term $e\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{r}_\mu)$, where $\mathbf{A}=-\frac{1}{2}\mathbf{r}_\mu\times\mathbf{B}$ is the vector potential due to a constant magnetic field $\mathbf{B}$ and $-e$ is the charge of the muon. The result is

$$g_0+g_3(FG)=\frac{8k}{(2k+1)(2k-1)}\int_0^\infty FG(r/\lambda_c)dr, \tag{5}$$

where $\lambda_c=\hbar/mc$ is the Compton wavelength of the reduced mass; $r$ is the muon-nucleus relative coordinate; $k$ is the angular quantum number ($-1$ for an $s_{1/2}$ state, $+1$ for a $p_{1/2}$ state, $-2$ for a $p_{3/2}$ state, etc.); $G$ is the large component and $F$ the small component of the radial wave function, normalized according to

$$\int (F^2+G^2)dr=1. \tag{6}$$

The sign in (5) rests on a choice of relative sign for $F$ and $G$, and our convention is based on the choice of four-component wave function,

$$\psi=\begin{pmatrix} r^{-1}G(r)\chi_{km} \\ ir^{-1}F(r)\chi_{-km}\end{pmatrix}, \tag{7}$$

where $\chi_{km}$ is a two-component angle and spin function.[8] This choice leads to the radial equations

$$\begin{aligned} dG/dr &= -kr^{-1}G+(E-V+m)F, \\ dF/dr &= kr^{-1}F-(E-V-m)G, \end{aligned} \tag{8}$$

which give $F$ and $G$ opposite signs in the $1s$ state.

We then define, for the $1s$ state,

$$g_3(FG)/g_0 = -(4/3)\int FG(r/\lambda\!\!\!^{-}_c)dr-1. \tag{9}$$

Although this formula is an accurate expression in principle for the direct binding correction, it is impractical for calculation at low $Z$ because of the near cancellation of the two terms on the right. A direct formula for $g_3$ itself is called for. This can readily be obtained by manipulation of the radial Eqs. (8). If the first of Eqs. (8) is multiplied by $rG$, the second by $rF$, and the equations are then added and integrated, one finds

$$\int FG(r/\lambda\!\!\!^{-}c)dr = \frac{2k-1}{4}\left[1-\frac{4k}{2k-1}\int F^2dr\right]. \tag{10}$$

Equation (9) for the $1s$ state ($k=-1$) may accordingly be written

$$g_3(F^2)/g_0 = -(4/3)\int F^2dr, \tag{11}$$

where the $F^2$ on the left designates the revised method of calculation.

Final predicted magnetic moments make use of Eq. (11), but calculations with Eqs. (3) and (9) are made for a test of accuracy. For the heavy elements ($Z \geqslant 24$), $g_3(F^2)$ and $g_3(FG)$ agree to a few parts in $10^4$, while the approximation $g_3(T)$ differs from these by a maximum amount of 3%. For the light elements $g_3(F^2)$ and $g_3(T)$ agree, while $g_3(FG)$ differs from them because of calculational errors. In the evaluation with Eq. (3), we define the mean kinetic energy by

$$\langle T\rangle = E-m-\langle V\rangle, \tag{12}$$

where

$$\langle V\rangle = \int (F^2+G^2)Vdr. \tag{13}$$

One may note, incidentally, that an interesting but less direct way to get from Eq. (9) to (11) is by means of the Foldy-Wouthuysen transformation.[9] The perturbing Hamiltonian is $H'=\frac{1}{2}e\mathbf{B}\cdot(\mathbf{r}_\mu\times\boldsymbol{\alpha})$, or in terms of relative and center-of-mass coordinates ($\mathbf{r}$ and $\mathbf{R}$), it is

$$H'=\tfrac{1}{2}[M/(M+m_\mu)]e\mathbf{B}\cdot(\mathbf{r}\times\boldsymbol{\alpha})+\tfrac{1}{2}e\mathbf{B}\cdot(\mathbf{R}\times\boldsymbol{\alpha}). \tag{14}$$

The second term has zero expectation value. The first term is transformed, to lowest order, into

$$H''=(e\hbar/2m_\mu c)\beta(\boldsymbol{\sigma}+\mathbf{L}_{\text{rel}})\cdot\mathbf{B}. \tag{15}$$

The expectation value of $H''$ leads to a $g$ factor given by

$$g=g_k\left[1-\left(1+\frac{g_{-k}}{g_k}\right)\int F^2dr\right], \tag{16}$$

where $g_k$ and $g_{-k}$ are the nonrelativistic $g$ factors for the large and small components. This result is the same as (9) and (10). In particular for $s$ states, $g_k=2$, $g_{-k}=2/3$, and (16) leads to (11). The result is precise because all higher terms in the complete Foldy-Wouthuysen transformation contain powers of $B$ higher than the first and do not contribute to the magnetic moment. (Note that our Foldy-Wouthuysen transformation differs from the usual one in that the $\boldsymbol{\alpha}\cdot\mathbf{p}$ term is not transformed. The transformed Hamiltonian remains relativistic with a four-component solution.)

An interpretation of Eq. (16) may be given by writing it in the following form:

$$g=g_k\int G^2dr-g_{-k}\int F^2dr. \tag{17}$$

The probability that the muon is in the nonrelativistic state $k$ (e.g., $s_{1/2}$) is $\int G^2dr$; the probability that it is in the nonrelativistic state $-k$ (e.g., $p_{1/2}$) is $\int F^2dr$. The sum of these two probabilities is, of course, 1[Eq. (6)]. The minus sign in (17) arises from the $\beta$ matrix in (15) and is familiar in problems involving negative energy states. The $\beta$ matrix changes the sign of the mass for the lower two components and therefore changes the sign of the Bohr magneton.

### II.1 Numerical Calculations

The direct binding correction and the binding correction to the radiative correction have been obtained from relativistic bound state muon wave functions calculated on an IBM 704 computer.[10] The radial Dirac equations were integrated numerically with trial eigenvalues, and an iteration scheme was used to converge on the correct eigenvalue. The wave functions were then normalized according to (6) and with these wave functions the quantities needed in (2), (3), (9), and (11) were calculated. Accuracy of the results was verified by alteration of the integration interval size, by agreement with perturbation theory for low $Z$, and by agreements among the values of $g_3$ calculated in different ways. Agreement with previous calculations of $2p\rightarrow 1s$ transition energies was also obtained.

[8] H. A. Bethe and E. E. Salpeter, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. XXV, p. 149.

[9] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[10] Most of the computations were carried out at the Los Alamo. Scientific Laboratory, and a few at the MIT Computation Centers We are indebted to the authorities at both places for the opportunity to use these computers.

The muon mass assumed was[11,12]

$$m_\mu = 206.84 m_e = 105.69 \text{ MeV}. \tag{18}$$

The reduced mass used in the calculations was $m=\mu m_\mu$, where $\mu$ is given in terms of the mass number $A$ by

$$\mu = A/(A+0.1135). \tag{19}$$

The muon was taken to move in a central potential arising from the nuclear charge distribution,

$$\begin{aligned}\rho(r) &= (Ze/4\pi r_1{}^3 N_0)(1-\tfrac{1}{2}e^{-n(1-x)}), \quad x<1 \\ &= (Ze/4\pi r_1{}^3 N_0)(\tfrac{1}{2}e^{-n(x-1)}), \quad x\geqslant 1\end{aligned}, \tag{20}$$

where $x=r/r_1$, and

$$N_0 = \tfrac{1}{3} + 2n^{-2} + n^{-3}e^{-n}. \tag{21}$$

The parameters $r_1$ and $n$, which characterize the nuclear radius and surface thickness, were chosen to be those which have been used to fit high energy electron scattering results,[13] or else were interpolated between such values. For the lightest element considered, helium, the form (20) is inappropriate, and a Gaussian charge density was used instead.[14]

Detailed calculations have been carried out for 35 elements, of which a representative sample of 13 are included in this paper.[15] Table I lists these elements and the assumed parameters of the nuclear charge distribution. The equivalent uniform radius, $R_{eq}$, is defined by $R_{eq}=[5\langle r^2\rangle/3]^{1/2}$. Table II gives calculated energies and the integral, $\int F^2 dr$.

Table III shows the binding correction $g_3/g_0$ calculated in three different ways, together with the nonrelativistic point-nucleus value of $g_3/g_0$. The value $g_3(F^2)/g_0$ is accurate over the whole periodic table and is adopted for the final predicted magnetic moment. The error appears to be a few parts in $10^4$. The error in $g_3$ contributes an error of a few parts in $10^6$ to $g$ for the heavy elements, and less than 1 part in $10^6$ for the very light elements.

Table IV shows the sensitivity of the calculated $g_3$ to variation of the nuclear parameters, and permits $g_3$ to be readily corrected for different nuclear radii or surface thicknesses. (This calculation happened to be carried out for the odd nuclei $_{51}$Sb and $_{83}$Bi, but interpolation for even nuclei is possible.)

TABLE I. Elements and parameters.[a]

| $Z$ | Element | $A$ | $r_1/A^{1/3}$ | $n$ | $R_{eq}/A^{1/3}$ |
|---|---|---|---|---|---|
| 2 | He | 4.00 | 0.825[b] | [b] | 1.30 |
| 6 | C | 12.01 | 1.009 | 3.0 | 1.69 |
| 8 | O | 16.00 | 1.045 | 3.5 | 1.59 |
| 12 | Mg | 24.3 | 1.03 | 3.5 | 1.56 |
| 14 | Si | 28.1 | 1.03 | 3.8 | 1.49 |
| 16 | S | 32.1 | 1.04 | 4.0 | 1.46 |
| 20 | Ca | 40.1 | 1.063 | 4.1 | 1.47 |
| 24 | Cr | 52.0 | 1.07 | 4.8 | 1.38 |
| 30 | Zn | 65.4 | 1.07 | 5.5 | 1.31 |
| 42 | Mo | 96.0 | 1.08 | 6.8 | 1.24 |
| 50 | Sn | 118.7 | 1.08 | 7.3 | 1.22 |
| 64 | Gd | 157 | 1.095 | 7.5 | 1.23 |
| 82 | Pb | 207 | 1.11 | 10.0 | 1.19 |

[a] Distances are in fermis ($10^{-13}$ cm).
[b] Gaussian charge distribution used for helium, $\rho = Z\pi^{-3/2}r_1^{-3}\exp(-r^2/r_1^2)$.

TABLE II. Some calculated quantities for muon ground states.[a]

| $Z$ | Element | $\langle V\rangle$ | $\langle T\rangle$ | $\int F^2 dr$ |
|---|---|---|---|---|
| 2 | He | −0.0219 | 0.0109 | 0.000053 |
| 6 | C | −0.1987 | 0.0989 | 0.00047 |
| 8 | O | −0.3512 | 0.1741 | 0.00083 |
| 12 | Mg | −0.7700 | 0.3766 | 0.00178 |
| 14 | Si | −1.0347 | 0.5029 | 0.00238 |
| 16 | S | −1.3287 | 0.6407 | 0.00303 |
| 20 | Ca | −1.984 | 0.9366 | 0.00441 |
| 24 | Cr | −2.741 | 1.270 | 0.00596 |
| 30 | Zn | −4.014 | 1.805 | 0.00845 |
| 42 | Mo | −6.789 | 2.852 | 0.0133 |
| 50 | Sn | −8.669 | 3.469 | 0.0161 |
| 64 | Gd | −11.772 | 4.313 | 0.0199 |
| 82 | Pb | −15.881 | 5.286 | 0.0244 |

[a] Energies are in MeV.

TABLE III. Direct binding correction.

| $Z$ | Element | $-g_3(FG)/g_0$ | $-g_3(F^2)/g_0$ | $-g_3(T)/g_0$ | $-g_3(pt)/g_0$ |
|---|---|---|---|---|---|
| 2 | He | 0.000074 | 0.000071 | 0.000071 | 0.000071 |
| 6 | C | 0.000638 | 0.000629 | 0.000629 | 0.000639 |
| 8 | O | 0.001104 | 0.001104 | 0.001106 | 0.001136 |
| 12 | Mg | 0.002380 | 0.002379 | 0.002387 | 0.002556 |
| 14 | Si | 0.003290 | 0.003172 | 0.003185 | 0.003479 |
| 16 | S | 0.004069 | 0.004035 | 0.004056 | 0.004544 |
| 20 | Ca | 0.005883 | 0.005883 | 0.005925 | 0.007100 |
| 24 | Cr | 0.007952 | 0.007952 | 0.008026 | 0.01022 |
| 30 | Zn | 0.01130 | 0.01126 | 0.01140 | 0.01598 |
| 42 | Mo | 0.01770 | 0.01769 | 0.01801 | 0.03131 |
| 50 | Sn | 0.02148 | 0.02146 | 0.02191 | 0.04438 |
| 64 | Gd | 0.02660 | 0.02659 | 0.02723 | 0.07270 |
| 82 | Pb | 0.03249 | 0.03248 | 0.03336 | 0.1194 |

TABLE IV. Sensitivity to parameter variation.

| $Z$ | Element | $d(\ln g_3/g_0)/d(\ln r_1)$ | $d(\ln g_3/g_0)/d(\ln n)$ |
|---|---|---|---|
| 6 | C | −0.0277 | 0.0181 |
| 22 | Ti | −0.302 | 0.113 |
| 51 | Sb | −0.718 | 0.109 |
| 83 | Bi | −0.953 | 0.121 |

[11] E. R. Cohen, K. M. Crowe, and J. W. M. DuMond, Phys. Rev. **104**, 266 (1956); G. Shapiro and L. M. Lederman, *ibid.* **125**, 1022 (1962).

[12] The precise value of the muon mass used in the calculations is unimportant. It affects the calculated $g$ factors only through its influence on the ratio of nuclear radius to muon Compton wavelength, and the uncertainty in nuclear radius is considerably greater than the error in the chosen mass value in (18).

[13] U. Meyer-Berkhout, K. W. Ford, and A. E. S. Green, Ann. Phys. (New York) **8**, 119 (1959).

[14] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[15] More extensive tables of numerical results are available in Los Alamos Scientific Laboratory Report LAMS-2387. See also K. W. Ford and J. G. Wills, Nucl. Phys. **35**, 295 (1962).

### III. NUCLEAR POLARIZATION EFFECT

A nuclear polarization correction term, $g_4$, arises in second order from $H_2(\mu N)$ and $H(NB)$ each acting once. The nucleus is virtually excited from its 0+ ground state to a 1+ state, and the muon is left alone:

$$g_4=-\frac{4m_\mu}{eBm_j}\,\mathrm{Re}\,\sum_k{}'\,\frac{\langle 0|H(NB)|k\rangle\langle k|H_2(\mu N)|0\rangle}{E_k-E_0}, \quad (22)$$

where $m_j=\pm\frac{1}{2}$ is the projection of the total angular momentum along the direction of the magnetic field. The form to adopt for $H(NB)$ is fairly unambiguous, since there is no reason to do better than a nonrelativistic approximation. We use

$$H(NB)=-\frac{eB}{2M_N}\sum_i \mu_{iz}, \quad (23)$$

where $M_N$ is the mass of a nucleon, $\boldsymbol{\mu}_i$ is the total magnetic moment (dimensionless) of the $i$th nucleon, and $\mathbf{B}$ lies along the $z$ axis. The matrix element $\langle 0|H(NB)|k\rangle$ connecting the 0+ ground state to the 1+ excited state is proportional to the M1 gamma-ray matrix element between these levels.

For the spin-dependent muon-nucleus interaction, $H_2(\mu N)$, the most natural choice is a form relativistic for the muon and nonrelativistic for the nucleons:

$$H_2(\mu N)=e^2\boldsymbol{\alpha}\cdot\left[\frac{1}{2M_N}\sum_i\frac{\boldsymbol{\mu}_i{}^s\times\mathbf{r}_i{}'}{(r_i{}')^3}+\sum_p\frac{\mathbf{v}_p}{r_p{}'}\right], \quad (24)$$

where $i$ labels all nucleons, and $p$ the protons, and the last term, containing noncommuting factors, is understood to be symmetrized. In this formula, $\boldsymbol{\mu}_i{}^s$ is the nucleon spin moment, $\mathbf{v}_p$ is the proton velocity, and $\mathbf{r}_i{}'=\mathbf{r}-\mathbf{R}_i$ is the muon-nucleon relative coordinate.

The expansion of (24) to find the part which contributes to Eq. (22) for $g_4$ is sketched in the Appendix. The result for the matrix element of $H_2(\mu N)$ is

$$\begin{aligned}\langle k|H_2(\mu N)|0\rangle=&(e^2m_j/2M_N)\\ &\times\{\langle k|\textstyle\sum_i \mathfrak{M}_1(\mu_{iz}{}^s+g_{li}l_{iz})|0\rangle\\ &+\langle k|\textstyle\sum_i\mathfrak{M}_2[(2\pi)^{1/2}(\mu^sY_2)_{iz}-g_{li}l_{iz}]|0\rangle\},\end{aligned} \quad (25)$$

where $g_{li}$ is the nucleon orbital $g$ factor; $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are functions of the nucleon radial coordinates,

$$\begin{aligned}\mathfrak{M}_1(R_i)&=-(8/3)\int_{R_i}^{\infty}FGr^{-2}dr\\ \mathfrak{M}_2(R_i)&=(8/3R_i{}^3)\int_0^{R_i}FGrdr,\end{aligned} \quad (26)$$

with $F$ and $G$ being the muon radial functions; and $(\mu^sY_2)_{iz}$ designates a tensor product,

$$(\mu^sY_2)_{iz}=\sum_q C(121;q,-q,0)(\mu_i{}^s)_q{}^1Y_{2,-q}(\theta_i,\phi_i). \quad (27)$$

In the low-$Z$ limit, $\mathfrak{M}_1\gg\mathfrak{M}_2$, and $\mathfrak{M}_1\cong(8/3)(m_\mu{}^2)(\alpha Z)^3$, independent of $R_i$. Even for high $Z$, the nuclear radial integral over $\mathfrak{M}_2$ is no greater than 1/5 the radial integral over $\mathfrak{M}_1$. In view of the general uncertainty in making a numerical estimate of $g_4$, we ignore the second term on the right of (25) for all $Z$. In this approximation,

$$g_4=\frac{8}{3}\frac{\alpha^4Z^3m_\mu{}^3}{M_N{}^2}\gamma(Z)\sum_k{}'\,\frac{|\langle 0|\sum_i\mu_{iz}|k\rangle|^2}{E_k}, \quad (28)$$

where $\mu_{iz}=\mu_{iz}{}^s+g_{li}l_{iz}$. The factor $\gamma(z)$ in (28) corrects for the fact that $\mathfrak{M}_1$ is smaller than $(8/3)m_\mu{}^2(\alpha Z)^3$ at high $Z$, and varies with $R_i$. The function $\mathfrak{M}_1$ has been evaluated numerically for several elements, and $\gamma$ is arbitrarily set equal to $\mathfrak{M}_1(c)$, where $c$ is the half density radius of the nucleus. Some sample values of $\gamma(Z)$ are $\gamma(0)=1$, $\gamma(6)=0.81$, $\gamma(22)=0.38$, $\gamma(37)=0.17$, $\gamma(82)=0.018$. For $Z\lesssim 15$, $\gamma(Z)=1-Z/32$. In Eq. (28) and the following equations we take $E_0=0$.

We have also examined an alternative choice for $H_2(\mu N)$, in which the orbital and normal spin moments are treated by means of a Breit interaction, and the anomalous moments are treated by a nonrelativistic term like that which appears in (24). The result for the matrix element $H_2(\mu N)$, is exactly the same as (25), except that $\mu^s$ is replaced by $\mu^A$ (anomalous moment), and $l_{iz}$ is replaced by $M_N(\mathbf{R}_i\times\boldsymbol{\alpha}_i)_z$. This modified form of $H_2(\mu N)$ therefore leads to no significant difference, and, in the approximation of ignoring the second term in (25), Eq. (28) again results.

The nuclear polarization contribution $g_4$, in the approximation of Eq. (28), may be related to $M1$ gamma transition rates by

$$g_4\cong 8(\alpha Z)^3m_\mu{}^3\gamma(Z)\sum_k{}'\,(E_k)^{-4}T_{M1}(k\rightarrow 0), \quad (29)$$

where $T_{M1}$ is the $M1$ transition rate (sec$^{-1}$) from the state $k(1+)$ to the ground state (0+). The same formula may be rewritten

$$g_4\cong\frac{32\pi}{9}\frac{\alpha^4Z^3m_\mu{}^3}{M_N{}^2}\gamma(Z)\sum_k{}'\,\frac{B_{M1}(k\rightarrow 0)}{E_k}, \quad (30)$$

where $B_{M1}$ is the reduced matrix element as defined by Bohr and Mottelson.[16] These formulas are, in fact, not too useful since so few $M1$ rates are known experimentally in any one nucleus, but they provide a convenient way of putting a lower limit on $g_4$.

For the numerical estimates of $g_4$ finally included in Table V, we use the radial correction factor $\gamma$ as defined above and use a closure approximation for the $M1$ matrix elements, ignoring all correlation terms. Specifically, we take

$$\sum_k|\langle 0|\sum_i\mu_{iz}|k\rangle|^2=7.80Z+3.65N+\tfrac{1}{3}Z\langle l(l+1)\rangle_p, \quad (31)$$

in which $7.80=\mu_p{}^2$, $3.65=\mu_N{}^2$, and $\langle l(l+1)\rangle_p$ means the

[16] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **27**, No. 16 (1953).

mean square orbital angular momentum of the protons. Finally then,

$$g_4/g_0 = \frac{4}{3}\frac{\alpha^4 Z^3 m_\mu^3 \gamma(Z)}{M_N^2 \langle E\rangle} \times[7.80Z+3.65N+\tfrac{1}{3}Z\langle l(l+1)\rangle_p]. \quad (32)$$

The quantity $\langle l(l+1)\rangle_p$ is set equal to zero for He, 1.33 for C, 1.50 for O, and is smoothly varied from 2.1 for Mg through 5.1 for Zn, 7.5 for Sn, to 10.0 for Pb. For the numerical estimates in Table V, $\langle E\rangle$ is taken to be 15 MeV.

Within the $1p$ shell, the estimate (32) for $g_4$ may be checked roughly in two ways—by an explicit $jj$ coupling shell model calculation from Eq. (28), and by the substitution of known strong $M1$ transition rates into Eq. (30). For $O^{16}$, the simple shell model picture gives $g_4=0$, and indeed $M1$ transitions are known to be inhibited in this nucleus.[17] The estimate (32) is therefore likely to be a substantial overestimate for $O^{16}$, by perhaps a factor of 10. For $C^{12}$, the $jj$ coupling shell model gives $\sum_k|\langle 0\sum_i \mu_{iz}|k\rangle|^2=16$, to be compared with the value 71 from (31). Again the estimate (32) appears to be too large. Using the $M1$ width of the 12.76 MeV state in $C^{12}$ to be less than 44 eV, and the width of the 15.11 MeV state to be 55 eV,[17,18] we find that these two transitions contribute to $g_4/g_0$ at most $1.05\times10^{-6}$. The closure estimate (32) is $4\times10^{-6}$.

## IV. EFFECT OF ATOMIC ELECTRONS

The important effects of the orbital electrons may be classified as electronic polarization, $g_5$, and diamagnetic shielding, $g_6$. Since the electrons are very distant from the nucleus compared to the muon, the muon may be treated as fixed at the origin, and the already existing theory of these effects of electrons on nuclear moments[19] may be taken over intact.

The electronic polarization correction for an isolated atom is given by

$$g_5/g_0=(\alpha/m_e^2)\sum_k{}'\,[1/(E_k-E_0)]\langle 0|\sum_i l_{iz}+2s_{iz}|k\rangle \times\langle k|\sum_i r_i^{-3}[l_{iz}-s_{iz}+3(\mathbf{s}_i\cdot\mathbf{r}_i)z_i/r_i^2]|0\rangle, \quad (33)$$

where $i$ labels the electrons, $m_e$ is the electron mass ($\hbar=c=1$), $|0\rangle$ is the electronic ground state, and $|k\rangle$ is an electronic excited state. This is a generalization to include spin of a result of Ramsey.[19] For a $^1S_0$ state, the polarization correction $g_5$ vanishes. For $^3P_0$, $^5D_0$, $^7F_0$, $\cdots$ states, $g_5$ could be quite large, as much as $+0.05$, since the matrix elements in (33) do not vanish and the energy denominator $(E_k-E_0)$ is a fine structure splitting, e.g., between $^3P_1$ and $^3P_0$.

The electronic polarization correction arises primarily from virtual excitation of outer electrons and hence is quite sensitive to the chemical[20] or solid state[21] environment of the mesonic atom. For this reason we have not attempted a careful evaluation of (33), since the perturbing effect of neighboring atoms would invalidate the result. In the experiments reported so far,[2] there is no evidence for a large electronic polarization effect, suggesting that in the mesonic atoms studied, electron spins are paired. For unpaired spins, e.g., a triplet state, the correction $g_5$ should be so large as to be immediately obvious experimentally. For paired spins, $g_5$ should be small and positive (a paramagnetic effect), and sensitive to the environment of the mesonic atom. In one case, for silicon, the polarization correction (the Knight shift) has been estimated[2,21] to be $g_5/g_0=+0.00018$. The correction $g_5$ is not included in Table V.

The diamagnetic shielding correction[22] is

$$g_6/g_0=-\tfrac{1}{3}\alpha^2\sum_i\langle a_0/r_i\rangle, \quad (34)$$

where $a_0$ is the Bohr radius and the sum is taken over all electrons. Numerical values of $g_6/g_0$ included in Table V are taken from Dickinson's work,[23] assuming that the mesonic atom exists as a positive ion with $Z$-2 electrons. This correction is nearly independent of the chemical state of the atom since the shielding arises largely from inner electrons.

Since the *average* muon-proton Coulomb interaction has been included in the zero-order Hamiltonian but the average muon-electron Coulomb interaction has not, there is an extra magnetic moment correction to consider for the electrons, which occurs in second order and has, schematically, the following form:

$$g_5\sim\langle 0|H(\mu B)|k\rangle\langle k|H_1(\mu e)|0\rangle. \quad (35)$$

We have estimated this term and find $g_5=7.6\times10^{-6}Z^{-1}$, negligible for all $Z$. The effect of $H_1(\mu e)$ on the muon energy level differences is also negligible.

## V. CENTER-OF-MASS CORRECTION

The magnetic moment of a muon in its lowest state is predominantly a spin moment, which requires no center-of-mass correction. The orbital contribution to the moment is of order $(\alpha Z)^2$, and the center-of-mass correction to the moment is therefore of order $(m_\mu/M)(\alpha Z)^2=(\alpha Z)^2/9A$. This is negligible for all $A$.

In the Dirac equations (8) we follow the prescription of Breit and Brown,[24] simply using the nonrelativistic reduced mass $m=Mm_\mu/(M+m_\mu)$. The first term in (14), whose expectation value determines the magnetic moment, has a mass dependence $m/m_\mu$. The $m_\mu^{-1}$ is absorbed in the Bohr magneton, leaving a proportionality to the reduced mass $m=\lambda_c^{-1}$, which appears, for example, in Eq. (5). This direct proportionality to $m$ is apparent

[17] D. A. Bromley (private communication).
[18] E. Almquist, D. A. Bromley, A. J. Ferguson, H. E. Gove, and A. E. Litherland, Phys. Rev. **114**, 1040 (1959).
[19] N. F. Ramsey, Phys. Rev. **78**, 699 (1950).
[20] N. F. Ramsey, Phys. Rev. **86**, 243 (1952).
[21] W. D. Knight, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1956), Vol. 2, p. 93.
[22] W. E. Lamb, Jr., Phys. Rev. **60**, 817 (1941).
[23] W. C. Dickinson, Phys. Rev. **80**, 563 (1950).
[24] G. Breit and G. E. Brown, Phys. Rev. **74**, 1278 (1948).

only, and not real. For a pure Coulomb field, the Dirac equations (8) depend only on the dimensionless variable $mr$. Similarly, the expression (5) for $g$ depends only on $mr$ and is independent of the length scale, except insensitively through its dependence on $mR$, where $R$ is the nuclear radius.

A crude procedure for estimating the error in our procedure is the following. In Eq. (16) there occur $g_k$ and $g_{-k}$, which, for the lowest state, are

$$g_k = g(s_{1/2}) = g_s = 2,$$
$$g_{-k} = g(p_{1/2}) = \tfrac{4}{3}g_l - \tfrac{1}{3}g_s = \tfrac{2}{3}. \tag{36}$$

If, arbitrarily, $g_l$ is modified nonrelativistically, $g_l \to 1-(m_\mu/M)$, and $g_s$ is unchanged, the expression (11) for $g_3$ becomes

$$(g_3+g_7)/g_0 = -(4/3)[1-1/2(m_\mu/M)]\int F^2 dr, \tag{37}$$

or

$$g_7/g_0 = (2m_\mu/3M_NA)\int F^2 dr, \tag{38}$$

which varies from $10^{-6}$ for He to $10^{-5}$ for Pb. The coefficient on the correction term may be incorrect, but it seems clear that $g_7$ is of order $(m_\mu/M)\int F^2 dr$. For present purposes this is negligible; the exact magnitude of the correction would require a more searching investigation of the relativistic two-body problem.

### VI. THEORETICAL RESULTS AND ERRORS

The gyromagnetic ratio of the bound negative muon is given by

$$g = 2[1+\sum_{i=1}^{7}(g_i/g_0)], \tag{39}$$

where $g_1/g_0$ is given by (1); $g_2/g_0$ is given by (2); $g_3/g_0$ is given by (11); $g_4/g_0$ is given formally by (22), approximately by (28), and crudely by (32); $g_5/g_0$ is given formally by (33); $g_6/g_0$ is given by (34); and $g_7/g_0$ is given crudely by (38). Numerical values for various contributions to $g/g_0$ are given in Table V, $g_2$, $g_3$, and $g_6$ being accurate. The final numerical estimate for $\sum_i (g_i/g_0)$ omits the electronic polarization correction $g_5/g_0$, because of its dependence on chemical and solid state effects. (For an isolated atom in a $^1S_0$ state, $g_5$ vanishes.)

Table V also includes some information on the electronic structure of the once-ionized mesonic atom. In all but two cases, the mesonic atom is expected to stabilize as a positive ion with $Z-2$ electrons because the last electron in the odd element $Z-1$ is more loosely bound than the last electron in the even element $Z$. Neighboring atoms can, therefore, not furnish the last electron to the mesonic ion. For the lightest two elements in the table, results are included for the $2s$ state as well as for the $1s$ state, since the $2s$ state may be metastable. The $2s$ states are designated by He* and C*.

The theoretical error in $g_0+g_1+g_2+g_3$ is less than $1\times10^{-6}$ for He and ascends gradually to about $10^{-3}$ for Pb. The error in $g_4$ is roughly $g_4$, itself, about $8\times10^{-6}$ for C up to $1\times10^{-2}$ for Pb. The error in $g_5$ is negligible only if the atom is in a singlet state.

### VII. CONCLUSIONS

As seen in Table V the agreement between theory and experiment is close, particularly in view of uncertainties about the electronic state of the $\mu-$mesonic atom (the state might even have electronic angular momentum $J$ nonzero and then the hfs interaction between muon and electrons would be important), about chemical effects, and about solid-state effects such as the Knight shift. The chemical shift and the Knight shift are paramagnetic effects having the same sign as that required to account for the small remaining discrepancies.

TABLE V. Corrections to gyromagnetic ratio of bound negative muon.

| $Z$ | Element | Radiative binding $g_2/g_0$ | Direct binding[a] $g_3/g_0$ | Nuclear polarization[b] $g_4/g_0$ | Diamagnetic shielding $g_6/g_0$ | $(\Delta g/g)_{\text{theor}} = \sum_{i=1}^{6} g_i/g_0$ [c] | $\frac{1}{2}(g-g_+)$ $10^4_{\text{theor}}$ [d] | $\frac{1}{2}(g-g_+)$ $10^4_{\text{expt}}$ [e] | Ion stable? | Ion electron configuration | Lowest electron 1+ state (eV) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | He | −0.000001 | −0.000071(0) | ∼0 | 0 | +0.001093 | −0.72 | | yes | no electrons | ... |
| 2 | He* | −0.000000 | −0.000018(0) | ∼0 | 0 | +0.001147 | −0.18 | | yes | no electrons | ... |
| 6 | C | −0.000008 | −0.000629(0) | +0.000004 | −0.00019 | +0.00034 | −8.2(1) | −7.5(7) | yes | $2s^2$ $^1S_0$ | 16.08[h] |
| 6 | C* | −0.000002 | −0.000159(0) | ∼0 | −0.00019 | +0.00081 | −3.5 | | yes | $2s^2$ $^1S_0$ | 16.08[h] |
| 8 | O | −0.000013 | −0.001104(1) | +0.000012 | −0.00032 | −0.00026 | −14.3(2) | −9.3(10) | no | $2s^2 2p^2$ $^3P_0$ | 0.00609 |
| 12 | Mg | −0.000029 | −0.002379(6) | +0.000053 | −0.00062 | −0.00181 | −29.8(6) | −26.3(7)[f] | yes | $2s^2 2p^6$ $^1S_0$ | 36.34 |
| 14 | Si | −0.000040 | −0.003172(10) | +0.000090 | −0.00079 | −0.00275 | −39.1(10) | −36.1(11) | yes | $3s^2$ $^1S_0$ | 11.31[h] |
| 16 | S | −0.000051 | −0.004035(15) | +0.00014 | −0.00096 | −0.00374 | −49.1(15) | −48.1(16) | no | $3s^2 3p^2$ $^3P_0$ | 0.02065 |
| 20 | Ca | −0.000076 | −0.005883(31) | +0.00029 | −0.00133 | −0.00583 | −70.0 | | yes | $3s^2 3p^6$ $^1S_0$ | 22.71 |
| 24 | Cr | −0.000105 | −0.00795(5) | +0.00051 | −0.00171 | −0.00809 | −92.6 | | yes[g] | $3d^4$ $^5D_0$ | 0.004468 |
| 30 | Zn | −0.000153 | −0.01126(10) | +0.00089 | −0.00238 | −0.01173 | −129.0 | | yes | $3s^2 3p^6 3d^{10}$ $^1S_0$ | 2.975 |
| 42 | Mo | −0.000259 | −0.01769(22) | +0.0020 | −0.00384 | −0.0186 | −197.9 | | yes | $4d^4$ $^5D_0$ | 0.019707 |
| 50 | Sn | −0.000331 | −0.02146(30) | +0.0028 | −0.0049 | −0.0227 | −238.9 | | yes | $(5s)^2$ $^1S_0$ | 11.64 |
| 64 | Gd | −0.000449 | −0.02659(44) | +0.0040 | −0.0069 | −0.0288 | −299.4 | | yes | $4f^6 6s^2$? $^7F_0$? | ? |
| 82 | Pb | −0.000605 | −0.03248(62) | +0.0046 | −0.0098 | −0.0371 | −382.9 | | yes | $(6s)^2$ $^1S_0$ | 13.04 |

[a] Numbers in parentheses are uncertainties arising from 2% uncertainty in nuclear radius.
[b] Crude estimates based on Eq. (32) with average energy denominator of 15 MeV.
[c] Does not include electron polarization correction $g_5$.
[d] The numbers in parenthesis are estimated theoretical errors for the cases for which experimental data are available.
[e] See reference 2. Experimental errors in parentheses.
[f] For Mg in $MgH_2$, the measured value is −29.6(7). Private communication from J. Menes.
[g] Ion stable by only 0.02 eV.
[h] Lowest 1+ state with single particle excitation. There is a somewhat lower 1+ state with a two-particle excitation.

Thus far, all evidence indicates that the muon and the electron have the same interactions with other particles. The most precise test of this is given by the muon $g$-2 experiment[25] whose result is $(g-2)/2 = 0.001162(5)$ in excellent agreement with the theoretical value, $g_1$, given in Eq. (1). Still it is of importance to look for some intrinsic difference between the electron and the muon other than their mass, which may possibly be of electromagnetic origin. It is conceivable that the muon is a polarizable structure whose magnetic moment will be altered by the strong electric field in the vicinity of the nucleus.

A crude estimate for the alteration of the magnetic moment of a muon due to polarization by the Coulomb field is given by:

$$\Delta g/g \sim (\alpha Z)^6 (l/\lambdabar_c)^2, \tag{40}$$

where $l$ is the "size" of the muon, and $\lambdabar_c$ is its Compton wavelength. Because of the finite size of the nucleus, this will be an overestimate for very high $Z$. The result (40) arises from picturing the electron and muon as $1s$ and $2s$ states, respectively, of some hydrogen-like structure whose dipole matrix elements are of order $l$, and whose energy level differences are of the order $m_\mu$, the muon mass. This method of estimating a muon polarization contribution to $g$ was suggested by H. Primakoff.

It is interesting to consider what limit the present experimental and theoretical knowledge of the $g$ values of mu-mesonic atoms places on such a muon polarizability. The most sensitive test is given by the $g$ value of $\mu-$ with $S$ for which the experimental error is about 1 part in $10^4$ and the theoretical error is also about 1 part in $10^4$ due to the uncertainty in the nuclear polarization correction $g_4$. If $\Delta g/g$ is taken to be $10^{-4}$ for $S$, we find from Eq. (40) that $l$ can be no larger than 10 F.

## APPENDIX. REDUCTION OF SPIN DEPENDENT INTERACTIONS

Consider the term appearing in Eq. (24),

$$T_i \equiv (r_i')^{-3} \boldsymbol{\alpha}\cdot\boldsymbol{\mu}_i \times \mathbf{r}_i', \tag{A1}$$

where $\mathbf{r}_i' = \mathbf{r} - \mathbf{R}_i$, the muon-nucleon relative coordinate. It is desired to find the part of $T$ which is the product of a pseudovector in the muon variables and a pseudovector in the nucleon variables. This may be done conveniently by working with the spherical components of the vectors, using relationships such as

$$(\boldsymbol{\mu}\times\mathbf{r})_q{}^1 = -i(8\pi/3)^{1/2} \times r \sum_{q'} C(111; q', q-q', q)\mu_{q'}{}^1 Y_{1,q-q'}, \tag{A2}$$

and expanding $(r')^{-3}$ as

$$(r')^{-3} = \sum_l \sum_m \frac{4\pi(-1)^m}{2l+1} g_l(r,R_i) Y_{lm}(\Omega) Y_{l,-m}(\Omega_i). \tag{A3}$$

After a good deal of algebra, the contributing part of $T_i$ reduces to

$$T_i{}^c = -i(8\pi/3)\{(-rg_0 + \tfrac{1}{3}R_i g_1)[\mu_i{}^1\cdot(\alpha^1 Y^1)^1] + (2\pi)^{1/2}(\tfrac{1}{3}R_i g_1 - \tfrac{1}{5} r g_2)[(\mu^1 Y^2)_i{}^1 \cdot (\alpha^1 Y^1)^1]\}, \tag{A4}$$

where the tensor product notation is defined by

$$(A^r B^s)_q{}^t = \sum C(rst; q', q-q', q) A_{q'}{}^r B_{q-q'}{}^s, \tag{A5}$$

and the quantities within square brackets are ordinary scalar products. If we define

$$\begin{aligned} f_1(r,R_i) &= -g_0 + \tfrac{1}{3}(R_i/r) g_1, \\ f_2(r,R_i) &= \tfrac{1}{3}(R_i/r) g_1 - \tfrac{1}{5} g_2, \end{aligned} \tag{A6}$$

and note that only the $z$ components of the vectors can contribute to the matrix elements of interest, then $T_i{}^c$ may be written

$$T_i{}^c = [f_1 \mu_{iz} + (2\pi)^{1/2} f_2 (\mu Y_2)_{iz}](\boldsymbol{\alpha}\times\mathbf{r})_z. \tag{A7}$$

The radial functions are

$$\begin{aligned} f_1 &= -r^{-3}, & r > R_i, \\ &= 0, & r < R_i \end{aligned} \tag{A8}$$

$$\begin{aligned} f_2 &= 0, & r > R_i \\ &= +R_i^{-3}, & r < R_i \end{aligned} \tag{A9}$$

and $(\mu Y_2)_{iz}$ is defined by (27). Finally, one takes the diagonal matrix element of $T_i$ with respect to the muon ground state to give the effective nucleon operator appearing in (25). The functions $\mathfrak{M}_1(R_i)$ and $\mathfrak{M}_2(R_i)$ given by (27) are defined by

$$\mathfrak{M}_k(R_i) = m_j^{-1} \langle 0_\mu | f_k(r,R_i)(\boldsymbol{\alpha}\times\mathbf{r})_z | 0_\mu \rangle. \tag{A10}$$

A similar reduction may be applied to the orbital term

$$U_p = (r_p')^{-1} \boldsymbol{\alpha}\cdot\mathbf{v}_p \tag{A11}$$

in Eq. (24), where it is understood that the term should be symmetrized between $\mathbf{r}_p$ and $\mathbf{v}_p$. The result of this reduction to the contributing part of $U_p$ is

$$U_p{}^c = \frac{1}{2M_N}(f_1 - f_2) L_{pz} (\boldsymbol{\alpha}\times\mathbf{r})_z. \tag{A12}$$

The results (A7) and (A12) are combined in Eq. (25).

[25] G. Charpak, F. J. M. Farley, R. L. Garwin, T. Muller, J. C. Sens, V. L. Telegdi, and A. Zichichi, Phys. Rev. Letters 6, 128 (1961); Phys. Letters 1, 16 (1962).