High-Frequency Region of the Spectrum of Electron and Positron Bremsstrahlung*

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The cross section for a bremsstrahlung process in which almost all the energy of an incident high-energy electron is transferred to the photon is calculated analytically, using techniques previously developed for atomic photoeffect. Only terms of relative orders a^4, a^2q^2, and q^4 are neglected, where $a \equiv Ze^2$ and q is the momentum of the outgoing low-energy electron (with $\hbar = c = m_e = 1$). Similar results are obtained for positron bremsstrahlung and pair production; previous work on the photoeffect is extended. The accuracy of these predictions is discussed, and they are compared with experiments.

I. INTRODUCTION

CONTRARY to the Bethe-Heitler formula, the cross section remains finite for a bremsstrahlung process in which the momentum q of the outgoing electron is zero. A low-energy electron in the Coulomb field does not look like a plane wave, but rather like a bound state, and (as was noted by Fano, Koch, and Motz[1]) the process may be viewed as an approximate inverse of the atomic photoelectric effect. In recent theoretical calculations[1–5] the leading terms of an expansion in $a \equiv Ze^2$ have been obtained for this limiting bremsstrahlung cross section σ_{tip}.[6] For example, from previous photoeffect results, σ_{tip} from electrons of very high energy k is immediately given as

$$\sigma_{\text{tip}} = (4\pi e^2 a^3/k) e^{-\pi a}[1-(4\pi/15)a], \qquad (1)$$

if terms of relative order a^2 are omitted.[3] The convergence of such a series for large a is evidently uncertain. Complications also arise since experiments[1,7,8] are performed for small but finite q. In the present paper we attack both of these problems, by calculating to several more orders in a and retaining dependence of the cross section on q; our results neglect only relative orders a^4, a^2q^2, and q^4 in σ_{tip} from very high energy electrons. Only the leading term in $1/k$ has been retained in the expression for the cross section. Furthermore, as in Eq. (1), this expression is not a complete series expansion in a, since some normalization factors are left unexpanded. These factors, as discussed later, are not entirely unique; they are chosen to improve the apparent convergence of the series of the total cross section at the tip.

We use a formalism previously developed for the photoelectric effect[9,10]; the same calculation will also give us new results for other related processes. An improved version of the formalism is outlined in Sec. II. The method requires calculation of successive terms of an expansion in the angular momentum l of the low-energy particle; each term contributes in relative order a^{2l}. We obtain an expression for the s state cross section of any of these related processes, valid neglecting only relative $O(a^4)$, etc.; all the remaining conclusions of this paper follow directly from this result. First, we have new analytical results for s-state photoeffect; these are discussed in Sec. III and compared with previous work.[9,10] For other processes we also need the results previously obtained for p-state cross sections.[3,10] Then in Sec. IV we give predictions for the bremsstrahlung cross section σ_{tip} from very high energy electrons, with the assumption that the low-energy outgoing electron is not observed. After discussing the extrapolation to finite incident electron energy we compare with experiments. In Sec. V we give the analogous results for positron bremsstrahlung and for pair production in which one member of the pair is of low energy.

II. GENERAL FORMALISM

It has been shown[3,9] that for any very high energy Coulomb-field vertex process in which there is almost complete transfer of momentum between a photon and an electron or positron, the cross section, integrated over all states of the outgoing high-energy particle, is determined from the integral[11]

$$I = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \int_0^{\infty} d\rho\rho\, e^{-i\epsilon(z-z')} \left(\frac{r-z}{r'-z'}\right)^{ia} F(r,r'), \qquad (2)$$

where ϵ is the total energy of the low-energy electron in the process. If we sum over the polarizations of the high-energy particle, then[11]

$$F(r,r') = \sum \psi^{\dagger}(r') \left(\frac{1-\alpha_z}{2}\right) \psi(r), \qquad (3)$$

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[1] U. Fano, H. W. Koch, and J. W. Motz, Phys. Rev. **112**, 1679 (1958).

[2] K. W. McVoy and U. Fano, Phys. Rev. **116**, 1168 (1959).

[3] R. H. Pratt, Phys. Rev. **120**, 1717 (1960).

[4] W. R. Johnson and C. J. Mullin, Phys. Rev. **119**, 1270 (1960).

[5] R. T. Deck, C. J. Mullin, and C. L. Hammer (to be published).

[6] Unrationalized charge units are used throughout the paper. Also $\hbar = c = m_e = 1$; and $x = O(y)$ shall mean x is of order y.

[7] H. E. Hall and A. O. Hanson, Research in Nuclear Physics, University of Illinois Technical Report No. 26, ONR1834(05) (unpublished).

[8] C. E. Dick, I. Granacher, and W. C. Miller, Bull. Am. Phys. Soc. **7**, 339 (1962).

[9] R. H. Pratt, Phys. Rev. **117**, 1017 (1960).

[10] R. H. Pratt, Phys. Rev. **119**, 1619 (1960).

[11] These differ from the corresponding expressions of reference 9 in that none of the r dependence of the low-energy wave function is explicitly extracted from ψ.

where $\mathbf{r}$ and $\mathbf{r}'$ are expressed in cylindrical coordinates (ϱ,z) and (ϱ',z'), and $\varrho=\varrho'$; under these circumstances F depends only on $|\varrho|$, not ϱ itself. In Eq. (3), ψ is the wave function of the low-energy electron and the sum is over all states of this electron which will not be distinguished. In Sec. V we will generalize these results to include the cases for which the low-energy particle in the process is a positron.

For the purposes of this paper it is sufficient to take ψ as a partial-wave solution of the Dirac equation of definite j, l, m, and to sum over m. In photoeffect this gives us the cross section σ_{jl} from a given shell, which is the quantity of direct physical interest. In bremsstrahlung, if we do not observe the outgoing low-energy electron, the cross section can be written simply as the sum of the cross sections for emission of the electron into definite partial-wave states, without any interference terms: $\sigma=\sum\sigma_{jl}$. Since each σ_{jl} is, in principle, a physically measurable cross section, the series for σ (which resembles a Born series since $\sigma_{jl}\sim a^l$) should converge even if the Born series does not. Thus, if in Eq. (3) we sum over m for fixed j, l, we obtain

$$\begin{aligned}4\pi F_{\pm}(r,r')&=kA(r)A(r')P_l(\cos\omega)+ka^2B(r)B(r')P_{l'}(\cos\omega)\\&\quad+iaA(r')B(r)\left[k\cos\theta P_l(\cos\omega)\mp\sin^2\theta\frac{d}{d\cos\theta}P_l(\cos\omega)\right]-iaA(r)B(r')\\&\quad\times\left[k\cos\theta' P_l(\cos\omega)\mp\sin^2\theta'\frac{d}{d\cos\theta'}P_l(\cos\omega)\right],\end{aligned}\tag{4}$$

where $A(r)$ and $aB(r)$ are, respectively, the large- and the small-component radial wave functions of the low-energy particle; the choice of signs $(\pm)$ corresponds to the possible cases $j=l\pm\frac{1}{2}$, $l'=l\pm1$, $k=j+\frac{1}{2}$, $\omega=\theta-\theta'$, $\cos\theta=z/r$, $\sin\theta=\rho/r$.

A technique for evaluating integrals of the form of Eq. (2) as a power series in a was developed for the photoeffect.[9] The method is also valid for bremsstrahlung at the tip and the other related processes,[3] giving a series in a and q. It was shown that if the radial wave functions, after removing a factor $r^{\gamma-1}$ [where $\gamma=(k^2-a^2)^{1/2}$] and a normalization factor N, are expanded in series in (ar) and (qr),[12]

$$A(r)=Nr^{\gamma-1}\sum A_n(ar)^n,\quad B(r)=Nr^{\gamma-1}\sum B_n(ar)^n,\tag{5}$$

where A_n and B_n are constants, then the nth term of the expansion first contributes to the cross section in relative order a^n or higher, and hence to any desired order it is sufficient to replace the sum in (5) by finite polynomials in r. Making the transformations

$$z=\rho\sinh\omega,\quad z'=\rho\sinh\omega',\tag{6}$$

performing the ρ integration, making the transformations

$$x=\omega+i\tfrac{1}{2}\pi,\quad y=\omega'-i\tfrac{1}{2}\pi,\tag{7}$$

and returning the contours to the real axis, the integral (2) becomes

$$\begin{aligned}I=\frac{\Gamma(2\gamma+2)}{\epsilon^{2\gamma+2}}e^{-\pi a}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,(\sinh^2x)^{(\gamma-k)/2}\\\times(\sinh^2y)^{(\gamma-k)/2}\frac{e^{-i[\theta(x)-\theta(y)]}}{(\cosh x+\cosh y)^{2\gamma+2}}\\\times\sum_n\frac{\Gamma(2\gamma+2+n)}{\Gamma(2\gamma+2)\epsilon^n}\frac{\sinh x^k\sinh y^kF_n(x,y)}{(\cosh x+\cosh y)^n},\end{aligned}\tag{8}$$

where

$$\begin{aligned}F(x,y)&=(rr')^{\gamma-1}\textstyle\sum_n\rho^nF_n(x,y),\\\theta(x)&=ax+\tfrac{1}{2}\pi(\gamma-k)S_x,\end{aligned}\tag{9}$$

with S_x the usual step function: ±1 according as $x>0$ or $x<0$.[13]

To calculate an s-state cross section neglecting only relative $O(a^4)$, etc., we need in Eq. (5) the first four terms of the expansion of $A(r)$ and the first three of $B(r)$. Systematically expanding (8) in a and evaluating the integrals, we get

$$I=\frac{e^{-\pi a}\Gamma(2\gamma+2)}{4\pi\epsilon^{2\gamma+2}}\sum_{n=0}^{4}\frac{1}{\epsilon^n}\frac{\Gamma(2\gamma+2+n)}{\Gamma(2\gamma+2)}I_n,\tag{10}$$

where $\gamma=(1-a^2)^{1/2}$ and

$$\begin{aligned}\frac{3}{2a^2}I_n=\sum_{i+j=n,i\le j}a_{ij}A_iA_j+\sum_{i+j=n,i\le j}b_{ij}B_iB_j\\+\sum_{i+j=n}c_{ij}A_iB_j,\end{aligned}\tag{11}$$

and to the desired order the only nonvanishing coeffi-

[12] In the form of Eq. (5), the series is written as in ar so that the nth coefficient may contain terms like (a/q) to powers as high as the nth.

[13] $\theta(x)$ differs from that used in the photoeffect papers, since there the factors $e^{-\delta r}$ of the bound wave functions were not expanded. The present procedure is simpler, even for photoeffect.

cients are

$$a_{00}=\frac{1}{2}-\frac{\pi}{6}a+\left(1-\frac{71\pi^2}{576}\right)a^2-\left(\frac{5}{18}-\frac{25\pi^2}{576}\right)\pi a^3,$$

$$a_{01}=\frac{1}{4}-\frac{\pi}{20}a+\left(\frac{7}{48}-\frac{7\pi^2}{120}\right)a^2+\left(\frac{41}{400}+\frac{\pi^2}{120}\right)\pi a^3,$$

$$a_{02}=-\frac{1}{4}a^2+\frac{23}{300}\pi a^3, \quad a_{11}=\frac{1}{30}-\left(\frac{1}{100}+\frac{\pi^2}{180}\right)a^2+\frac{23}{900}\pi a^3,$$

$$a_{03}=-\frac{1}{15}a^2+\frac{\pi}{70}a^3, \quad a_{12}=-\frac{1}{20}a^2+\frac{\pi}{140}a^3, \quad a_{13}=-\frac{1}{70}a^2,$$

$$b_{00}=1+\left(\frac{25}{12}-\frac{\pi^2}{6}\right)a^2+\frac{5}{12}\pi a^3,$$

$$b_{01}=-\frac{3}{4}a^2+\frac{3}{20}\pi a^3, \quad b_{02}=-\frac{4}{15}a^2, \tag{12}$$

$$c_{00}=-1+\frac{\pi}{5}a-\left(\frac{11}{6}-\frac{7\pi^2}{30}\right)a^2-\left(\frac{2}{75}+\frac{\pi^2}{30}\right)\pi a^3,$$

$$c_{01}=\frac{1}{2}a^2-\frac{\pi}{6}a^3, \quad c_{02}=\frac{2}{15}a^2-\frac{1}{35}\pi a^3,$$

$$c_{10}=-\frac{1}{3}-\left(\frac{1}{4}-\frac{\pi^2}{18}\right)a^2-\frac{37}{180}\pi a^3, \quad c_{11}=\frac{7}{60}a^2-\frac{3\pi}{140}a^3,$$

$$c_{12}=\frac{1}{30}a^2, \quad c_{20}=\frac{3}{10}a^2-\frac{3}{70}\pi a^3, \quad c_{30}=\frac{1}{10}a^2.$$

Equations (10)–(12) are the major result of this paper; substitution of specific choices for the radial wave function coefficients A_n and B_n for the low-energy particle immediately gives the corresponding s-state cross section.

III. PHOTOEFFECT

For K-shell photoeffect we may take the radial coefficients of the bound-state wave function as

$$\begin{aligned} &A_0=2+O(a^4), && B_0=-(1+\tfrac{1}{4}a^2)+O(a^4),\\ &A_1=-2+O(a^4), && B_1=1+O(a^2),\\ &A_2=1+O(a^2), && B_2=-\tfrac{1}{2}+O(a^2),\\ &A_3=-\tfrac{1}{3}+O(a^2). \end{aligned} \tag{13}$$

This choice is not entirely unique, since it depends on the definition of N in Eq. (5). Then from (10)

$$\sum_{n=0}^{4}\frac{1}{\epsilon^n}\frac{\Gamma(2\gamma+2+n)}{\Gamma(2\gamma+2)}I_n=\frac{2}{3}a^2\left(1-\frac{4\pi}{15}a+\frac{37}{12}a^2-\frac{139}{720}\pi^2a^2+\frac{77}{720}\pi^3a^3-\frac{7681}{6300}\pi a^3\right). \tag{14}$$

The total cross section for K-shell photoeffect in the limit of high photon energy k may then be written as

$$\sigma_K=\sigma_0\frac{a^{2\gamma_1-2}}{\epsilon^{2\gamma_1+2}}e^{-\pi a}(1-0.837a-0.100a^2+0.552a^3), \tag{15}$$

where

$$\sigma_0=4\pi e^2a^5/k, \quad \gamma_1=\epsilon=(1-a^2)^{1/2}. \tag{16}$$

The term of relative $O(a^2)$ was earlier obtained in this way,[9] and has recently been verified by Weber and Mullin.[14]

In the result (15) most, but not all, dependence on a has been expanded in series. Factors which have not been expanded converge slowly, or appear to worsen convergence of the series used, or worsen agreement with the exact results obtained numerically.[9] Justification for retaining the factor $e^{-\pi a}$, which appears in a natural way in the calculations of all these related processes, seems quite strong; there is less justification for the other factors and we have here made a somewhat different choice than previously,[9] in order to be able to generalize to the bremsstrahlung problem.

The unscreened cross section for photoeffect from the L_I subshell may be obtained in the same way. To the same orders as before, the radial coefficients become

$$\begin{aligned} &A_0=2-\tfrac{1}{4}a^2, && B_0=-1-\tfrac{1}{8}a^2,\\ &A_1=-2-\tfrac{3}{8}a^2, && B_1=\tfrac{3}{4},\\ &A_2=\tfrac{3}{4}, && B_2=-\tfrac{1}{4},\\ &A_3=-\tfrac{1}{6}, \end{aligned} \tag{17}$$

and then

$$\sum_{n=0}^{4}\frac{1}{\epsilon^n}\frac{\Gamma(2\gamma+2+n)}{\Gamma(2\gamma+2)}I_n=\frac{2}{3}a^2\left(1-\frac{4\pi}{15}a+\frac{10}{3}a^2-\frac{139}{720}\pi^2a^2+\frac{77}{720}\pi^3a^3-\frac{7561}{6300}\pi a^3\right). \tag{18}$$

The corresponding cross section is

$$\sigma_{2s}=\sigma_0\frac{\delta^{2\gamma_1+1}e^{-\pi a}}{a^3\epsilon^{2\gamma_1+2}}(1-0.837a+0.214a^2+0.561a^3), \tag{19}$$

where

$$\gamma_1=(1-a^2)^{1/2}, \quad \epsilon=\left(\frac{1+\gamma_1}{2}\right)^{1/2}, \quad \delta=(1-\epsilon^2)^{1/2}. \tag{20}$$

From (15) and (19) we also obtain for the ratio of cross sections

$$\sigma_{2s}/\sigma_{1s}=\tfrac{1}{8}(1-0.120a^2+0.269a^3), \tag{21}$$

which shows, in agreement with the numerical calculations,[9,10] that the ratio is nearly $\frac{1}{8}$ for all elements. The numerical and analytical results for the cross sections themselves are shown in Table I.

[14] T. A. Weber and C. J. Mullin, Phys. Rev. **126**, 615 (1962).

TABLE I. High-energy limit of $1s$ and $2s$ photoeffect. σ/σ_0 is given as a function of a: (1) as determined by Eq. (15), (2) by machine calculation,[a,b] and (3) by Eq. (19).

	$1s$		$2s$	
a	(1)	(2)	(3)	(2)
0.1	0.696	0.699	...	...
0.3	0.397	0.394	0.0496	0.0491
0.4	0.317	0.315	0.0403	0.0392
0.5	0.268	0.260	0.0333	0.0329
0.6	0.242	0.222	0.0289	0.0286

[a] See reference 9.
[b] See reference 10.

IV. ELECTRON BREMSSTRAHLUNG

We now proceed to our main task, to calculate the cross section for the high-frequency region of the bremsstrauhlung spectrum, in the limit of high incident electron energy, neglecting only relative orders a^4, a^2q^2, q^4, etc., where $q\ll 1$ is the final electron momentum. In this approximation, the cross section is simply the sum of the partial cross sections for final s and p electrons

$$\sigma_{\text{brem}}=\sigma_s+\sigma_{p_{1/2}}+\sigma_{p_{3/2}}. \tag{22}$$

The s-state cross section σ_s can be obtained to the desired order from Eqs. (10)–(12); the p-state cross sections, which are of relative $O(a^2)$, are already known to the needed order from the L-shell photoeffect work.[3,10]

For the s state we have

$$\sigma_s=[(2\pi e)^2/k]I, \tag{23}$$

with I as defined in Eq. (2). The radial functions A and B of (4) and (5) may be expanded to the needed order in r as

$$A(r)=\frac{Gr^{\gamma-1}2^\gamma}{\Gamma(2\gamma+1)}\left(1+\frac{3}{8}q^2\right)\times[A_0+A_1ar+A_2(ar)^2+A_3(ar)^3], \tag{24}$$

$$B(r)=\frac{Gr^{\gamma-1}2^\gamma}{\Gamma(2\gamma+1)}\left(1+\frac{3}{8}q^2\right)[B_0+B_1ar+B_2(ar)^2],$$

where

$$G=\exp\left(\frac{\pi a\epsilon}{2q}\right)\left|\Gamma\left(\gamma+i\frac{a\epsilon}{q}\right)\right|q^{\gamma-1/2}(2\pi)^{-1/2}, \tag{25}$$

and $\epsilon=(1+q^2)^{1/2}$ is the total energy of the final electron and, $\gamma=\gamma_1=(1-a^2)^{1/2}$. Note that[15]

$$\lim_{q\to 0} G=a^{\gamma-1/2}. \tag{26}$$

With this choice of normalization N, the coefficients in the radial expansion are given to needed order as

$$\begin{aligned} A_0&=2-\tfrac{1}{2}a^2-\tfrac{1}{2}q^2, & B_0&=-1+\tfrac{1}{4}q^2,\\ A_1&=-2-\tfrac{1}{3}a^2-\tfrac{1}{3}q^2, & B_1&=\tfrac{2}{3}-\tfrac{1}{3}(q/a)^2,\\ A_2&=\tfrac{2}{3}-\tfrac{1}{3}(q/a)^2, & B_2&=-\tfrac{1}{6}+\tfrac{1}{3}(q/a)^2,\\ A_3&=-\tfrac{1}{9}+(2/9)(q/a)^2. \end{aligned} \tag{27}$$

The terms in (q/a) arise, as already remarked, from writing the series in q and a formally as a series in a only. Inserting (27) in (10)–(12), and also expanding ϵ and γ_1 in a, we have

$$\sum_{n=0}^{4}\frac{1}{\epsilon^n}\frac{\Gamma(2\gamma_1+2+n)}{\Gamma(2\gamma_1+2)}I_n=\frac{2}{3}a^2\left(1-\frac{4\pi}{15}a+\frac{13}{4}a^2-\frac{139}{720}\pi^2a^2+\frac{77}{720}\pi^3a^3-\frac{7241}{6300}\pi a^3+\frac{1}{6}q^2+\frac{22}{315}\pi aq^2\right). \tag{28}$$

We then obtain for the s-state cross section

$$\sigma_s=\frac{8\pi e^2a^2}{k\epsilon^3}\frac{G^2e^{-\pi a}}{\Gamma(2\gamma_1+1)}(1-0.837a+0.318a^2+0.564a^3+0.417q^2+0.012aq^2). \tag{29}$$

The normalization in front of this expansion has been chosen in such a way that under the formal substitution $q^2\to -a^2$ the expansion (28) reduces to that of Eq. (14) for the K-shell photoeffect.[16] The good convergence of the photoeffect result (in the sense that it agrees well with the numerical calculations even for large a) encourages the hope that the bremsstrahlung result in the form of Eq. (29) may also converge fairly well.

For p-state bremsstrahlung the cross sections, to the desired order, may be written[3]

$$\begin{aligned} \sigma_{p_{1/2}}&=F(\nu)\frac{2\pi e^2a^{2\gamma_1+3}}{k\Gamma(2\gamma_1+1)}e^{-\pi a}\left(1+\frac{4\pi}{9}a\right),\\ \sigma_{p_{3/2}}&=F(\nu)\frac{256\pi e^2a^{2\gamma_2+1}}{3k\Gamma(2\gamma_2+1)}e^{-\pi a}\left(1-\frac{33\pi}{140}a\right), \end{aligned} \tag{30}$$

where

$$\gamma_1=(1-a^2)^{1/2},\quad \gamma_2=(4-a^2)^{1/2},$$

and

$$F(\nu)=(1+\nu^{-2})(1-e^{-2\pi\nu})^{-1},\quad \nu=a/q. \tag{31}$$

These results were check by an independent calculation using the formalism of Sec. II; the dependence on a has also been verified by Gavrila[17] in his work on L-shell photoeffect.

From Eq. (22) we conclude that the cross section for bremsstrahlung near the tip can be described by the

[15] Also note that in the approximation

$$\gamma=\epsilon=1,\quad G=a^{1/2}(1-e^{-2\pi a/q})^{-1/2}$$

in agreement with the result of reference 3 for the behavior of the cross section near the tip.

[16] It is easy to see that it is possible to connect the two expansions in this way to *all* orders. For looking at the Dirac equation for large and small components, under this substitution the continuum wave function with $\epsilon=(1+q^2)^{1/2}$ becomes the bound wave function of total energy $(1-a^2)^{1/2}$, except for differences in normalization. Hence under this substitution, I_{brem} differs from I_{photo} only by a simple normalization factor.

[17] Mihai Gavrila, Phys. Rev. **124**, 1132 (1961).

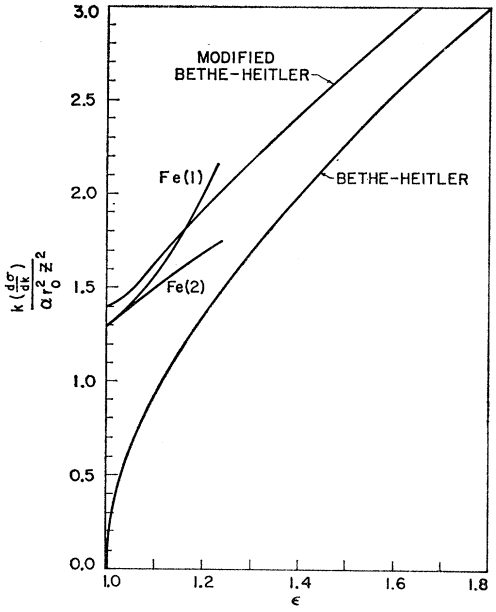

FIG. 1. High-frequency region of extremely relativistic electron bremsstrahlung spectrum for Fe as calculated by (1) Eqs. (22), (29), and (30); (2) Eq. (33). The original and modified (see reference 18) Bethe-Heitler predictions are also shown. $k(d\sigma/dk)/\alpha r_0^2 Z^2$ is plotted against the energy ϵ of the low-energy outgoing electron, where $\epsilon=k_{\max}-k+1$ in units of m_e.

sum of (29) and (30), each of which has a physical meaning. However, it is also instructive to combine these terms into one series to facilitate comparison with other work[5] and for simplicity of calculation. Making the replacements

$$\gamma_2 \rightarrow 2\gamma_1, \quad F(\nu) \rightarrow (1+\nu^{-2})a^{-1}G^2, \tag{32}$$

the total cross section may be written

$$\sigma_{\text{brem}}=\frac{8\pi e^2a^2}{k\epsilon^3}\frac{G^2e^{-\pi a}}{\Gamma(2\gamma_1+1)}(1-0.837a+1.457a^2 +0.255a^3+1.556q^2-0.297aq^2), \tag{33}$$

which neglects only relative orders a^4, a^2q^2, q^4, etc.

In Fig. 1 we show the predictions of our theory for the shape of the Fe spectrum near the tip for highly relativistic electrons calculated in two different ways: (1) using Eqs. (22), (29), and (30); (2) using Eq. (33). The difference between these is mainly due to the added ϵ^{-3} dependence which Eq. (33) gives to the p-state cross sections. For reference we show the corresponding Bethe-Heitler prediction, which, of course, is not valid in this high-frequency region. We also show the result of an analytic extrapolation[18] which reduces to the Bethe-Heitler formula away from the tip and is also moderately accurate in the tip limit. The results are given for the dimensionless quantity $k(d\sigma/dk)/\alpha r_0^2Z^2$, where $(d\sigma/dk)$ is the cross section, α is the fine structure constant, r_0 is the classical electron radius, k is the photon energy, and Z is the nuclear charge; they are plotted against the energy ϵ of the outgoing low energy electron, i.e., against $k_{\max}-k+1$.

At the tip, $q\rightarrow 0$, and the total high-energy cross section is from (33)

$$\sigma_{\text{brem}}(q=0)=\frac{8\pi e^2a^3a^{2\gamma_1-2}}{k\Gamma(2\gamma_1+1)}e^{-\pi a}(1-0.837a +1.457a^2+0.255a^3). \tag{34}$$

The equivalent of the a^2 term in the above expansion (34) has recently been calculated[19] by Deck, Mullin, and Hammer[5] (to be called the DMH result). The difference in the results obtained by (22), (29), and (30) and from (33) and (34) and the DMH formula provide some measure of the uncertainties in these expansion procedures. For the tip, these three methods are nearly equal for low-Z elements, differ by a maximum of 2% for Ag($Z=47$), and less than 4% for Th($Z=90$). The closeness of these results suggests the choice of Eq. (34) for actual calculations because of its relative simplicity. It is instructive to rewrite Eq. (34) with the choice of normalization factor of DMH, thus changing both of the relative $O(a^2)$ and $O(a^3)$ coefficients. The predictions for high-energy electron bremsstrahlung cross section at the tip obtained from (34) and its modified form are shown in Fig. 2 as a function of Z. The difference for Ag is only 3%, whereas for Th it is 35%.

There are two sources of uncertainty in this calculation of the total cross section at the tip: (1) the accuracy with which we have calculated the s and p cross sections, and (2) the contribution from higher partial waves. We argue that we have taken expressions which well represent the s and p cross sections; for medium-weight elements they are certainly quite good, but in the heavy elements it is possible that the errors are large (30% or more). Again, in medium weight elements the contributions of higher states of relative $O(a^4)$ are certainly small. But in heavy elements, for which the p states contribute about half of the s-state contribution, it has been argued[3] that the higher shells are also important. A calculation of the d-state contribution, to lowest order, indicates that its relative magnitude is approximately $2a^4$, and so is large. In Fig. 2 we show the modification to the predictions of Eq. (34) due to this d-state contribution, and it is fairly sizable for large Z. It thus appears that realistic quantitative predictions cannot yet be made for the very heavy

[18] The extrapolation is performed by multiplying the Bethe-Heitler prediction (at high energy) by the factor $\exp[\pi a(\epsilon/q-1)] \times|\Gamma(1+ia\epsilon/q)|^2/|\Gamma(1+ia)|^2$ which tends to unity for high q where the original prediction holds. For $q\ll 1$ this factor is proportional to $1/q$, and hence the modified Bethe-Heitler formula becomes finite at the tip, with the correct value to lowest order in a, and also the correct large characteristic $e^{-\pi a}$ factor of the tip cross section. However, the $-4\pi a/15$ term of the tip is not obtained.

[19] This result, which is only valid through relative $O(a^2)$, has a different appearance than Eq. (34) because the choice of unexpanded normalization factors is not the same. The two results are, of course, equivalent if the same factors are retained in the normalization.

elements, but that accurate results are now available for medium-weight and light elements.

We now wish to compare these results with the recent experiments on the high-frequency region of the bremsstrahlung spectrum. To do this requires a further approximation, since the discussion to this point has kept only the leading order $1/k$ energy dependence of the cross section on the energy of the high-energy incident electron, and all present experiments are at low enough energies so that the deviations from this high-energy approximation are sizable. Now, the energy dependence of bremsstrahlung and photoeffect are the same through order a, and hence from Gavrila's[20] work we know the complete energy dependence of the bremsstrahlung cross section through this order.[3] A function $P(\beta)$, such that $P(1)=1$, multiplies the entire cross section, and a function $Q(\beta)$, where $Q(0)=0$ and $Q(1)=-4\pi/15$, further multiplies the term of relative order a. Both of these changes tend to increase the cross section as β, the velocity of the incident electron, decreases. Our approximation in using this procedure for bremsstrahlung, as discussed in reference 3, is the neglect of the further energy dependence of the terms of relative $O(a^2)$, etc., and we are again encouraged by the good results obtained in this way for photoeffect cross sections. Our confidence will of course be greater for high-energy incident electrons, and we will not discuss bremsstrahlung from electrons of energy less than 2 MeV.

Using electrons of kinetic energy 15.1 MeV, Fano, Koch, and Motz[1] measured σ_{tip} for W and Hall and Hanson measured it for Th. For W the result for $k(d\sigma/dk)/Z^2$ was 1.38±0.41 mb and for Th 1.6±0.16 mb. The present theory, including d states (extrapolated to finite energies), gives 1.41 mb and 1.66 mb with rather large uncertainties; the agreement seems acceptable. For 4.54-MeV electrons on Au, Fano, Koch, and Motz[1] found 1.8±0.3 mb, to be compared with 1.7 mb from our theory.

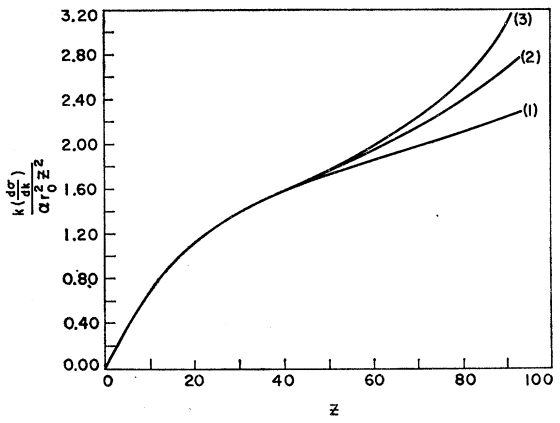

FIG. 2. Z dependence of high-energy electron bremsstrahlung cross section at the tip as calculated by (1) Eq. (34); (2) Eq. (34) plus d-state contribution; (3) the equivalent of (34) but with the choice of normalization of DMH. $k(d\sigma/dk)/\alpha r_0^2 Z^2$ is plotted against the nuclear charge Z.

[20] Mihai Gavrila, Phys. Rev. **113**, 514 (1959).

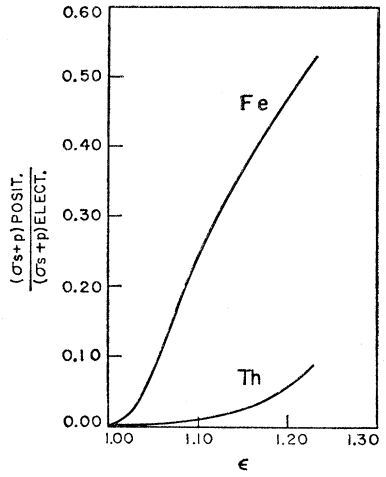

FIG. 3. Ratio of positron to electron bremsstrahlung cross sections for Fe and Th in the high-frequency region; d states are not included. The figure may also be interpreted as giving the ratio of production of low-energy positrons to production of low-energy electrons in high-energy pair production.

V. POSITRON BREMSSTRAHLUNG AND AND PAIR PRODUCTION

It is easily verified that the integral I of Eqs. (2) and (3) also describes processes involving a low-energy positron, if we make the substitution $a\rightarrow -a$. Hence the corresponding results are obtained for the high-frequency region of positron bremsstrahlung simply by making this substitution in Eqs. (29)–(33). One must note, however, that $G(-a)=G(a)\exp(-\pi a\epsilon/q)$ and so Eq. (26) becomes

$$\lim_{q\rightarrow 0} G(-a)=e^{-\pi a\epsilon/q}|a|^{\gamma-1/2}. \tag{35}$$

Low-energy positrons do not get near the nucleus, and the cross section for positron processes is accordingly suppressed, vanishing at the tip $q=0$. For this reason, Eq. (34) in the positron case becomes

$$\sigma_{\text{pos.brem}}(q\rightarrow 0)=\frac{8\pi e^2 a^3 a^{2\gamma_1-2}}{k\Gamma(2\gamma_1+1)}e^{\pi a-2\pi a\epsilon/q}$$
$$\times(1+0.837a+1.457a^2-0.255a^3), \tag{36}$$

with $a>0$. Note that well away from the tip, when $\epsilon/q\approx 1$, electron and positron bremsstrahlung have the same characteristic factor $e^{-\pi a}$. The terms in $4\pi a/15$, etc., remain opposite in sign, but the work of Davies, Bethe, and Maximon[21] indicates that they actually

[21] Handel Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954). In this paper, the correction to Born approximation for high-energy initial and final electron bremsstrahlung is proportional to a^2 for small Z. Indeed, the expression they obtain is *even* in a. This implies that for large q, electron and positron bremsstrahlung are equal.

vanish for large q, when a series expansion in q is no longer legitimate. We show in Fig. 3 how the ratio of positron and electron bremsstrahlung cross sections for Fe and Th vary in the high-frequency region of the spectrum. The results for Th are, of course, less accurate, but they show how the factor $e^{2\pi\alpha(1-\epsilon/q)}$ of the ratio keeps the cross sections significantly different much further back from the tip.

Predictions for the two limiting regions of the pair production spectrum require no further work. It has previously been shown[9] that the cross section for a pair production process in which the positron takes almost all the energy is identical to the cross section for the high-frequency region of the electron bremsstrahlung spectrum, under the usual assumption of a very high energy incident particle. Hence, all results for electron bremsstrahlung may be taken over without modifications, understanding q again as the momentum of the low-energy electron. In the same way, pair production in which the electron takes almost all the energy is identical with the high-frequency region of the positron bremsstrahlung spectrum, with q the momentum of the low-energy positron. Figure 3 may hence also be interpreted as the ratio of pair production cross sections for these two cases, production of low-energy positrons is suppressed.

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Excitation of Atomic Hydrogen by Fast Protons

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The excitation of the metastable state ($2S$) of atomic hydrogen by fast protons is discussed. The direct excitation cross section of $2S$ is calculated from a coupled set of equations ($1S$, $2S$, $2P$). The indirect population of the $2S$ state by excitation to higher states with subsequent cascade to the $2S$ state is described by Born approximation. The calculation is shown to apply to a fast crossed beam experiment.

I. INTRODUCTION

THE virtue of the proton-hydrogen scattering system is that it is the simplest of the atom-atom[1] type. It is worthwhile understanding this system as a prototype of a more general atom-atom collision thereby gaining some confidence in the theoretical methods which we would like to apply to these more complicated collisions.

It would appear that the general atom-atom scattering problem breaks into two natural subgroups. Firstly, at high energies (where the relative velocity of the atoms is larger than the circulating electron velocities) we have the problem of direct collisions (elastic and inelastic) and the problem of rearrangement collisions. The direct collisions are treated here neglecting rearrangement. The wave functions thus obtained can then be used to calculate the rearrangement collisions. The justification for this procedure is that rearrangement probabilities are small at high energies.[2]

Secondly, at low energies a molecular description is more appropriate and direct and rearrangement collisions can be treated on almost the same footing.[2] Only direct high-energy collisions will be treated here.

We have in mind the following experiment. A slow beam of ground-state atomic hydrogen is crossed with a high-energy proton beam. The slow hydrogen beam is then allowed to proceed sufficiently long for excited states to decay ($\sim 10^{-6}$ sec) leaving only the metastable $2S$ state. The beam then passes through a small electric field where the $2S$ state is quenched. The resultant Lyman α radiation is then detected. This then measures the $2S$ population due to the collisions and cascade from higher states.

In the next section we describe the population of the $2S$ level by a calculation by the method of coupled states ($1S$,$2S$,$2P$). This method yields a result in sharp disagreement with so-called distorted-wave methods[2,3] but in qualitative agreement with the second Born approximation.[4] The reasons for this are discussed. The method of closely coupled states is known to be an approximation to a more exact optical potential method.[5] The modifications introduced by this last method are also briefly discussed.

In the third section the population of the higher

[1] By atom-atom collision we mean here the collision between two heavy bodies. One or both may be ions.
[2] M. H. Mittleman, Phys. Rev. **122**, 499 (1961).
[3] D. R. Bates, Proc. Phys. Soc. (London) **73**, 227 (1959).
[4] A. E. Kingston, B. L. Moiseiwitsch, and B. G. Skinner, Proc. Roy. Soc. (London) **258**, 273 (1960). The second Born approximation is not calculated exactly here. Only the $1S$, $2S$, and $2P$ states are allowed of the infinite, set of intermediate states.
[5] M. H. Mittleman and R. Pu, Phys. Rev. **126**, 370 (1962).