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Angular Momentum Poles of the Nonrelativistic S Matrix for Spin $\frac{1}{2}$ Particles*

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The study of the structure of the two-particle S matrix as a function of the angular momentum in potential theory is extended to spin-dependent interactions between two spin $\frac{1}{2}$ particles, including the tensor force. The results reveal considerable similarity with the spin zero case, including the symmetry properties. The main differences are two branch points at $j=0$ and $j=-1$ and a pole at $j=-\frac{1}{2}$. It is shown that judicious combinations of S-matrix elements contain none of these singularities, and, as a result, neither do the scattering amplitudes. Certain modifications of the canonical situation are found in the presence of spin-orbit forces or other orbital angular momentum-dependent potentials. The factorization of the residue of the S matrix is also discussed.

1. INTRODUCTION

LATELY, it has become of great interest, both for theoretical and for practical phenomenological purposes, to consider the partial-wave scattering amplitude for two particles as an analytic function of the angular momentum.[1] In the restricted context of nonrelativistic potential scattering, the resulting properties, the existence of Regge poles and their motion as functions of the energy, can be studied in great detail and statements about them can be proved.[2,3] In the more general case of relativistic scattering, where the most important applications are found, proofs are much more difficult and one usually resorts to postulation by analogy from the low-energy case. This adds to the necessity of exploring the region accessible to proof quite thoroughly.

The present paper stays entirely within the realm of nonrelativistic quantum mechanics with interparticle potentials. We are extending previous work, which has been confined to particles of spin zero, to the case of scattering of spin $\frac{1}{2}$ particles.[4] If both have spin $\frac{1}{2}$, then it is the presence of the tensor force that adds new complications by giving rise to coupling between triplet $l=j\pm1$ states. These complications are investigated and cleared up.

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[1] T. Regge, Nuovo Cimento **14**, 951 (1959); **18**, 947 (1960).

[2] R. G. Newton, J. Math. Phys. **3**, 867 (1962).

[3] At the time of this writing we are aware of the following papers concerning Regge poles in the potential context: R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962); A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962); E. Predazzi and T. Regge, *ibid.* **24**, 518 (1962); A. Bottino and A. M. Longoni, *ibid.* **24**, 353 (1962); G. M. Prosperi, *ibid.* **24**, 957 (1962); S. Mandelstam, Ann. Phys. (N. Y.) (to be published); H. Cheng, Phys. Rev. **127**, 647 (1962); E. J. Squires, Nuovo Cimento (to be published); V. Singh, Phys. Rev. **127**, 632 (1962); J. M. Charap and E. J. Squires, *ibid.* **127**, 1387 (1962); A. Ahmadzadeh, P. G. Burke, and C. Tate (to be published); A. Martin (to be published); J. R. Taylor, Phys. Rev. **127**, 2257 (1962); J. M. Cornwall and M. A. Ruderman, *ibid.* **128**, 1474 (1962); H. Cheng and R. Nunez-Lagos, Nuovo Cimento **24**, 177 (1962); A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962); P. E. Kaus and C. J. Pearson, Bull. Am. Phys. Soc. **7**, 300 (1962).

[4] The case in which only one of the two interacting particles has spin $\frac{1}{2}$, the other spin zero, has been considered by L. Favella and M. T. Reineri, Nuovo Cimento **23**, 616 (1962). The multichannel spin zero problem is also discussed here.

The most important of our new results can be summarized as follows[5]: The S matrix has, in its off-diagonal elements, branch points at $j=0$ and $j=-1$. The partial-wave amplitudes, however, contain compensating radicals so that they are analytic there. In addition, the triplet-state S matrix has, in general, a pole at $j=-\frac{1}{2}$. We prove that, nevertheless, there exist specific simple linear combinations of S-matrix elements called $\bar{S}$ in which this pole always cancels and that they are directly related to the helicity S matrices; moreover, the amplitudes are expressible in terms of these combinations without the appearance of the pole. As a result, the partial-wave amplitudes are as well behaved in the complex j plane as they are in the spin zero case, and the Watson transformation can be performed under similar conditions on the potentials. In addition, these linear combinations of S-matrix elements obey the same symmetry property with respect to j and $-j-1$ as does the spin zero S matrix.

In Sec. 2 we introduce the main tools of our analysis. Section 3 generalizes Regge's limitations on the difference between successive phase shifts to the sum of the eigenphase shifts. In Sec. 4 we generalize both the limitation on the number of "right-hand" trajectories, and the way they leave the real axis. Section 5 deals with the factorization of the residue of the S matrix, and Sec. 6 is concerned with the symmetry properties with respect to interchange of j and $-j-1$. We show that the transformed S matrix, $\bar{S}$ of (6.12), has the same symmetry found in the spin zero case. Section 7 treats the point $j=-\frac{1}{2}$ in detail. At this point the triplet state S matrix has, in general, a pole. We prove here the absence of the pole in $\bar{S}$.

[5] After this work was finished a preprint by J. M. Charap and E. J. Squires "On Complex Angular Momentum in Many-Channel Potential-Scattering Problems, I" came to our attention. There is considerable overlap between their work and ours, theirs treating also the multichannel case but being based on more restrictive assumptions on the potential. There are a number of problems which we treat and they do not. The problem arising from the point $j=-\frac{1}{2}$ and its generalization to their more general angular momentum coupling is not treated correctly, as a comparison of the statements following their (3.26) with our Sec. 7 will reveal. Note also that their definition (3.8) of $N(Jts)$ contains a misprint. The factor $2J+2s+1$ should be replaced by $2J+2t+1$ [E. J. Squires (private communication)].

In Sec. 8 we relax our previous restriction to local potentials and add a spin-orbit term. Nothing essential is changed, except for possible changes in the shape of the pole trajectories and the asymptotic behavior of the amplitudes. The trajectories need no longer turn over to the left under conditions which formerly forced them to do so. We also briefly discuss other types of angular momentum dependent forces. In Sec. 9 we write down the scattering amplitudes explicitly. We show that all appearances of inconvenient radicals are spurious, that the amplitudes can be expressed most simply in terms of $\bar{S}$ without presence of poles, and that, therefore, the Watson transformation can be carried out as it could in the spin-zero case.

2. GENERAL PROCEDURE

We assume the most general local spherically symmetric potential between two spin-$\frac{1}{2}$ particles

$$V(\mathbf{r})=V_c(r)+V_\sigma(r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+V_t(r)S_{12},$$

where S_{12} is the tensor operator

$$S_{12}=3\boldsymbol{\sigma}_1\cdot\mathbf{n}\boldsymbol{\sigma}_2\cdot\mathbf{n}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

with $n\equiv\mathbf{r}/r$. Partial-wave analysis then leads to a set of ordinary radial Schrödinger equations for the singlet states and for the triplet states of parity $(-)^j$. These states can be treated just as the spin zero case; there is nothing new to be learned here. For the triplet states of parity $(-)^{j+1}$ we have, however, a set of coupled radial Schrödinger equations with $l=j\pm1$, which in matrix notation read[6]

$$-\psi''+C(j)r^{-2}\psi+V\psi=k^2\psi. \tag{2.1}$$

The centrifugal term contains the diagonal matrix

$$C(j)=\begin{pmatrix}(j-1)j & 0\\ 0 & (j+1)(j+2)\end{pmatrix},$$

and the potential matrix is, with $V_d\equiv V_c+V_\sigma$

$$V^{(j)}=\frac{1}{2j+1}\times\begin{pmatrix}(2j+1)V_d-2(j-1)V_t & 6[j(j+1)]^{1/2}V_t\\ 6[j(j+1)]^{1/2}V_t & (2j+1)V_d-2(j+2)V_t\end{pmatrix}. \tag{2.2}$$

An irregular matrix solution $f(j,k;r)$ is defined in the standard manner by the integral equation

$$f(j,k;r)=f_0(j,k;r)-\int_r^\infty dr'\,g(j,k;r,r')V^{(j)}(r')f(j,k;r'), \tag{2.3}$$

where

$$f_0(j,k;r)=e^{-i\frac{1}{2}\pi j}(\tfrac{1}{2}\pi kr)^{1/2}\times\begin{pmatrix}H_{j-\frac{1}{2}}^{(2)}(kr) & 0\\ 0 & -H_{j+\frac{3}{2}}^{(2)}(kr)\end{pmatrix}, \tag{2.4}$$

and

$$g(j,k;r,r')=\begin{pmatrix}g_{j-\frac{1}{2}}(k;r,r') & 0\\ 0 & g_{j+\frac{3}{2}}(k;r,r')\end{pmatrix}, \tag{2.5}$$

$$g_\lambda(k;r,r')=\tfrac{1}{2}\pi(rr')^{1/2}[J_\lambda(kr')Y_\lambda(kr)-J_\lambda(kr)Y_\lambda(kr')],$$

J_λ and Y_λ being the Bessel functions of the first and second kind, respectively. The function f satisfies the boundary condition

$$\lim_{r\to\infty}e^{ikr}f(j,k;r)=\mathbf{1}. \tag{2.6}$$

The regular solution is difficult to define by a boundary condition at $r=0$ because of the coupled angular momenta. We avail ourselves instead directly of the integral equation[7–9]

$$\begin{aligned}\varphi(j,k;r)={}&\varphi_0(j,k;r)\times\left\{\mathbf{1}+6[j(j+1)]^{1/2}\int_1^r dr'\,r'^{-1}V_t(r')P\right\}\\&+\int_0^r dr'\{g(j,k;r,r')V^{(j)}(r')\varphi(j,k;r')\\&-6[j(j+1)]^{1/2}r'^{-1}\varphi_0(j,k;r)V_t(r')P\},\end{aligned} \tag{2.7}$$

where[10]

$$\varphi_0(j,k;r)=(\tfrac{1}{2}\pi kr)^{1/2}k^{-j}\begin{pmatrix}J_{j-\frac{1}{2}}(kr) & 0\\ 0 & J_{j+\frac{3}{2}}(kr)\end{pmatrix} \tag{2.8}$$

and

$$P=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

A Jost matrix function is defined by

$$F(j,k)\equiv f^T\varphi'-f'^T\varphi\equiv W(f,\varphi), \tag{2.9}$$

so that

$$\varphi(j,k;r)=(2ik)^{-1}[f(j,-k;r)F(j,k)-f(j,k;r)F(j,-k)], \tag{2.10}$$

and the unitary and symmetric S matrix is given by[11]

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[6] We use units in which $\hbar=2m=1$.

[7] See R. G. Newton, Phys. Rev. **100**, 412 (1955).

[8] See R. G. Newton, J. Math. Phys. **1**, 319 (1960).

[9] The purpose of the somewhat cumbersome "counter-term" is to eliminate an otherwise present divergence. Its presence is unnecessary if $r^{-1}V_t$ is integrable at $r=0$. However, we want to include it in order not to have to make unphysically strong assumptions about the tensor force.

[10] It is convenient for later purposes to have no k^{-2} factor in front of the $J_{j+\frac{3}{2}}$. Since we are not primarily concerned with the behavior near $k=0$ this produces no difficulties.

[11] The phase factor is designed to assure that S is unitary and tends to unity as $k\to\pm\infty$, even for nonintegral values of j.

$$S(j,k)=e^{i\pi(j+1)}F(j,k)[F(j,-k)]^{-1}. \quad (2.11)$$

Here the superscript T denotes transpose.

The series of successive approximations to f and φ converge under the usual conditions on $V^{(j)}$ and the proofs of the standard properties go through as usual, including that of the analyticity of φ as a function of j in the left half of the complex j plane. As shown in reference 2 that depends only on the number of finite derivatives of rV at $r=0$. Of course, φ has, in general, the usual simple poles at $j=0, -1, -2\cdots$.[12] A look at $V^{(j)}$, however, seems to indicate trouble at $j=-\frac{1}{2}$. Since a simple pole in the potential is iterated infinitely many times, it looks as though there will be an essential singularity in f and φ at $j=-\frac{1}{2}$. That this is spurious is shown by diagonalizing $V^{(j)}$ with the matrix

$$U_j\equiv\begin{pmatrix}(j+1)^{1/2} & j^{1/2}\\ j^{1/2} & -(j+1)^{1/2}\end{pmatrix}, \quad (2.12)$$

so that

$$U_j^{-1}=U_j/(2j+1),$$

and

$$U_jV^{(j)}U_j^{-1}\equiv W=\begin{pmatrix}V_d+2V_t & 0\\ 0 & V_d-4V_t\end{pmatrix}. \quad (2.13)$$

Writing

$$\begin{aligned}\phi(j,k;r)&\equiv U_j\varphi(j,k;r),\\ \mathfrak{f}(j,k;r)&\equiv U_jf(j,k;r), \quad (2.14)\\ G(j,k;r,r')&\equiv U_jg(j,k;r,r')U_j^{-1},\end{aligned}$$

we get from (2.3) and (2.7)

$$\mathfrak{f}(j,k;r)=\mathfrak{f}_0(j,k;r)-\int_r^\infty dr'G(j,k;r,r')W(r')\mathfrak{f}(j,k;r'), \quad (2.15)$$

$$\begin{aligned}\phi(j,k;r)=\phi_0(j,k;r)&\times\left\{1+6[j(j+1)]^{1/2}\int_1^r dr'r'^{-1}V_t(r')P\right\}\\ &+\int_0^r dr'\,\{G(j,k;r,r')W(r')\phi(j,k;r')\\ &-6[j(j+1)]^{1/2}r'^{-1}\phi_0(j,k,r)V_t(r')P\}. \quad (2.16)\end{aligned}$$

Now it is easily shown by means of the Bessel function recurrence relations that the $(2j+1)$ in the denominator of G, coming from U_j^{-1}, cancels out and G has no singularity at $j=-\frac{1}{2}$. As a result $\mathfrak{f}(j,k,r)$ and $\phi(j,k,r)$ contain no singularity there either. Consequently,

$$\begin{aligned}F(j,k)&=\mathfrak{f}^TU_j^{-1}U_j^{-1}\phi'-\mathfrak{f}'^TU_j^{-1}U_j^{-1}\phi\\ &=(\mathfrak{f}^T\phi'-\mathfrak{f}'^T\phi)/(2j+1) \quad (2.17)\end{aligned}$$

has, in general, a *simple pole* at $j=-\frac{1}{2}$. We shall return to the detailed consideration of the point $j=-\frac{1}{2}$ in Sec. 7.

[12] The pole at $j=0$ comes from $l=-1$.

The presence of the tensor force introduces additional singularities in the S matrix. Both the original form (2.2) of $V^{(j)}$ and the "diagonalized" form of the integral equations (2.15) and (2.16) (the latter in G) contain the factor $[j(j+1)]^{1/2}$. As a result both f and φ, and hence F and S, acquire a branch line running from $j=-1$ to $j=0$. In the region $-1<j<0$ the potential matrix $V^{(j)}$ is not Hermitian and as a result $S(j,k)$ is not unitary in that region, even for real k.[13] Because only the off-diagonal elements of $V^{(j)}$ contain the factor $[j(j+1)]^{1/2}$, and since the diagonal elements of S must be even functions of those off-diagonal elements of $V^{(j)}$, only the off-diagonal elements of the S matrix contain the branch line. The diagonal S-matrix elements are, in general, regular at $j=0$ and $j=-1$ (if the potentials are sufficiently well behaved).

3. THE PHASE-SHIFT DERIVATIVE

Differentiating the Schrödinger equation (2.1) with respect to j yields with the standard Wronskian technique

$$\left[\frac{\partial\varphi'^T}{\partial j}\varphi-\frac{\partial\varphi^T}{\partial j}\varphi'\right]'=r^{-2}\varphi^TC'(j)\varphi,$$

where

$$C'(j)\equiv\frac{dC(j)}{dj}=\begin{pmatrix}2j-1 & 0\\ 0 & 2j+3\end{pmatrix}.$$

Integration from 0 to ∞ gives by (2.10) for real k

$$\tfrac{1}{2}i\left[\frac{\partial F^T(j,k)}{\partial j}F(j,-k)-\frac{\partial F^T(j,-k)}{\partial j}F(j,k)\right]=k\int_0^\infty dr\,r^{-2}\varphi^TC(j)\varphi. \quad (3.1)$$

We multiply by $F^T(j,k)^{-1}$ on the left and by $F(j,-k)^{-1}$ on the right, then take the trace. The result is

$$\tfrac{1}{2}i\frac{\partial}{\partial j}\ln\det F(j,k)[F(j,-k)]^{-1}=k\int_0^\infty dr\,r^{-2}\,\mathrm{tr}[F^T(j,k)]^{-1}\varphi^TC'(j)\varphi[F(j,-k)]^{-1},$$

where the symmetry of S has been used and the fact that

$$\frac{\partial}{\partial j}\ln\det F=\mathrm{tr}F^{-1}\frac{\partial}{\partial j}F.$$

Now for real k and real $j>\frac{1}{2}$ the right-hand side is positive. Hence if we write $\Delta(j,k)$ for the sum of the eigenphase shifts

$$\Delta(j,k)=-\tfrac{1}{2}i\ln\det S(j,k),$$

[13] Since $V^{(j)}$ remains symmetric, S does too.

we obtain by (2.11)

$$\partial\Delta/\partial j<\pi,$$

and therefore

$$\Delta(j+1,k)-\Delta(j,k)<\pi.$$

This is the generalization of Regge's formula for spin 0[1]. It must be remembered, though, that it holds, in general, only for $j>\frac{1}{2}$. For $j=0$ the physical S matrix has, of course, only one element, that for $l=1$. There does not appear to be any restriction on the difference between that phase shift and the sum of the $j=1$ eigenphase shifts.

4. THE NUMBER OF TRAJECTORIES AND THEIR BEHAVIOR NEAR $E=0$

In order to generalize the bound on the number of zero trajectories in the right half plane to the present case, we first replace $V^{(j)}$ by a negative definite potential matrix $-\mathcal{V}^{(j)}$ whose eigenvalves are nowhere bigger than those of the matrix $V^{(j)}$. In other words,

$$\mathcal{V}^{(j)}=U_j\mathcal{W}U_j^{-1}, \tag{4.1}$$

where

$$\mathcal{W}=\begin{pmatrix}\mathcal{W}_1 & 0\\ 0 & \mathcal{W}_2\end{pmatrix},$$

and

$$\begin{aligned}\mathcal{W}_1&=-V_d-2V_t \quad \text{if} \quad V_d+2V_t<0,\\ &=0 \quad \text{otherwise},\\ \mathcal{W}_2&=-V_d+4V_t \quad \text{if} \quad V_d-4V_t<0,\\ &=0 \quad \text{otherwise}.\end{aligned}$$

Then we look for the number of $E=0$ bound states introduced when $-\mathcal{V}^{(j)}$ is replaced by $-\sigma\mathcal{V}^{(j)}$ and σ increases from 0 to 1. That leads directly to the generalized Bargmann formula[14]

$$n_j\leqslant\int_0^\infty dr\, r\mathcal{V}_{11}^{(j)}(r)/(2j-1)+\int_0^\infty dr\, r\mathcal{V}_{22}^{(j)}(r)/(2j+3), \tag{4.2}$$

which can be used, of course, only for $j>\frac{1}{2}$. The $j=\frac{1}{2}$ case can be treated as in reference 2. The result is

$$n_{1/2}\leqslant 1+\tfrac{1}{4}\int_0^\infty dr\, r\mathcal{V}_{22}^{(1/2)}(r)+\frac{\int_0^\infty dr\int_0^r dr'\, rr'\mathcal{V}_{11}^{(1/2)}(r)\mathcal{V}_{11}^{(1/2)}(r')\ln(r/r')}{\int_0^\infty dr\, r\mathcal{V}_{11}^{(1/2)}(r)}, \tag{4.3}$$

where

$$\mathcal{V}_{11}^{(1/2)}=\tfrac{1}{4}(3\mathcal{W}_1+\mathcal{W}_2),\quad \mathcal{V}_{22}^{(1/2)}=\tfrac{1}{4}(\mathcal{W}_1+3\mathcal{W}_2).$$

[14] V. Bargmann, Proc. Natl. Acad. Sci. U. S. **38**, 961 (1952); J. Schwinger, *ibid.* **47**, 122 (1961).

In order to get down to $j=-\frac{1}{2}$, it would be necessary to take the $l=j-1$ part to $l=-\frac{3}{2}$. No simple way of doing that is known as yet. At $j=0$ the equations uncouple and only the $l=j+1$ part has physical significance. Therefore there is a simple Bargmann inequality for the number of *physical* bound states with $j=0$, but not for the number of S-matrix poles.[15]

We now wish to generalize the previous results[2,16] concerning the way in which trajectories leave the real axis at $E=0$. The starting point is the motion of a zero j_0 of $\det F$ along the real j axis for negative energy. Straightforward generalization of the Wronskian condition preceding (4.3) of the reference 2 yields for negative energies

$$\frac{dj_0}{dk^2}=1\bigg/\left\langle r^{-2}\begin{pmatrix}2j_0-1 & 0\\ 0 & 2j_0+3\end{pmatrix}\right\rangle, \tag{4.4}$$

which can be used only for $j_0>\frac{1}{2}$. Otherwise the zero entails no normalizable state.

Now $\det F=0$ implies the existence of a nonzero, k-dependent vector a such that $Fa=0$. The physical significance of the components of a is that they determine the "mixture parameter" in the bound or shadow state. If one of the components of a is zero, it is a pure $l=j-1$ or $l=j+1$ state. If that happens at $E=0$ then the previous results are immediately applicable. On the other hand, if we take the more general case in which at $E=0$ neither component of a vanishes, so that the "bound state" is a mixture of $l=j-1$ and $l=j+1$, then the energy derivatives of the real and imaginary parts of j_0 are dominated by the $l=j-1$ term. As a consequence we get as in reference 2 for $j_0\geq\frac{1}{2}$ as $E\to 0+$

$$\frac{d\,\mathrm{Im}j_0}{dE}=O(E^{j_0-\frac{3}{2}}), \tag{4.5}$$

$$\begin{aligned}d\,\mathrm{Re}j_0/dE&=O(1) &&\text{for } j_0>\tfrac{3}{2},\\ &=O(\log|E|) &&\text{for } j_0=\tfrac{3}{2},\\ &=O(E^{j_0-\frac{3}{2}}) &&\text{for } j_0<\tfrac{3}{2},\end{aligned} \tag{4.6}$$

and hence for the angle γ of the trajectory with the real j axis

$$\begin{aligned}\cot\gamma&=O(E^{\frac{3}{2}-j_0})=\infty &&\text{for } j_0>\tfrac{3}{2},\\ &=O(\log|E|)=\infty &&\text{for } j_0=\tfrac{3}{2},\\ &=O(1) &&\text{for } j_0<\tfrac{3}{2}.\end{aligned} \tag{4.7}$$

[15] It has been remarked both in Secs. 3 and 4 that at $j=0$ the S matrix is diagonal and only one of the two terms, that for $l=j+1$, has physical significance. This is a special example of Gell-Mann's discussion of "sense" and "nonsense" terms [M. Gell-Mann (to be published)]. For $j=0$ the functions $\det F(j,k)$ factors, F being diagonal. If a zero of $\det F$ passes through $j=0$ it must then be a zero of either the "sense" or the "nonsense" factor. If it is a zero of the $l=-1$ factor then it does not correspond to a physical bound state. We then have a Regge trajectory that leaves the real axis at $j>0$ and yet it is not connected with a bound state.

[16] Barut and Zwanziger, reference 3.

Moreover, the trajectory leaves the real axis in the forward direction if it leaves at $j_0>1$, and in the backward direction if at $j_0<1$.

It must be recognized, though, that the significance of the foregoing statements depends on the value of the mixture parameter. They are true if the "bound state" at $E=0$ contains any admixture at all of $l=j-1$. If that admixture is small the trajectory will follow a general shape appropriate to $l=j+1$ and only at very small energies will it revert to its proper $l=j-1$ behavior.

5. FACTORIZATION OF THE RESIDUE

The possibility of "factorizing" the residue of the S matrix[17] at a Regge pole is an expression of the fact that, although in general each element of S has the pole, there is (except in the case of accidental degeneracy) only *one* vector (or more exactly, one *ray*) which when multiplied by S, has a pole. That implies that the residue R is a singular matrix whose null space[18] is one dimensional, and since it is symmetric it must be writable as

$$R_{ij}(k)=a_i(k)a_j(k),$$

so that $Rb=0$ for all the $n-1$ linearly independent vectors b (if we are dealing with an $n\times n$ S matrix) orthogonal to a. In the present case, of course, $n=2$.

In order to see the relation to the zero of $\det F(\lambda,-k)$, we realize that $\det F(\lambda_0,-k_0)=0$ implies that the null space of the matrix $F(\lambda_0,-k_0)$ is at least one dimensional. We assume it is exactly that; otherwise we would call it accidentally degenerate and since $n=2$ that would imply $F(\lambda_0,-k_0)=0$. The range[18] of $F(\lambda_0,-k_0)$ is therefore one dimensional (i.e., $n-1$ dimensional) and that range must be equal to the null space of the residue of $[F(\lambda_0,-k)]^{-1}$ at (λ_0,k_0). In other words, the vector $a(k_0)$ is orthogonal to the range of $F(\lambda_0,-k_0)$.

Another connection comes from the symmetry of S which implies that a spans the range of R. But the range of the residue of $[F(\lambda,-k)]^{-1}$ must equal the null space of $F(\lambda_0,-k_0)$, and hence by (2.11)

$$a\propto F(\lambda_0,k_0)c,$$

if

$$F(\lambda_0,-k_0)c=0.$$

6. SYMMETRY PROPERTIES

The transformation U_j of (2.12) applied to (2.1) yields the equation

$$-\psi''+D(j)r^{-2}\psi+W\psi=k^2\psi, \tag{6.1}$$

which is satisfied by $\phi(j,k;r)$ as well as $\mathfrak{f}(j,k,r)$. Here

$$D(j)=U_jC(j)U_j^{-1}=\begin{pmatrix}\lambda^2-\frac{1}{4} & -2(\lambda^2-\frac{1}{4})^{1/2}\\ -2(\lambda^2-\frac{1}{4})^{1/2} & \lambda^2+(7/4)\end{pmatrix},$$

with $\lambda=j+\frac{1}{2}$. Thus the transformed equation is again a function only of λ^2, just as in the spin 0 case. The function

$$\bar{f}(\lambda,k;r)\equiv U_jf(j,k;r)U_j^{-1}=\mathfrak{f}(j,k;r)U_j^{-1} \tag{6.2}$$

satisfies (6.1) and the boundary condition

$$\lim_{r\to\infty}e^{ikr}\bar{f}(\lambda,k;r)=1. \tag{6.3}$$

Hence it is a function only of λ^2.

Similarly we introduce

$$\bar{\varphi}(\lambda,k;r)\equiv U_j\varphi(j,k;r)U_j^{-1}=\phi(j,k;r)U_j^{-1}. \tag{6.4}$$

In order to obtain the analog of the symmetry relation obeyed by F in the spin zero case we calculate the Wronskian of $\bar{\varphi}(\lambda,k;r)$ and $\bar{\varphi}(-\lambda,k;r)$ by means of the integral equation (2.16):

$$W[\bar{\varphi}(\lambda,k;r),\ \bar{\varphi}(-\lambda,k;r)]=-\sin\pi\lambda. \tag{6.5}$$

The function $\bar{\varphi}$ is expressible in terms of $\bar{f}$ as

$$\bar{\varphi}(\lambda,k;r)=(2ik)^{-1}[\bar{f}(\lambda,-k;r)\bar{F}(\lambda,k)-\bar{f}(\lambda,k;r)\bar{F}(\lambda,-k)], \tag{6.6}$$

where

$$\bar{F}(\lambda,k)=U_jF(j,k)U_j^{-1}. \tag{6.7}$$

Insertion in (6.5), together with the evenness of $\bar{f}$ as a function of λ gives

$$\bar{F}^T(\lambda,k)\bar{F}(-\lambda,-k)-\bar{F}^T(\lambda,-k)\bar{F}(-\lambda,k)=-2ik\sin\pi\lambda. \tag{6.8}$$

In order to eliminate the poles from $\bar{F}$ we may then define

$$\mathfrak{F}(\lambda,k)\equiv\bar{F}(\lambda,k)\lambda/\Gamma(\tfrac{1}{2}+\lambda), \tag{6.9}$$

and get

$$\mathfrak{F}^T(\lambda,k)\mathfrak{F}(-\lambda,-k)-\mathfrak{F}^T(\lambda,-k)\mathfrak{F}(-\lambda,k)=(ik/\pi)\lambda^2\sin2\pi\lambda. \tag{6.10}$$

Since

$$S(j,k)=e^{i\pi(j+1)}U_j^{-1}\mathfrak{F}(\lambda,k)[\mathfrak{F}(\lambda,-k)]^{-1}U_j=e^{i\pi(j+1)}U_j^{-1}\bar{F}(\lambda,k)[\bar{F}(\lambda,-k)]^{-1}U_j, \tag{6.11}$$

this means that the function which satisfies a simple symmetry relation is

$$\bar{S}(\lambda,k)\equiv U_jS(j,k)U_j^{-1}, \tag{6.12}$$

rather than S itself. $\bar{S}$ too is symmetric and unitary.[19]

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[17] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); J. M. Charap and E. J. Squires, Phys. Rev. **127**, 1387 (1962).

[18] The null space of a matrix M is the space of all vectors a which it annihilates: $Ma=0$. The range of M is the space *onto* which M maps; i.e., the space of all vectors b that can be written $b=Mc$.

[19] Explicit calculation of $\bar{S}$ in terms of S, as in (9.3), shows that $\bar{S}$ has the same branch point properties as S. The diagonal elements are free from branch points at $j=0$ and $j=-1$, but the off-diagonal elements contain the factor $[j(j+1)]^{1/2}$.

The symmetry relation is

$$e^{-i\pi\lambda}\bar{S}(\lambda,k)-e^{i\pi\lambda}\bar{S}(-\lambda,k)$$
$$=-(k/\pi)\lambda^2\sin 2\pi\lambda[\mathfrak{F}(-\lambda,-k)\mathfrak{F}^T(\lambda,-k)]^{-1}$$
$$=-2k\sin\pi\lambda[\bar{F}(-\lambda,-k)\bar{F}^T(\lambda,-k)]^{-1}. \quad (6.13)$$

The discussion given in reference 2 concerning the symmetries of S-matrix poles for positive and negative integral or half-integral j values applies again. It will be clear in Sec. 9 that U_j is connected to the transformation to helicity states.

7. THE POINT $j=-1/2$

We now want to determine the nature of the point $j=-\frac{1}{2}$ in the S matrix. The fact that F has a pole there independently of k no longer allows us, as it does in the spin zero case, to conclude that there is no pole in S.[5] We first determine whether F^{-1} has a pole at $j=-\frac{1}{2}$.

Let us examine the nature of the pole of $\bar{F}(\lambda,k)=W(\bar{f},\bar{\varphi})$ at $\lambda=0$. The pole of $\bar{\varphi}$ comes from the right-hand U_j^{-1} in $\bar{\varphi}_0$

$$\bar{\varphi}_0(\lambda,k;r)=(2\pi r)^{1/2}k^{1-\lambda}[\lambda(kr)^{-1}J_\lambda(kr)\mathbf{1}+\tfrac{1}{2}J_\lambda'(kr)\lambda^{-1}A(\lambda)], \quad (7.1)$$
$$A(\lambda)=\begin{pmatrix}1 & (4\lambda^2-1)^{1/2}\\ (4\lambda^2-1)^{1/2} & -1\end{pmatrix}.$$

The residue at $\lambda=0$ is proportional to the singular matrix

$$A(0)\equiv A=\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}. \quad (7.2)$$

Consequently the residue of $\bar{\varphi}$ at $\lambda=0$ is a left matrix multiple of A, and so is that of $\bar{F}$

$$\bar{F}(\lambda,k)=R(k)\lambda^{-1}+\cdots, \quad (7.3)$$

where

$$R(k)=M(k)A, \quad (7.4)$$

and it can be assumed without loss of generality that $M(k)$ is not singular.

Consider now the determinant of $\bar{F}$. Since R is singular, $\det\bar{F}$ has at most a *simple pole* at $\lambda=0$. We may expand it in a convergent power series in the "potential strength" and find that to first order there is no pole, but the constant term at $\lambda=0$ is in general different from zero. This is, of course, the same result as in the absence of the tensor force. Although we cannot, at this point rule out a pole of $\det\bar{F}$ at $\lambda=0$, the constant term at $\lambda=0$ can vanish at most for specific values of k, not identically in k. Higher orders in V cannot alter this state of affairs. It follows that $\bar{F}^{-1}$ has at most a *simple pole* at $\lambda=0$ (except possibly for specific values of k), i.e., that $(\lambda\bar{F})^{-1}$ has *at most* a double pole there.

The next observation to make is that the residue $R(k)$ annihilates the constant vector

$$a=\begin{pmatrix}1\\ i\end{pmatrix}.$$

Furthermore, a look at $\bar{\varphi}_0$ shows that not only does $(\lambda\bar{\varphi}_0)$ at $\lambda=0$ annihilate a, but so does its λ derivative

$$\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}(\lambda\bar{\varphi}_0)a=0.$$

The same therefore holds for $\lambda\bar{F}$. From this and the fact that $(\lambda\bar{F})^{-1}$ has *at most* a double pole at $\lambda=0$, we conclude by the theorem of Appendix C of reference 7 that $(\lambda\bar{F})^{-1}$ has exactly a *double pole* there, i.e., that $\bar{F}^{-1}$ has a *simple pole* there

$$\bar{F}^{-1}(\lambda,k)=R'(k)\lambda^{-1}+\cdots \quad (7.5)$$

(except for isolated values of k).

An immediate consequence of the fact that both $\bar{F}$ and $\bar{F}^{-1}$ have *simple poles* at $\lambda=0$ is that $\det\bar{F}$ *cannot* have a pole there.[20]

We now want to see if $\bar{S}$ has a pole at $\lambda=0$. Direct computation of $(\det F)S$, which can be expanded in a convergent power series in the "potential strength," shows that to first order S in general *does* have a pole there.

Since $\bar{F}(k)\bar{F}(k)^{-1}=\mathbf{1}$ we must have by (7.3) and (7.5)

$$R(k)R'(k)=0,$$

and since $A^2=0$ it follows from (7.4) that

$$R'(k)=AM'(k).$$

Consequently we also have

$$R(k)R'(-k)=0$$

and therefore $\bar{S}(\lambda,k)$ has *at most* a simple pole at $\lambda=0$.

In order to examine the residue of $\bar{S}$ at $\lambda=0$ we look at the symmetry relation (6.13) near there. We get

$$S_R(k)=\tfrac{1}{2}\pi kR'^T(k)R'(-k)=\tfrac{1}{2}\pi kM'^T(-k)A^2M'(-k)=0.$$

This proves that $\bar{S}(\lambda,k)$, in contrast to the S matrix itself, has *no pole* at $j=-\frac{1}{2}$. It is, therefore, of great importance that the scattering amplitudes can be expressed in terms of $\bar{S}$ without explicit introduction of a pole, as we have done in (9.2) and (9.4) below.

8. THE SPIN-ORBIT FORCE

Dropping our previous restriction to local potentials, we now want to include a spin-orbit coupling term in the potential

$$V_{LS}=\mathbf{L}\cdot\mathbf{S}V_0(r),$$

[20] That means that since a Regge trajectory is defined by a zero of $\det\bar{F}$, $\lambda=0$ is not a possible trajectory end point, as it would be if $\det\bar{F}$ had a pole there, following arguments given in reference 2. A remark made to the contrary, at the 1962 Midwest Theoretical Physics Conference, before we realized that $\det\bar{F}$ has no pole at $\lambda=0$, should therefore be withdrawn.

since such a force is clearly indicated in the low-energy nucleon-nucleon system. V_{LS} manifests itself in the potential matrix (2.2) as an additional term

$$\Delta V^{(j)}(r)=\begin{pmatrix} j-1 & 0 \\ 0 & -j-2 \end{pmatrix} V_0(r). \quad (8.1)$$

After application of the transformation U_j of (2.12) which previously diagonalized the potential, W is now changed to

$$W \to W+U_j\Delta V^{(j)}U_j^{-1}$$
$$=W+\begin{pmatrix} -1 & [j(j+1)]^{1/2} \\ [j(j+1)]^{1/2} & -2 \end{pmatrix} V_0, \quad (8.2)$$

which is a function only of λ^2 and contains no pole at $j=-\frac{1}{2}$. Hence the previous results apply again, except for one.

There is a change owing to the fact that $\Delta V^{(j)}$ is a linear function of j. As $|j| \to \infty$ the potential therefore becomes effectively stronger and stronger. Since its growth is only linear, it is for large $|j|$ still small relative to the centrifugal term and hence the proof of Bottino, Longoni, and Regge,[3] that $S \to 1$ as $|j| \to \infty$ along any ray toward the right still holds. However, Regge's proof that under certain specific conditions on the potential each pole trajectory must turn back to the left and either cross the line $\mathrm{Re}j=-\frac{1}{2}$ or else approach it asymptotically, now breaks down. It cannot now be ruled out that a trajectory approaches an asymptote parallel to and to the right of the imaginary j axis. Now, if for large energy all trajectories move to the left half of the complex j plane, then there exists an energy beyond which the scattering amplitude always asymptotically vanishes with increasing momentum transfer. The presence of a spin-orbit force may prevent that from happening. No matter how large the energy, the scattering amplitude may now always increase to infinity with increasing momentum transfer.

It is clear that what is happening in the presence of a spin-orbit force may happen to an even larger degree with other nonlocal potentials, such as $\mathbf{L}\cdot\mathbf{L}$ forces or higher powers of the angular momentum, which may be present even if the particles have spin zero. In fact, it is then in general impossible to perform the customary change of integration path in the Watson integral since the contributions from infinite $|j|$ need no longer vanish. Furthermore, even if that were possible in specific cases, pole trajectories could now move off to infinity toward the *right*. That would imply that the larger the energy the more strongly the scattering amplitude would increase with the momentum transfer.

It should be remarked though, that the foregoing conclusions for spin-orbit (and $\mathbf{L}\cdot\mathbf{L}$, etc.) forces need not be true if the potential V_0 in (8.1) is assumed to depend on j in such a way that $\Delta V^{(j)}$ remains bounded as $|j| \to \infty$. There is at present practically no experimental evidence which would decide that. Nor is it clear what the prediction of field theory would be. In any case, such possible effects should be kept in mind.

9. THE WATSON TRANSFORMATION

Writing down the various scattering amplitudes for spin-$\frac{1}{2}$ particles on spin-$\frac{1}{2}$ particles with the aim of applying the Watson transformation, we are faced with a number of problems. First, each spherical harmonic as well as each Clebsch-Gordan coefficient that appears contains various radicals such as $j^{1/2}$, $(j+1)^{1/2}$, $(j-1)^{1/2}$, etc.; second, as we have seen in Sec. 2, the off-diagonal elements of the triplet state S matrix contain the factor $[j(j+1)]^{1/2}$, but are otherwise regular, apart from the Regge poles; third, there is a pole in S at $j=-\frac{1}{2}$. As for the first and second points, explicit calculation shows that all the radicals cancel out, except for a factor of $[j(j+1)]^{1/2}$ multiplying the off-diagonal triplet state S-matrix elements. There are therefore no branch points in the partial amplitudes.[21] The third point will be eliminated if we can write the amplitudes in terms of the elements of the matrix $\bar{S}$ of (6.12) without explicitly introducing a pole at $j=-\frac{1}{2}$. That is, in fact, possible. Indeed, in so doing we actually acquire an additional factor of $(2j+1)$ that makes the partial amplitude vanish at $j=-\frac{1}{2}$. Moreover, the use of $\bar{S}$ simplifies the expressions.

In order to exhibit these features explicitly we introduce the following functions of θ:

$$\pi_j(\cos\theta)\equiv P_j'(\cos\theta)/j(j+1),$$
$$\tau_j(\cos\theta)\equiv P_j(\cos\theta)-\cos\theta\pi_j(\cos\theta),$$
$$\alpha_j(\cos\theta)\equiv[P_{j+1}(\cos\theta)-P_{j-1}(\cos\theta)]/(2j+1)$$
$$=\int_1^{\cos\theta} dz\, P_j(z), \quad (9.1)$$
$$\beta_j(\cos\theta)\equiv[(j+1)P_{j-1}(\cos\theta)+jP_{j+1}(\cos\theta)]/(2j+1)$$
$$=1+j(j+1)\int_1^{\cos\theta} dz\, z\pi_j(z),$$

which are all analytic functions of j everywhere. (Zeros of the numerators cancel those of the denominators.) Now there are two kinds of center of mass scattering amplitudes that can be usefully written down. One is a set classified by the total spin of the two particles. The only case of interest here is that of the triplet state; the singlet is no different from the spin zero case. If we choose the z axis along the direction of the incident beam and write $\Theta_{\nu',\nu}$ for the triplet scattering amplitude in which the initial and final z

[21] It was shown in reference 5 that this happens under more general conditions with angular momentum coupling.

components of the total spin are ν and ν', respectively, then we get[22]

$$\Theta_{0,0}=(2ik)^{-1}\sum_j(2j+1)(T_{j++}\cos\theta P_j-T_{j+-}\alpha_j),$$

$$\Theta_{1,1}=(4ik)^{-1}\sum_j(2j+1)(T_{j--}\beta_j-T_{j+-}\alpha_j+T_jP_j),$$

$$\Theta_{-1,0}=(2^{3/2}ik)^{-1}e^{i\varphi}\sin\theta\sum_j(2j+1)(T_{j++}P_j-T_{j+-}\cos\theta\pi_j),\quad(9.2)$$

$$\Theta_{0,1}=(2^{3/2}ik)^{-1}e^{i\varphi}\sin\theta\sum_j(2j+1)[T_{j--}j(j+1)\tau_j-T_{j+-}\cos\theta\pi_j+T_j\pi_j],$$

$$\Theta_{-1,1}=-(4ik)^{-1}e^{2i\varphi}\sin^2\theta\sum_j(2j+1)(T_{j--}P_j+T_{j+-}\pi_j+T_j\pi_j'),$$

and the remaining amplitudes are obtained by

$$\Theta(\theta,\varphi)_{\nu',\nu}=(-)^{\nu'-\nu}\Theta(\theta,-\varphi)_{-\nu',-\nu}.$$

We have used here the abbreviations

$$\begin{aligned}1+T_{j++}&\equiv\bar{S}_{+,+}{}^j=\{jS_{j-1,j-1}{}^j+(j+1)S_{j+1,j+1}{}^j-2[j(j+1)]^{1/2}S_{j-1,j+1}{}^j\}/(2j+1),\\ 1+T_{j--}&\equiv\bar{S}_{-,-}{}^j=\{(j+1)S_{j-1,j-1}{}^j+jS_{j+1,j+1}{}^j+2[j(j+1)]^{1/2}S_{j-1,j+1}{}^j\}/(2j+1),\quad(9.3)\\ T_{j+-}&\equiv[j(j+1)]^{1/2}\bar{S}_{+,-}{}^j=[j(j+1)/(2j+1)]\{S_{j-1,j-1}{}^j-S_{j+1,j+1}{}^j-[j(j+1)]^{1/2}S_{j-1,j+1}{}^j\},\\ 1+T_j&\equiv S_{j,j}{}^j.\end{aligned}$$

The other set of amplitudes is classified according to the spin directions of each of the two particles relative to their direction of motion (i.e., "helicity amplitudes"). Writing $\Theta_{\mu_1'\mu_2',\mu_1\mu_2}$ for the amplitude for initial and final spin projections on the momenta μ_1, μ_2, μ_1', μ_2', respectively, with $\mu=+,-$ indicating forward or backward spin, we get[23]

$$\Theta_{++,++}=\Theta_{--,--}=(4ik)^{-1}\sum_j(2j+1)(T_{j++}+S_j-1)P_j,$$

$$\begin{aligned}\Theta_{--,-+}&=\Theta_{+-,++}=\Theta_{++,-+}=\Theta_{+-,--}=-\Theta_{++,+-}\\&=-\Theta_{-+,--}=-\Theta_{-+,++}=-\Theta_{--,+-}\\&=(4ik)^{-1}\sin\theta\sum_j(2j+1)T_{j+-}\pi_j,\end{aligned}$$

$$\begin{aligned}\Theta_{+-,+-}&=\Theta_{-+,-+}\\&=(4ik)^{-1}\sum_j(2j+1)(T_{j--}+T_j)(\pi_j+\tau_j),\end{aligned}\quad(9.4)$$

$$\begin{aligned}\Theta_{+-,-+}&=\Theta_{-+,+-}\\&=(4ik)^{-1}\sum_j(2j+1)(T_{j--}-T_j)(\pi_j-\tau_j),\end{aligned}$$

$$\begin{aligned}\Theta_{++,--}&=\Theta_{--,++}\\&=(4ik)^{-1}\sum_j(2j+1)(T_{j++}+1-S_j)P_j,\end{aligned}$$

where S_j stands for the singlet state S-matrix element. Notice here that $\bar{S}$ together with S_j is essentially the helicity S matrix.

The form of these amplitudes shows that there is no difficulty in applying the Watson transform. All arguments that have been used in the spin zero case can be taken over directly.

[22] These are obtained by explicit calculation from reference 8. The notation is the same as there.

[23] These are obtained by explicit calculation from M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).