PHYSICAL REVIEW VOLUME 129, NUMBER 3 1 FEBRUARY 1963

Unitarity and High-Energy Behavior of Scattering Amplitudes*

A. MARTIN†
University of Washington, Seattle
and
CERN, Geneva, Switzerland
(Received 27 August 1962)

Upper and lower bounds of the imaginary part of a scattering amplitude are obtained for physical and unphysical values of the scattering angle, respectively, from unitarity alone. This imposes rather stringent conditions on the high-energy behavior of the scattering amplitude. In particular unitarity alone rules out Regge poles with value larger than unity for zero momentum transfer. On the other hand, it is shown that the total cross section cannot increase faster than the logarithm squared of the energy under assumptions appreciably more general than Mandelstam representation. Finally, in the light of the preceding results, we give a few comments on the problem of the shrinking of the diffraction peak and its connection with the decrease of the elastic cross section.

I. INTRODUCTION

IT is well known that the total cross section, including inelastic processes, for a two-body collision determines the imaginary part of the corresponding forward scattering amplitude as a consequence of unitarity. It seems, however, that for angles or transfers different from zero the restrictions imposed by unitarity have not been fully exploited. In the present paper we shall present an upper bound of the imaginary part of the scattering amplitude in the physical region and a lower bound of the same quantity in the unphysical region defined by $1<\cos\theta<\cos\theta_0$, where θ is the scattering angle and $\cos\theta_0$ is the major axis of the ellipse in which the Legendre expansion of the imaginary part of the scattering amplitude is convergent.

The bound obtained in the physical region provides some limitations on the high-energy behavior of the scattering amplitude. In particular, it is shown that a Regge behavior[1] of the scattering amplitude, namely, $f(t)s^{\alpha(t)}$, where t is the momentum transfer and s the square of the c.m. energy, is inconsistent with unitarity alone unless $\alpha(0)$ is less than or equal to unity. More generally it is shown that if the diffraction peak shrinks when energy increases, the total cross section should not increase faster than $1/\Delta t$, where Δt is the width of the diffraction peak.

The bound obtained in the unphysical region enables us to derive a result previously obtained by Froissart,[2] in a more general way. It is shown that if for some positive unphysical transfer the scattering amplitude can be expanded in Legendre polynomials and is bounded by some arbitrary power of the energy, the total cross section cannot increase faster than $\ln^2 s$, which is precisely the result of Froissart who assumed the validity of the Mandelstam representation.

In the last section we study the relation between the shape of the diffraction peak and the behavior of the total and elastic cross section.

II. INVESTIGATION OF THE PHYSICAL REGION

We define the imaginary part of the scattering amplitude for two spinless particles as

$$\mathrm{Im} f(s,\cos\theta)=(s^{1/2}/k)\sum(2l+1)a_l(s)P_l(\cos\theta), \quad (1)$$

where the normalization is defined in such a way that by application of the optical theorem the total cross section is given by

$$\sigma_t=(4\pi/k^2)\sum(2l+1)a_l(s), \quad (2)$$

k being the c.m. momentum. Then a necessary condition due to unitarity is

$$0\leqslant a_l(s)\leqslant 1. \quad (3)$$

In other terms, in a given angular momentum state, the amplitude of the outgoing wave should be less or equal to the amplitude of the ingoing wave.

We now consider the following problem: the total cross section at a given energy is supposed to be known. What can we say then about $\mathrm{Im} f$ at a given physical angle? First, it is obvious that $|\mathrm{Im} f(s,\cos\theta)|<\mathrm{Im} f(s,1)$ because the Legendre polynomial $P_l(\cos\theta)$ has a modulus smaller than unity in the physical region. In order to improve this bound it is convenient to replace $P_l(\cos\theta)$ in (1) by some smooth upper bound $B_l(\cos\theta)$ which, as will appear necessary later, has the property

$$B_l(\cos\theta)>B_L(\cos\theta) \quad \text{for} \quad L>l. \quad (4)$$

In addition, this bound should make sense in the neighborhood of $\theta=0$, which is not the case for the bounds proposed in the literature. We propose

$$|P_l(\cos\theta)|\leqslant B_l(\cos\theta)=[1+l(l+1)\sin^2\theta]^{-1/4}. \quad (5)$$

This bound, which is established in Appendix A, satisfies condition (4), is less than unity in the physical region, and is such that for $\cos\theta=1$, $P_l=B_l$ and $P_l'=B_l'$; for large l, θ fixed, it exceeds the best possible bound by a factor $(\pi/2)^{1/2}$.

* A preliminary account of this work has been presented to the *Geneva High-Energy International Conference* (CERN, Geneva, 1962).
† Permanent address: CERN, Geneva 23, Switzerland.
[1] See, for instance, G. F. Chew and S. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[2] M. Froissart, Phys. Rev. **123**, 1054 (1961).

Equation (1) is replaced by the following inequality:

$$|\mathrm{Im} f(s, \cos\theta)| < (s^{1/2}/k)\sum(2l+1)B_l(\cos\theta)a_l = \phi(\cos\theta). \quad (6)$$

We want to find the maximum possible value of the right-hand side of (6) for a given total cross section, and to do so we are free to vary the a_l's in the limits imposed by Eq. (2) and inequalities (3). We shall prove that $\phi(\cos\theta)$ is maximized by the choice[3]

$$a_0 = a_1 = \cdots = a_L = 1, \quad a_{L+1} = \epsilon < 1, \quad a_{L+n+1} = 0. \quad (7)$$

Let $\phi_0(\cos\theta)$ be the function associated with this choice, and let $\phi(\cos\theta)$ be another function, with arbitrary a_l's corresponding to the same cross section. Since the cross sections are the same we have

$$\sum_0^L (2l+1)(1-a_l) + \epsilon(2L+3) - \sum_{L+1}^{\infty}(2l+1)a_l = 0. \quad (8)$$

Since B_l decrease with increasing l, we can say

$$\phi(\cos\theta) < \sum_0^L (2l+1)a_l B_l + B_{L+1}\sum_{L+1}^{\infty}(2l+1)a_l.$$

Taking (8) into account, we find

$$\phi_0(\cos\theta) - \phi(\cos\theta) \geqslant \sum_0^L (2l+1)(1-a_l)(B_l - B_{l+1}) \geqslant 0.$$

The choice (7) maximizes $\phi(\cos\theta)$. L is obtained from the total cross section from

$$k^2\sigma_t/4\pi = (L+1)^2 + \epsilon(2L+3), \quad 0 \leqslant \epsilon < 1. \quad (9)$$

So

$$|\mathrm{Im} f(s, \cos\theta)| \leqslant \frac{\sqrt{s}}{k}\left[\sum_0^L (2l+1)B_l(\cos\theta) + \epsilon(2L+3)B_{L+1}(\cos\theta)\right].$$

It is possible to present this result in a more concise and practical form (see Appendix B):

$$\left|\frac{\mathrm{Im} f(s, \cos\theta)}{\mathrm{Im} f(s,1)}\right| \leqslant \frac{4}{3}\frac{[1+\bar{L}(\bar{L}+1)\sin^2\theta]^{3/4} - 1}{\bar{L}(\bar{L}+1)\sin^2\theta}, \quad (10)$$

where $k^2\sigma_t/4\pi = (\bar{L}+1)^2$, provided $\bar{L} > 0$.

It is satisfactory to see that the right-hand side of (10) reduces to unity as $\theta \to 0$ and also as $\bar{L} \to 0$; the latter case corresponds to pure S wave and therefore isotropic scattering amplitude.

An alternative form of (10) is obtained by expressing the angle in terms of the momentum transfer $t = -2k^2 \times (1-\cos\theta)$. Defining $F(s,t) \equiv f(s, \cos\theta)$, we obtain

$$\left|\frac{\mathrm{Im} F(s,t)}{\mathrm{Im} F(s,0)}\right| < \frac{4}{3}\frac{[1+x]^{3/4} - 1}{x},$$

with

$$x = \left(\frac{\sigma_t}{4\pi}\right)^{1/2}\left[\left(\frac{\sigma_t}{4\pi}\right)^{1/2} - \frac{1}{k}\right]|t|\left[1 - \frac{|t|}{4k^2}\right], \quad -4k^2 < t < 0 \quad (11)$$

and, in particular,

$$\frac{d}{dt}[\ln \mathrm{Im} F(s,t)]_{t=0} > \frac{1}{8}\left(\frac{\sigma_t}{4\pi}\right)^{1/2}\left[\left(\frac{\sigma_t}{4\pi}\right)^{1/2} - \frac{1}{k}\right]. \quad (12)$$

We want now to use (11) and (12) to impose some restrictions on the possible high-energy behaviors of scattering amplitudes. Let us notice first that if the cross section becomes constant at very high energy, the diffraction peak according to (11) can only have a finite extension in t. Now in the case where the diffraction peak shrinks with increasing energy, $(d/dt) \times[\ln \mathrm{Im} F(s,t)]$ can be considered as some measurement of the inverse of the width. Therefore, according to (12) the total cross section should not increase faster than the inverse of the width of the diffraction peak. In particular, if the scattering amplitude has a Regge behavior, i.e.,

$$F(s,t) \simeq g(t)s^{\alpha(t)}, \quad (13)$$

then the width of the diffraction peak is of the order of $1/\ln s$ and the cross section must increase slower than $\ln s$; however since the total cross section, by application of the optical theorem to (13), behaves like $s^{\alpha(0)-1}$, we see that we are forced to take $\alpha(0) \leqslant 1$. This qualitative argument, based on inequality (12), can be made more rigorous by using (11); we assume that $g(t)$ and $\alpha(t)$ are continuous in the nieghborhood of $t=0$ (they need not have derivatives). Then for t small enough, $\mathrm{Im} g(t) \neq 0$ because $\mathrm{Im} g(0) \neq 0$. Let us show that if $\alpha(0) > 1$ one arrives at a contradiction. The left-hand side of (11) behaves like $s^{\alpha(t)-\alpha(0)}$. The right-hand side behaves for fixed t as $s^{-(1/4)[\alpha(0)-1]}$ if $\alpha(0) > 1$. Then one can choose in advance t small enough so that $\frac{1}{4}[\alpha(0)-1] > \alpha(0) - \alpha(t)$ and arrive at a contradiction.

We shall discuss again the question of the diffraction peak in the last section.

III. INVESTIGATION OF THE UNPHYSICAL REGION

We restrict ourselves to transfers $0 < t < t_0(s)$, where $t_0(s)$ is the maximum value of t for which the Legendre expansion of $\mathrm{Im} F(st)$ is convergent. It is related to the semimajor axis of the ellipse in which the Legendre expansion is convergent by

$$\cos\theta_0 = 1 + t_0(s)/2k^2.$$

Then we can write

$$\mathrm{Im} F(s,t) = (s^{1/2}/k)\sum(2l+1)a_l P_l(1+t/2k^2).$$

[3] It was pointed out to me by Dr. Bell that this corresponds to the rather extreme situation where the L first phase shifts are real, equal to $\pi/2$, in which case there is no room for inelastic processes.

Here, since $P_l(x)>1$ for $x>1$, we have, obviously,

$$\operatorname{Im}F(s,t)>\operatorname{Im}F(s,0), \tag{14}$$

and we want to improve this inequality, i.e., to find the strict minimum of $\operatorname{Im}F(s,t)$ for a given total cross section. This problem is extremely simple because in this region the Legendre polynomials increase with l:

$$1<P_l(x)<P_L(x), \quad x>1, \quad L>l. \tag{15}$$

Following the same lines as in the preceding section, one finds that the minimum of $\operatorname{Im}F(s,t)$ is obtained by taking

$$a_0=a_1=\cdots=a_L=1, \quad a_{L+1}=\epsilon<1, \quad a_{L+n+1}=0,$$

with $k^2\sigma_t/4\pi=(L+1)^2+\epsilon(2L+3)$, i.e., the same choice of a_l's as in the preceding section. Hence we have

$$\operatorname{Im}F(s,t)\geqslant\frac{\sqrt{s}}{k}\left[\sum_0^L(2l+1)P_l\left(1+\frac{t}{2k^2}\right)+(2L+3)\epsilon P_{L+1}\left(1+\frac{t}{2k^2}\right)\right]. \tag{16}$$

A very crude lower limit of the right-hand side will be sufficient for our purpose: We take

$$\operatorname{Im}F(s,t)\geqslant(s^{1/2}/k)(2\bar{L}-1)P_{\bar{L}-1}(1+t/2k^2),$$

where $\bar{L}$ is again defined by $k^2\sigma_t/4\pi=(\bar{L}+1)^2$.

Now using a lower limit of P_l derived in Appendix C, we find

$$\operatorname{Im}F(s,t)>\frac{\sqrt{s}}{k}\left(\frac{2}{\pi}\right)^{1/2}\left[1+\frac{\sqrt{t}}{k}\right]^{k(\sigma_t/4\pi)^{\frac{1}{2}}-2}.$$

Let us now assume that $t_0(s)$, which gives the limit up to which the Legendre expansion converges, can be choosen energy independent beyond a certain energy.[4] Then one sees that the right-hand side of (17) is dominated, for large k, by the factor $\exp(t\sigma_t/4\pi)^{1/2}$. It follows that if one now requires that $\operatorname{Im}F(s,t)$ should be less than some arbitrary polynomial in s for fixed t, the total cross section cannot increase faster than $\ln^2 s$. This is precisely the result obtained by Froissart[2] on the basis of the Mandelstam representation. Our assumptions: boundedness by some power of s, dimensions of the ellipse in the t plane, are of course contained in the Mandelstam representation. They constitute, however, a much weaker requirement, as can be illustrated by potential scattering; our assumptions are satisfied by all potentials decreasing faster than an exponential at infinity; however, only superpositions of Yukawa potentials satisfy the Mandelstam representation. Our result could probably also have been derived by following the method of Low and Greenberg.[5]

[4] This assumption has not yet been proven in axiomatic field theory; see H. Lehmann, Nuovo Cimento **10**, 579 (1958).

[5] F. E. Low and M. Greenberg, Phys. Rev. **124**, 2047 (1961).

In the limiting case $\sigma_t=C\ln^2 s$ it is interesting to notice that a lower limit of the number of subtractions in s for fixed t can be obtained from the knowledge of C. In particular, when t takes its largest possible value we see that the number of subtractions in s is certainly larger than $(Ct_0/4\pi)^{1/2}$. Hence, increasing C from zero to infinity, we see clearly the transition from a finite number of subtractions to an infinite number of subtractions.

IV. DIFFRACTION PEAK AND CONNECTION WITH ELASTIC AND TOTAL CROSS SECTIONS

In Sec. II we have seen that if the total cross section becomes constant at high energy, unitarity, as we used it, is unable to predict a narrowing of the diffraction peak. On the other hand, the narrowing is automatic if the total cross section increases with energy. In particular, if the limit obtained in Sec. III, $\sigma_t\sim\ln^2 s$, is reached, then the width of the diffraction peak is of the order of $1/\ln^2 s$ (for comparison a Regge behavior gives a width of the order of $1/\ln s$).

Now it is intuitively clear that if the cross section becomes dominantly inelastic the diffraction peak will get narrower, and we want to show this in a rigorous fashion. The result we shall obtain is not the best possible one but it is sufficient for our purpose.

The proof is as follows: Let us apply Schwarz's inequality to the imaginary part of the scattering amplitude:

$$\left[\sum(2l+1)a_lP_l(\cos\theta)\right]^2<\sum(2l+1)a_l\sum(2l+1)a_lP_l^2(\cos\theta).$$

We notice that in the right-hand side we have a sum over $P_l^2(\cos\theta)$ which is more rapidly convergent than the one we started with, which was over $P_l(\cos\theta)$. We can repeat the process till $\sum(2l+1)[P_l]^{2^n}$ is convergent. Since $P_l(\cos\theta)\sim1/\sqrt{l}$ for fixed θ, we have to repeat the operation twice more:

$$\left[\sum(2l+1)a_lP_l^2\right]^2<\sum(2l+1)a_l\sum(2l+1)a_lP_l^4,$$
$$\left[\sum(2l+1)a_lP_l^4\right]^2<\sum(2l+1)a_l^2\sum(2l+1)P_l^8.$$

Now, since

$$\sigma_t=(4\pi/k^2)\sum(2l+1)a_l,$$
$$\sigma_e>(4\pi/k^2)\sum(2l+1)a_l^2,$$

we get

$$\left|\frac{\operatorname{Im}f(s,\cos\theta)}{\operatorname{Im}f(s,1)}\right|<\left(\frac{\sigma_e}{\sigma_t}\right)^{1/8}\left(\frac{4\pi\sum(2l+1)P_l^8(\cos\theta)}{k^2\sigma_t}\right)^{1/8}.$$

Using the bound (5), one can show that

$$\sum(2l+1)P_l^8(\cos\theta)<2/\sin^2\theta.$$

Hence

$$\left|\frac{\operatorname{Im}F(s,t)}{\operatorname{Im}F(s,0)}\right|<\left(\frac{\sigma_e}{\sigma_t}\right)^{1/8}\left(\frac{8\pi}{\sigma_t|t|[1-|t|/4k^2]}\right)^{1/8}. \tag{18}$$

This equation shows that if in the high-energy region the total cross section remains larger than some fixed positive number and if the ratio σ_e/σ_t goes to zero, the diffraction peak shrinks. In a way the problem is displaced but it seems easier to find theories in which the nonelastic channels dominate the elastic channel than to attack directly the problem of the narrowing of the diffraction peak.

V. EXTENSION TO PARTICLES WITH SPIN

All the results so far derived have been obtained under the assumption that the particles are spinless. However, it is not difficult to guess that this can be extended to the case of particles with spin. Let us consider, for example, the case of spin 0–spin $\frac{1}{2}$ scattering (e.g., meson–nucleon). Then the imaginary part of the no-spin-flip amplitude is

$$\sum[(l+1)a_l^+ + la_l^-]P_l(\cos\theta),$$

where a_l^+ and a_l^- refer to angular momenta $l+\frac{1}{2}$ and $l-\frac{1}{2}$, respectively. All the considerations made in the the spinless case can be extended here: The optical theorem applies to the no-spin-flip amplitude, and the upper (lower) bound in the physical (unphysical) region will be obtained again by taking

$$a_0=a_1^+=a_1^-=\cdots a_L^+=a_L^-=1,\quad a_{L+n}^+=a_{L+n}=0.$$

Therefore the bounds are the same and the consequences for high-energy behavior are the same. One can also obtain, for given total cross section, an upper bound of the imaginary part of the spin-flip amplitude $\sin\theta \times\sum(a_l^+ - a_l^-)P_l'(\cos\theta)$ in the physical region, but this is less interesting. It is very likely that in the general spin case it will always be possible to find at least one scalar amplitude to which the above considerations can be applied.

VI. ACKNOWLEDGMENTS

The final version of the present paper was written during a stay at the Summer Institute of the University of Washington. It is a pleasure to acknowledge the kind hospitality of Professor Boris Jacobsohn and Professor Ronald Geballe. I also thank Dr. J. Loeffel for some simplications in Appendix A.

APPENDIX A

An Upper Bound for Legendre Polynomials

We want to establish the following upper bound for the Legendre polynomial $P_l(x)$ with $-1\leqslant x\leqslant 1$:

$$|P_l(x)|\leqslant[1+l(l+1)(1-x^2)]^{-1/4}=B_l(x).\quad\text{(A1)}$$

This bound has several advantages:

(i) It makes sense in the whole physical range $-1\leqslant x\leqslant 1$.

(ii) It has, up to a constant factor, the correct asymptotic behavior as $l\to\infty$.

(iii) It is excellent when x is close to 1 since

$$B_l(1)=P_l(1)=1$$

and

$$(dB_l/dx)_{x=1}=(dP_l/dx)_{x=1}=l(l+1)/2.$$

The proof will be made in two steps: We first establish that (A1) holds for $x_l\leqslant x\leqslant 1$, where x_l is the largest zero of $P_l(x)$. Then in $0\leqslant x\leqslant x_l$ we compare the bound (A1) with a known upper bound of $|P_l(x)|$ and we show that $B_l(x)$ is larger than this upper bound.

Let us define

$$f(x)=1-[P_l(x)]^4[1+l(l+1)(1-x^2)].$$

We want to show that $f(x)\leqslant 0$ in $x_l\leqslant x\leqslant 1$. A useful double inequality for this purpose is

$$[l(l+1)/(1+x)]P_l(x)<P_l'(x)<l(l+1)/(1+x),\quad\text{(A2)}$$

in $x_l<x<1$, which is easily deduced from

$$(1-x^2)P_l'(x)=l(l+1)\int_x^1 P_l(x)dx;$$

the latter comes directly from integration of the Legendre equation. Two interesting consequences of (A2) are:

$$x_l<2\exp\left[-\frac{1}{l(l+1)}\right]-1<\left[1-\frac{1}{l(l+1)}\right]^2\quad\text{(A3)}$$

for $l\geqslant 1$, and

$$f'(x)<2l(l+1)P_l^4(x)\times\{-[2/(1+x)][1+l(l+1)(1-x^2)]+x\}<0$$

for

$$x_l<x<1.$$

Since $f(1)=0$, $f(x)$ is positive for $x_l\leqslant x\leqslant 1$, which constitutes the first part of the proof.

Next we consider $0\leqslant x\leqslant x_l$. We compare (A1) with the bound given by Szegö[6]:

$$|P_l(x)|<(2/\pi l)^{1/2}(1-x^2)^{-1/4}=A_l.$$

Now

$$(1/A_l)^4-(1/B_l)^4\geqslant(1-x_l^2)l[\tfrac{1}{4}\pi^2 l-l-1]-1,\quad\text{for}\quad l\geqslant 1.$$

One easily sees that for $l\geqslant 2$, using (A3), B_l is larger than A_l. Since the bound B_l holds also for $l=0$ and $l=1$, as can be checked algebraically, it holds for any l in the whole range $-1\leqslant x\leqslant 1$.

APPENDIX B

Here we want to evaluate

$$\sum_0^L(2l+1)B_l(\cos\theta)+\epsilon(2L+3)B_{L+1}(\cos\theta),\quad\text{(B1)}$$

---

[6] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society Colloquim Publications, New York, 1959), revised ed., Vol. XXIII.

with

$$B_l(\cos\theta)=[1+l(l+1)\sin^2\theta]^{-1/4}.$$

Let us first consider the case $\epsilon=0$ which is appreciably simpler. Since $(2l+1)B_l(\cos\theta)$ is an increasing function of l with negative curvature, as can be checked easily, we have

$$2\int_0^L(2l+1)B_l(\cos\theta)dl>\left(\sum_0^{L-1}+\sum_1^L\right)(2l+1)B_l(\cos\theta). \quad \text{(B2)}$$

Hence

$$\sum_0^L(2l+1)B_l(\cos\theta)<\frac{2L+1}{2}[1+L(L+1)\sin^2\theta]^{-1/4}+\frac{1}{2}+\frac{4}{3}\frac{[1+L(L+1)\sin^2\theta]^{3/4}-1}{\sin^2\theta}. \quad \text{(B3)}$$

We want now to show that the right-hand side of (B3) is less than

$$\frac{4}{3}\frac{L+1}{L}\frac{[1+L(L+1)\sin^2\theta]^{3/4}-1}{\sin^2\theta}.$$

Therefore, we have to show that

$$\frac{4}{3}\frac{[1+L(L+1)\sin^2\theta]^{3/4}-1}{L\sin^2\theta}-\frac{1}{2}\left[\frac{2L+1}{[1+L(L+1)\sin^2\theta]^{1/4}}+1\right] \quad \text{(B4)}$$

is a positive quantity. Using the variable

$$z=[1+L(L+1)\sin^2\theta]^{1/4},$$

one can transform (B3) into

$$\frac{(z-1)[6L+3+(4L+1)z+(2L-1)z^2-3z^3]}{6z(z^3+z^2+z+1)}.$$

Now z lies certainly in the interval $1\leqslant z<(L+1)^{1/2}$ and it is easy to see that in this interval the bracket remains positive. Therefore,

$$\frac{\sum_0^L(2l+1)B_l(\cos\theta)}{\sum_0^L(2l+1)}<\frac{4}{3}\frac{[1+L(L+1)\sin^2\theta]^{3/4}-1}{L(L+1)\sin^2\theta}. \quad \text{(B5)}$$

In the case $\epsilon\neq 0$ the problem is slightly more difficult. Then $\bar{L}$, as defined by $k^2\sigma_t=4\pi(\bar{L}+1)^2$, is no longer an integer. One may notice, however, that (B1) is a linear function of $(\bar{L}+1)^2$ between $(\bar{L}+1)^2=(L+1)^2$ and $(\bar{L}+1)^2=(L+2)^2$, since ϵ is given by $(2L+3)\epsilon=(\bar{L}+1)^2-(L+1)^2$. One can convince oneself that the function $L^{-2}(L+1)\{[1+L(L+1)\sin^2\theta]^{3/4}-1\}$ has a negative curvature as a function of $(L+1)^2$ (not of L). Therefore, the relation

$$\left|\frac{\operatorname{Im}f(s,\cos\theta)}{\operatorname{Im}f(s,1)}\right|<\frac{4}{3}\frac{[1+\bar{L}(\bar{L}+1)\sin^2\theta]^{3/4}-1}{\bar{L}(\bar{L}+1)}$$

still holds for nonintegral values of $\bar{L}$.

APPENDIX C

Lower Bound For $P_l(x)$ for $x>1$.

One can write $P_l(x)$ in powers of z defined by

$$z+\frac{1}{z}=2x,\quad z=x+(x^2-1)^{1/2};$$

then $P_l(x)=\sum_{-l}^{+l}C_nz^n$. It is not very difficult to see, from the generating function for instance, that all the coefficients C_n are *positive*. The coefficient of z^l is naturally equal to the coefficient of x^l in the normal expansion divided by 2^l, i.e.,

$$(2l)!/2^{2l}l!l!>[4/\pi(2l+1)]^{1/2}.$$

Hence

$$P_l(x)>[4/\pi(2l+1)]^{1/2}[x+(x^2-1)^{1/2}]^l.$$