Particle Model of Scattering from Collective States of Nuclei*

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The differential cross sections for the excitation of the vibrational levels of nuclei are derived using the direct interaction between the projectile and the shell-model particles via a two-body force. The nuclear collective states are approximate solutions to the Hamiltonian consisting of a pairing plus a quadrupole interaction in a quasi-particle representation. The results obtained by this method are compared to those which follow from an interaction using collective coordinates for states of one and two phonons. Numerical results are given for the plane-wave first Born approximation scattering of alpha particles with a delta-function interaction between the projectiles and the nuclear particle and for the plane-wave Born approximation scattering of electrons, assuming a Coulomb interaction.

I. INTRODUCTION

EXPERIMENTS with medium and high-energy accelerators, in recent years, have produced considerable evidence that an important portion of reactions takes place through processes which can be described as "direct" rather than "compound" reactions; i.e., often the differential cross sections show patterns which can be successfully interpreted by assuming specific initial and final channels and explicitly following a few of the particles involved in the reaction. The influence of other open or virtual channels is neglected, except, perhaps, by introducing a mechanism for absorption or by explicitly assuming a particular form for the projectile wave functions or the interaction to try to include certain physical effects of the neglected processes.

The most successful application of these methods has been to the stripping and pickup processes, reactions in which one particle is exchanged between the projectile and the target. Often the angular distribution of scattered particles can be qualitatively understood by a simple semiclassical treatment of the kinematics.[1] Using methods which are essentially a cutoff Born approximation[2] with plane waves for the projectile, Butler has demonstrated that these reactions can often be correctly interpreted by assuming that the stripped (picked-up) particle is placed in (removed from) a particular single-particle shell-model state. Furthermore, reasonable extensions of this picture, including the distortion of the projectile wave function, Coulomb effects, etc., have generally improved the agreement with experiment.[3]

The direct interaction process which has perhaps attracted most interest recently is that of inelastic scattering. For a number of years such reactions have been used to identify single-particle states. That one can calculate cross sections which are in reasonable agreement with experiment using shell-model states with a model analogous to that of Butler's for stripping[4,5] and that these calculations can be improved by using distorted waves for the scattering particle[6,7] serves as further evidence of the possibility of successfully treating these processes by using a simple interaction linking the initial and final channels.

The excitation of collective states by inelastic scattering, first studied systematically by Cohen,[8] offers an exciting possibility for the study of nuclear structure and the nature of collective states. The empirical differential cross sections for the scattering of medium energy nuclear particles, especially alpha particles, show strong oscillations, in which often the maxima and minima occur at regular intervals in the scattering.[9] Theoretical cross sections with such a general shape have been derived by a diffraction model for scattering of particles from a nonspherical strongly-absorbing nucleus; this model was introduced by Drozdov,[10] extended by Inopin[11] to allow the introduction of collective coordinates for the excitation of quadrupole vibrational states, and further developed by Blair[12] to allow any type of surface phonon. Blair has emphasized the necessity to understand the inelastic and elastic scat-

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tering in terms of the same mechanism, and has demonstrated that certain qualitative relationships which have been found between the experimental elastic and inelastic cross sections can be derived by means of this model. These results can also be obtained by the more nearly general method of distorted-wave Born approximation[13]; however, since these latter calculations are quite complicated it is easier to learn of the qualitative features of the process in the adiabatic method.

Using a plane-wave Born approximation, Lemmer, de-Shalit, and Wall have recently extended the calculation of these reaction processes to the excitations of states of two phonons.[14] These authors have pointed out that in the region of Fe and Ni some of the 44-MeV alpha-particle reaction experiments seem to suggest a phase rule between the maxima and minima of the various cross sections according to which the angular distributions corresponding to the excitation of the one-phonon states of odd J (i.e., the octopole states) and to the two-phonon quadrupole vibrational states are in phase with angular distribution for the elastic scattering, while the maxima in the angular distribution corresponding to the excitation of the one-phonon quadrupole state occur approximately at the minima of these other distributions. Because of the simplicity and clarity of their formulation of the inelastic scattering in terms of collective coordinates, we compare certain limits of the derivations in terms of two-body forces, which is the subject of this paper, to the results of Lemmer *et al.*, and, therefore, give a brief sketch of their work.

They use the collective model of Bohr and Mottelson,[15] in which the nucleus surface is sharp, but nonspherical, with the deviation from spherical symmetry being expressed by collective coordinates, which are quantized to serve creation and destruction operators of phonons whose spin equals the order of the surface distortion. Assuming that the projectile experiences no interaction outside of the nuclear surface, they expand the interaction in terms of the distortion parameters, taking the zero order as a uniform potential well of radius R_0. The first term in the expansion is an optical potential for elastic scattering, which predicts an angular distribution of $[j_1(qR_0)/qR_0]^2$, q being the momentum transfer and j_1 representing the first-order spherical Bessel function. The second term is simply a surface interaction linear in phonon creation and annihilation operations, the interaction which had been used by Hayakawa and Yoshida[16] for the excitation of rotational states, which leads to one-phonon excitations in the first Born approximation with an angular distribution of $[j_L(qR_0)]^2$, with L being the spin of the phonon. This particular form is obtained because a derivative of a square well gives a delta function at the origin; a more realistic surface would result in an angular distribution which lacks this quasi-periodicity for large momentum transfer.

The two-phonon excitation results from the second term in the expansion in the second order and the third term in the first order, the latter terms apparently being most important for plane waves.[14] The first order calculation of the two-phonon excitation leads (in the large qR_0 limit) to a matrix element which contains a spherical Bessel function of order L–1, giving the phase rule stated above. One can expect this result to be altered by using a diffuse surface and distorted waves for the particle. Recent calculations have shown the two-step process to be as important as the one-step one,[17] but it is valuable to see how such a phase rule can arise from a calculation using a collective model for comparison with more nearly fundamental methods.

The development of linear accelerators which produce beams of electrons with energies greater than the π-meson rest mass has provided a valuable tool for the study of nuclear structure.[18] One can more easily formulate a theory of the excitation of collective nuclear states by electrons than by nuclear projectiles, since the interaction is more nearly accurately known for the former and since one knows better how to handle the system when the electrons are near the nucleus. Moreover, from the fact that electrons tend to penetrate the nucleus, while nuclear particles of intermediate energies are absorbed, one might expect to be able to learn more of the details of the nuclear structure from electron scattering.

There have been several treatments of electron scattering in terms of simple particle and collective coordinates,[19] and the shape of the differential cross sections can be qualitatively understood in terms of a simple model in the Born approximation. Still, one hopes that there is a great deal more to be learned about the nucleus from these reactions.

In the present work an attempt is made to understand the inelastic scattering from vibrational states in terms of two-body interactions, without introducing collective coordinates for the interaction. This is possible because of the success of the treatment of the collective states in terms of shell-model particles interacting with an interaction consisting of a short-range and a long-range component. In a quasi-particle interaction the short-range force is approximately diagonalized, resulting in a seniority spectrum for even nuclei with a ground state of seniority zero (no quasi-particles) well below the seniority two states (two quasi-particles).[20] Using the

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dilute fermion approximation,[21] which is accurate as long as the vibrational levels do not lie too close to the ground state, i.e., the number of quasi-particles in the ground state remain small, it has been shown that it is possible to approximately diagonalize a force which has the angular dependence of the second Legendre polynomial of the cosine of the angle between the interacting nuclear particles and that this leads to the quadrupole vibrational states.[22]

The vibrational levels for the single closed shell nuclei in this (Sawada) approximation have been calculated with a pairing plus a quadrupole force,[23] using the same parameters which had been used for the calculation in which the same force had been treated by a deformed field approximation,[24] with quite similar results except in the cases for which the adiabatic approximation is known to be a poor one. Similar results are obtained in the spherical region with both neutrons and protons outside of the closed shells.[25] While the deformed field calculation has the advantage of clearly revealing the physical content of the semiclassical collective motion, one knows precisely which terms are neglected in the calculations with two-body forces, and therefore, is in a better position to study those processes which are not properly treated in the adiabatic limit.

The assumption is made here that the projectile is an elementary particle which can be distinguished from the nuclear particles. This implies that the exchange terms are dropped, which might introduce considerable error for inelastic proton scattering,[6] especially at lower energies, but is not expected to be so important for alpha-particle scattering. It is then simple to transform the part of the interaction which involves the coordinates of the nuclear particles into the quasi-particle representation, and this result, along with certain other mathematical relationships, are given in Sec. II. Because of their intimate relationship to the deformed field approximation, these reactions are conveniently discussed in terms of the Sawada approximation and the corrections to this approximation. In this manner one can study in the Born approximation the elastic scattering, the one-phonon, and two-phonon excitations; and in Sec. III results, analogous to those obtained in the treatment with collective coordinates, are derived in terms of single-particle quantities. In particular, one can see how a phase rule is obtained in certain limiting cases, and how it depends upon the filling of the single-particle states.

In Sec. IV the method is applied to the scattering of alpha particles from Ni^{58} in the first Born approximation. Plane waves are used for the alpha-particle wave functions and a delta function is chosen for the interaction between the alpha particle and the nuclear particles. The relation between the magnitude of the force constant for the interaction of alpha particles and nuclear particles which results from this calculation and the optical potential for the elastic scattering of alpha particles from nuclei is discussed.

In Sec. V, the excitation of collective states by electrons is derived and numerical results are given for the scattering of 183-MeV electrons from the collective states of Ni^{58} in the first Born approximation.

II. INTERACTION HAMILTONIAN AND WAVE FUNCTIONS

A. Interaction Hamiltonian

The interaction between the projectiles and the nucleus is assumed to be of the form

$$\mathcal{U}=\sum_{kk'\nu\nu'}\langle k'\nu'|v|k\nu\rangle a_{k'}{}^{\dagger}b_{\nu'}{}^{\dagger}b_{\nu}a_{\nu}, \tag{1}$$

in which the $a_k{}^{\dagger}$ (a_k) are the creation (destruction) operators for the projectile particles and the $b_\nu{}^{\dagger}$ (b_ν) are the creation (destruction) operators for the nuclear particles, with k and ν standing for all of the quantum numbers of these particles. The $b_\nu{}^{\dagger}$ and b_ν obey fermion anticommutation rules and are assumed to commute with the $a_k{}^{\dagger}$ and the a_k, since the assumption is made that the scattered particles can be distinguished from the nuclear particles. Neglecting the internal excitations of the projectile particles, they can be treated approximately as bosons or fermions, depending upon their spin; but since only one-particle projectile states enter into calculation, the only property needed is that $a_k{}^{\dagger}a_{k'}=\delta_{kk'}$ when operating on one-particle states, so that the commutation or anticommutation rules do not have to be specified.

The two-body interaction between the projectile and the ith nuclear particle is taken as

$$v_{pi}=(1+a\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_i)v(r_{pi}), \tag{2}$$

in which the $\boldsymbol{\sigma}$'s are the Pauli spin operators and $\boldsymbol{r}_{pi}$ is the relative coordinate. To separate the interaction into nuclear and projectile part, the Slater expansion of the force and the spherical harmonic addition theorem are used, so that the two-body interaction can be written

$$v_{pi}=\sum_{LM}(-1)^{M}\frac{4\pi}{2L+1}f_L(r_p,r_i)[Y^{LM}(p)Y^{L-M}(i) +a\sum_{\mu}(-1)^{\mu}\sigma^{\mu}(p)\sigma^{-\mu}(i)Y^{LM}(p)Y^{L-M}(i)]. \tag{3}$$

B. Nuclear States

Let us restrict ourselves to the single closed shell isotopes for simplicity, although there is no difficulty

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in extending the derivation to nuclei in which both protons and neutrons are outside of the closed shells. (If the neutron-neutron, proton-proton, neutron-proton quadrupole interactions strengths are the same, the application of these results to nonsingle closed shell nuclei just involves extending the sums.) The Hamiltonian H_N from which the nuclear states are derived corresponds to particles moving in a shell-model well and interacting with a pairing force and a quadrupole force, a model which seems to account for many of the main features of the low-energy nuclear systematics.[24,25] For completeness, we briefly state the picture which is presented by such a model. In second quantized form the Hamiltonian is

$$H_N=\sum_{jm}\epsilon_j b_{jm}^{\dagger}b_{jm} -\tfrac{1}{2}G\sum_{jj'mm'}b_{j'm'}^{\dagger}b_{j'-m'}^{\dagger}b_{j-m}b_{jm}-\tfrac{1}{2}\chi\mathbf{Q}\cdot\mathbf{Q}, \quad (4)$$

in which $\mathbf{Q}$ is the nuclear quadrupole operator

$$\mathbf{Q}_{\mu}=\sum_{jj'mn}\langle j'm'|r^2Y_{\mu}^{2}|jm\rangle b_{j'm'}^{\dagger}b_{jm}. \quad (5)$$

In a quasi-particle representation, obtained by means of the transformation[26] (with $m>0$)

$$\begin{aligned}\alpha_{jm}&=U_j b_{jm}-V_j b_{j-m}^{\dagger},\\ \beta_{jm}&=U_j b_{j-m}+V_j b_{jm}^{\dagger},\end{aligned} \quad (6)$$

in which the transformation coefficients V_j^2, the probability of occupation of the jth level, are chosen to approximately diagonalize the first two terms in (4), for even-even nuclei the nuclear Hamiltonian can be approximately written as

$$H_N\cong E_0+\sum_{jm}E_{jm}\alpha_{jm}^{\dagger}\alpha_{jm} -\tfrac{1}{2}\chi\sum_M\Big|\sum_{jj'}[(U_jV_{j'}+U_{j'}V_j)/2(5)^{1/2}] \times\langle j'\|r^2Y^2\|j\rangle[A_{j'j}{}^{LM\dagger}+(-1)^M A_{j'j}{}^{L-M}]\Big|^2, \quad (7)$$

with

$$A_{j'j}{}^{LM\dagger}=(-1)^{l+l'}[\alpha_j^{\dagger}\alpha_{j'}^{\dagger}]_M{}^L \quad (8)$$

representing the operator formed by vector coupling the quasi-particle creation operators $(\alpha_j^{\dagger},\beta_j^{\dagger})$ with $(\alpha_{j'}^{\dagger},\beta_{j'}^{\dagger})$ to an operator of angular momentum L and z component M [with a phase $(-1)^{l+l'}$]. The first term, E_0 in Eq. (7) is the ground-state energy, the E_{jm} are the quasi-particle energies, and in the third term the sums and the quasi-particle operators are to be taken to give a scalar quantity. The accuracy of keeping only the first two terms to approximate the single-particle energies and the pairing force has been discussed in references 20 and 24 and has recently been checked by comparison with exact calculations.[27] The third term is the quadrupole interaction of Eq. (4) in the quasi-particle representation without terms which arise from the scattering of quasi-particles (see Sec. II C). In the deformed field approximation, i.e., replacing one of the factors in the last term by the average quadrupole moment and assuming that the particle motion is rapid with respect to the quadrupole vibration, this Hamiltonian fits the main systematics of the single-particle and collective states in the single closed shell isotopes.[24] The error of dropping of the terms in scattering of quasi-particles is of the same order as using the approximate commutation rules,

$$[A_{12}{}^{IM},A_{34}{}^{I'M'\dagger}] \cong\delta_{II'}\delta_{MM'}[\delta_{12}\delta_{34}-(-1)^{l_1+l_2+j_1+j_2+I}\delta_{14}\delta_{23}], \quad (9)$$

and analogous ones for the other combinations of quasi-particle operators. The eigenstates of the Hamiltonian (7) with the approximation (9) can easily be found.[21,22] The ground state can be expressed in terms of the quasi-particles by an expansion

$$\begin{aligned}\Psi_0&=\Psi_0{}^{(0)}+\Psi_0{}^{(4)}+\cdots\\ &=\Psi_0{}^{(0)}+\sum_{1234}c_{1234}[A_{12}{}^{2\dagger}A_{34}{}^{2\dagger}]_0{}^0\Psi_0{}^{(0)}+\cdots,\\ \alpha\Psi_0{}^{(0)}&=0.\end{aligned} \quad (10)$$

The energy of the phonon, the excitation energy of the first 2+ state, is given in the Sawada approximation as the smallest value of $\hbar\omega$ which is a solution to the relation

$$(2\chi/5)\sum_{jj'}(\mathcal{Q}_{j'j})^2(E_j+E_{j'})/[(E_j+E_{j'})^2-(\hbar\omega)^2] \quad (11)$$

with

$$\mathcal{Q}_{j'j}=\langle j'\|r^2Y^2\|j\rangle(U_jV_{j'}+U_{j'}V_j), \quad (12)$$

in which $\langle\|\ \|\rangle$ represents a reduced matrix element. In (10) the wave function $\Psi_0{}^{(0)}$ is the quasi-particle vacuum, except for normalization, and the constants c_{1234} are determined from the condition

$$B\Psi_0=0. \quad (13)$$

The excited collective states are obtained from the ground state by the phonon creation operator

$$\begin{aligned}B_M{}^{L\dagger}&=\sum_{j'j}(a_{j'j}A_{j'j}{}^{LM\dagger}+(-1)^M b_{j'j}A_{j'j}{}^{L-M})\\ &=N_\omega\sum_{jj'}\frac{\mathcal{Q}_{j'j}}{5^{1/2}}\Big[\frac{1}{E_j+E_{j'}-\hbar\omega}A_{j'j}{}^{LM\dagger} -(-1)^M A_{j'j}{}^{L-M}\frac{1}{E_{j'}+E_j+\hbar\omega}\Big]\end{aligned} \quad (14)$$

with

$$N_\omega^{-2}=(8\hbar\omega/5)\{\sum_{jj'}(\mathcal{Q}_{j'j})^2(E_j+E_{j'})/ [(E_j+E_{j'})^2-(\hbar\omega)^2]^2\}. \quad (15)$$

C. Corrections to the Sawada Approximation

As is shown in the next section, the Sawada approximation, described in the preceding paragraph, leads to the inelastic excitation of the one-phonon states, but does not allow the two-phonon states to be excited. In this section the various corrections to this approximation needed for the present work are given.

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Including the quasi-particle scattering terms, the operators which we need for the Hamiltonian given by Eqs. (1) and (2), under the Bogoliubov-Valatin transformation (5) are

$$\sum_{jj'mm'}\langle j'm'|Y_M{}^L|jm\rangle b_{j'm'}{}^\dagger b_{jm}$$
$$=\sum_{jj'}\Big\{\delta_{L0}(2j+1)^{1/2}V_j{}^2\delta_{j'j}+\frac{\langle j'\|V^L\|j\rangle}{(2L+1)^{1/2}}$$
$$\times\Big[\frac{(U_jV_{j'}+U_{j'}V_j)}{2}(A_{j'j}{}^{LM\dagger}+(-1)^MA_{j'j}{}^{L-M})$$
$$+(U_jU_{j'}-V_jV_{j'})\eta_{j'j}{}^{LM}\Big]\Big\}, \quad (16)$$

and

$$\sum_{jj'}\langle j'm'|\sigma^\mu Y_M{}^L|jm\rangle$$
$$=\sum_J C_{\mu,M,\mu+M}{}^{1LJ}\sum_{jj'}\frac{\langle j'\|T^{1L;J}\|j\rangle}{(2J+1)^{1/2}}$$
$$\times\{\tfrac{1}{2}(U_jV_{j'}-U_{j'}V_j)[A_{j'j}{}^{JM\dagger}+(-1)^MA_{j'j}{}^{J-M}]$$
$$+(U_jU_{j'}+V_jV_{j'})\eta_{j'j}{}^{LM}\}. \quad (17)$$

In (16) and (17) the quantity

$$\eta_{j'j}{}^{LM}=[\alpha_{j'}{}^\dagger\alpha_j]_M{}^L, \quad (18)$$

an operator carrying angular momentum L and z component M corresponding to the scattering of a quasi-particle from state j to j', has been introduced for convenience, and the quantity $t^{1L;J}$ is defined as

$$t^{1L;J}=[\sigma Y^L]^J, \quad (19)$$

the tensor of rank J formed from the vector coupling of the spin operator to the spherical harmonic of order L. In the dilute quasi-particle approximation the terms in η are dropped, so that the neglect of these terms introduces errors of the same order as the use of the commutation rules (9). For this reason, for the study of the excitation of the second vibrational state it is necessary to use the correct commutation rules. The basic commutation rule for the double quasi-particle operators is

$$[A_{12}{}^{IM},A_{34}{}^{I'M'\dagger}]$$
$$=\delta_{II'}\delta_{MM'}[\delta_{13}\delta_{24}-\delta_{23}(-1)^{l_1+l_2+j_1+j_2+I}]$$
$$+(-1)^{l_2+l_4+j_1+j_4+M}\sum_s(-1)^sC_{M',-M,M'-M}{}^{I'Is}$$
$$\times[(2I+1)(2I'+1)]^{1/2}\times\sum_{\substack{l=3,4=p\neq l\\ m=1,2=q\neq m}}(-1)^{I'\delta_{l3}+I\delta_{m1}}$$
$$\times\eta_{j_lj_m}{}^{sM'-M}\delta_{pq}W(I'j_lIj_m;j_ps), \quad (20)$$

with C and W being the Clebsch-Gordan and Racah coefficients, respectively. The last term in this relation represents the deviation of the fermion pairs from bosons and becomes small if the fermion density is small. Another commutator useful for this work is

$$[\eta_{12}{}^{sM'},A_{34}{}^{IM\dagger}]$$
$$=\sum_{s'}[(2s+1)(2I+1)]^{1/2}C_{M,M',M+M'}{}^{Iss'}$$
$$\times[\delta_{23}A_{14}{}^{s'}(-1)^{M+M'}W(sj_1Ij_4;j_2s')$$
$$+(-1)^{l_2+j_2+l_3+j_3+1-I}\delta_{24}A_{13}{}^{s'M+M'\dagger}$$
$$\times W(sj_1Ij_3;j_2s')]. \quad (21)$$

The commutation rules (20) and (21) are exact.

III. ELASTIC AND INELASTIC SCATTERING

A. Elastic Scattering

In the Born approximation the elastic scattering is calculated from the expectation value of V in the states given by (10). For a Wigner force there is a contribution from the part of the interaction independent of quasi-particle operators. From Eqs. (10) and (16), the matrix element from this part of the interaction is

$$\langle\Psi_0|\mathcal{U}|\Psi_0\rangle=\sum_j\varphi_{k'}{}^*(r_p)\varphi_k(r_p)f_0(r_p,r)$$
$$\times R_{nl}{}^2(r)V_j{}^2r^2drd^3r_p, \quad (21)$$

with $\varphi_k(r_p)$ the wave function of the projectile and R_{nl} the radial wave function of the nuclear particle in the state nlj.

To see the relation of this result to the derivation in terms of collective coordinates, let us consider the plane-wave Born approximation with a δ-function interaction between the projectile and the nuclear particles; i.e.,

$$\varphi_k(r_p)=(1/V^{1/2})e^{i\mathbf{k}\cdot\mathbf{r}_p},$$
$$f_L(r_p,r)=g[(2L+1)/4\pi][\delta(r-r_p)/rr_p], \quad (22)$$

with V the normalization volume and g the force strength. In this case the matrix element (21) becomes

$$\langle\Psi_0|\mathcal{U}|\Psi_0\rangle=\frac{4\pi}{V}g\int dr\,\frac{1}{qr}\sin qr\rho(r), \quad (23)$$

where q is the momentum transfer.

The quantity $\rho(r)$ appearing in (23), defined by

$$\rho(r)=\sum_j V_j{}^2R_{nl}{}^2(r) \quad (24)$$

is the radial density function. If one assumes a uniform distribution

$$\rho(r)=\rho_0, \quad r<R_0$$
$$=0, \quad r>R_0, \quad (25)$$

the elastic scattering differential cross section becomes

$$(d\sigma/d\Omega)^{el}=(4\mu^2/\hbar^4)(g\rho_0)^2R_0{}^6[j_1(qR_0)/qR_0]^2, \quad (26)$$

which is identical to the plane-wave Born approximation scattering from a square well of depth $g\rho_0$ and radius R_0 by a spinless particle with a reduced mass μ.

Because the ground-state wave functions consist of zero, four, etc., quasi-particles, the terms which correspond to the destruction and creation of two quasi-particles, the $A^\dagger$ and A terms in the interaction, do not contribute to the elastic scattering. Therefore, only the quasi-particle scattering parts of the interaction, the terms containing the operators $\eta_{j'j}$, which are neglected in the Sawada approximation, cause deviation from the above result. These corrections to the Sawada approximation are second order and higher in the expansion coefficients, since they connect only the parts of the wave function with four or more quasi-particles. The correction to the matrix element (21) arising from the four quasi-particle terms in the ground-state wave functions, with the neglect of the contribution from the quasi-particle scattering terms in the matrix element of $\langle\Psi_0|[AA]_0^0[A^\dagger A^\dagger]_0^0|\Psi_0\rangle$, is

$$\langle\Psi_0^{(4)}|\mathcal{U}|\Psi_0^{(4)}\rangle \cong\sum_j \varphi_{k'}^*(r_p)\varphi_k(r_p)f_0(r_p,r)R_{nl}^2(r)8[(U_j^2-V_j^2) +a\sum_\mu(-1)^\mu\sigma^\mu(p)\langle j\|\sigma\|j\rangle/\sqrt{3}(2j+1)^{1/2}] \times[\sum_{1234}c_{1234}^2(\delta_{1j}+\delta_{2j}+\delta_{3j}+\delta_{4j})/ 1+8\sum_{1234}c_{1234}^2]. \quad (27)$$

So long as the vibrational energy does not fall too low in the gap, as in the single closed shell nuclei, these contributions are small compared to the zero-order matrix element.

B. Excitation of the One-Phonon States

The matrix element needed in the Born approximation for the inelastic scatterings of the one-phonon state is

$$\langle k'\Psi_0 B|\mathcal{U}|\Psi_0 k\rangle. \quad (28)$$

Because of the relation $\langle\Psi_0|B|\Psi_0\rangle=0$, the interaction term independent of quasi-particle operators cannot lead to this inelastic process. Moreover, the quasi-particle scattering terms cannot lead to this process in any order since the operators $\eta_{j'j}$ do not change the number of quasi-particles while the state $B^\dagger\Psi_0$ has no component with the same number of quasi-particles as the ground state. Thus the matrix element is

$$\langle k'\Psi_0 B_M{}^L|\mathcal{U}|\Psi_0 k\rangle = \sum_{L'M'}\sum_{jj'}\frac{4\pi}{2L'+1}\int d^3r_p dr\, r^2 f_{L'}(r_p,r)\varphi_{k'}^*(r_p)\varphi_k(r_p)Y_{M'}{}^{L'}(\Omega_p)R_{n'l'}(r)R_{nl}(r)\Big\{\frac{(U_{j'}V_j+U_jV_{j'})}{2(2L'+1)^{1/2}} \times\langle j'\|Y^{L'}\|j\rangle\langle\Psi_0 B_\mu|(A_{j'j}{}^{L'-M^\dagger}+(-1)^{M'}A_{j'j}{}^{L'M})|\Psi_0\rangle+a\sum_\mu(-1)^\mu\sigma^\mu(p)(U_jV_{j'}-U_{j'}V_j)\sum_J C_{\mu,M'}{}^{1L'J} \times\frac{\langle j'\|T^{1L';J}\|j\rangle}{2(2J+1)^{1/2}}\langle\Psi_0 B_M{}^L|(A_{j'j}{}^{J-M^\dagger}+(-1)^{M'}A_{j'j}{}^{JM'})|\Psi_0\rangle\Big\}. \quad (29)$$

Using Eqs. (14) and (20), one easily finds that

$$\langle k'\Psi_0 B_M{}^L|\mathcal{U}|\Psi_0 k\rangle \cong\sum_{j'j}\frac{4\pi}{2L+1}\int\int d^3r_p dr\, r^2 f_L(r_p,r)\varphi_{k'}^*(r_p)\varphi_k(r_p)(a_{j'j}-b_{j'j})\frac{\langle j'\|Y^L\|j\rangle}{(2L+1)^{1/2}}(U_{j'}V_j+U_jV_{j'})R_{n'l'}(r)R_{nl}(r)Y_{-M}{}^L(\Omega_p) +a\sum_{jj'}\sum_{L'}\frac{4\pi}{2L'+1}\int\int d^3r_p dr\, r^2 f_{L'}(r_p,r)\varphi_{k'}^*(r_p)\varphi_k(r_p)Y^{L'-M}(\Omega_p)R_{n'l'}(r)R_{nl}(r)\sum_\mu(-1)^\mu\sigma^\mu(p)(a_{j'j}-b_{j'j}) \times(\langle j'\|t^{1L';L}\|j\rangle/5^{1/2})C_{\mu,-M}{}^{1L'L}(U_jV_{j'}-U_{j'}V_j). \quad (30)$$

In Eq. (30) terms in $\langle\Psi_0|\eta_{j'j}|\Psi_0\rangle$, which are of order c^2, have been dropped. In the second term, which arises from the Bartlett force, the part with $L'=2$ is expected to be most important for the excitation of the 2+ state. The matrix element (30) is precisely the Sawada approximation. For comparison with the derivation using collective coordinates, let us consider in the plane-wave Born approximation a spinless projectile interacting via a δ-function interaction with the nuclear particles. Using Eqs. (30) and (22), one finds that the cross section for the one-phonon excitation is

$$\Big(\frac{d\sigma}{d\Omega}\Big)^{1\text{ph}}\cong\frac{\mu^2g^2k}{\hbar^4\pi k}\Big[\sum_{j'j}\frac{\langle j'\|Y^2\|j\rangle}{(E_j+E_{j'})^2-(\hbar\omega)^2} \times(U_jV_{j'}+U_{j'}V_j)(a_{j'j}-b_{j'j}) \times\int R_{n'l'}(r)R_{nl}(r)r^2j_2(qr)dr\Big]^2. \quad (31)$$

Since V_j^2 and U_j^2 represent the probability of occupation and nonoccupation of the j level, respectively, the products U_jV_j are nonzero only for the levels which

are partly occupied. Furthermore, the E_j represent elementary excitations above the ground state, with the smallest E_j corresponding to the single-particle j levels nearest the Fermi energy, i.e., the partly filled levels which are in the outer orbits. Therefore, the coefficients of the radial integrals tend to be largest for the j level corresponding to the lowest quasi-particle, e.g., the $f_{5/2}$ level in the Ni^{62} isotope. If this is the case, the cross section is approximately given by

$$\left(\frac{d\sigma}{d\Omega}\right)^{1\mathrm{ph}} \cong \left| \int j_2(qr) r^2 R_f^2(r) \right|^2, \tag{32}$$

with $R_f(r)$ the radial wave function of the particle in the outer orbit. Since the quantity $r^2R_f^2(r)$ tends to be peaked near the surface, one has the qualitative result

$$(d\sigma/d\Omega)^{1\mathrm{ph}} \cong |j_2(qR_0)|^2. \tag{33}$$

This is the result of Lemmer *et al.*[14] and of Hayakawa and Yoshida[16] using collective coordinates with a uniform potential well as the zero-order approximation. In the extreme limits of a uniform density function and a particle in the outer orbit assumed to have a precise radial position, a comparison of Eqs. (32) and (26) shows that the phase rule is obtained and that the angular distribution for the excitation of the 2+ state is approximately out of phase with the elastic scattering.

This same derivation is applicable to other collective modes. For example, the octopole state would be described by a phonon operator B^3, and the result would be Eq. (33) with $j_2(qR_0)$ replaced by $j_3(qR_0)$ which gives the one-phonon phase rule mentioned in Sec. I. For alpha-particle scattering this phase rule should be quite good due to the strong absorption within the nucleus.

C. Excitation of the Two-Phonon States

In order to calculate in the first Born approximation the inelastic excitation of the two-phonon states, one needs the matrix element $\langle k'\Psi_0[BB]_M{}^I|\mathcal{U}|\Psi_0 k\rangle$, with I taking the values 0, 2, 4, corresponding to the three second vibrational states. The results are given only for the Wigner force, as the corresponding results with the spin-dependent force can be immediately obtained from these. Writing out this matrix element, one has

$$\begin{aligned}
&\langle k'\Psi_0[B^LB^L]_M{}^I|\mathcal{U}|\Psi_0 k\rangle \\
&= \sum_{M'}\sum_{jj'} \int d^3r_p dr\, r^2 \frac{4\pi}{2L+1} \varphi_k^*(\mathbf{r}_p)\varphi_k(\mathbf{r}_p) Y_M{}^L(\Omega_p) f_L(r_p,r) R_{n'l'}(r) R_{nl}(r) \Big\langle \Psi_0 \Big| \{ \sum_{1234} a_{12}a_{34}[A_{12}A_{34}]_M{}^I \\
&\quad + b_{12}b_{34}[A_{12}{}^\dagger A_{34}{}^\dagger]_M{}^I + a_{12}b_{34}[A_{12}A_{34}{}^\dagger]_M{}^I + b_{12}a_{34}[A_{12}{}^\dagger A_{34}]_M{}^I\} \\
&\quad \times \Big\{ \delta_{L0} V_j{}^2 \delta_{j'j}(2j+1)^{1/2} + \frac{\langle j'\|Y^L\|j\rangle}{(2L+1)^{1/2}} \Big[\frac{(U_jV_{j'}+U_{j'}V_j)}{2} (A_{j'j}{}^{L-M'\dagger} + (-1)^M A_{j'j}{}^{LM'}) \\
&\qquad + (U_jU_{j'} - V_jV_{j'})\eta_{j'j}{}^{LM} \Big] \Big\} \Big| \Psi_0 \Big\rangle.
\end{aligned} \tag{34}$$

In the Sawada approximation ($B\Psi_0=0, \eta\Psi_0=0$) the only terms in the approximation which occur are those in $A_{j'j}{}^\dagger$ and $A_{j'j}$. Since the operator $[BB]^I$ can change the number of quasi-particles by zero or four, the matrix elements which appear in the Sawada approximation vanish in all orders, i.e.,

$$\langle k\Psi_0[BB]^I|A^\dagger|\Psi_0\rangle = \langle k\Psi_0[BB]^I|A|\Psi_0\rangle = 0. \tag{35}$$

This can be expected from the fact that the dilute fermion approximation for a separable force, $T^L\cdot T^L$, is equivalent in the adiabatic limit to the deformed field approximation, i.e., $T^L\cdot T^L \approx T^L\cdot\langle T^L\rangle$ for the collective states arising from the interaction. In other words, in this approximation one has linearized the interaction in terms of the phonon operators, so that the excitation of the one-phonon state is allowed but the second vibrational state cannot be reached from the ground state. Thus, in the language of the particles properties the expansion in higher powers of the collective coordinates is explained as the interactions between the quasi-particles. Moreover, one can see that it is possible to find certain qualitative features, such as phase rules for the two-phonon excitation, in terms of the single-particle properties.

If one includes the terms which arise from scattering of quasi-particles, not only do new terms appear in the interaction, but also the relation $B\Psi_0=0$ is no longer satisfied, even though the matrix element $\langle\Psi_0 B\Psi_0\rangle$ vanishes. Using the commutation rules (20), one finds

$$B\Psi_0 = \sum a_{\nu\nu'}c_{1234}A_{\nu\nu'}{}^\dagger\Psi_0{}^0 + O(a_{\nu\nu'}c^3) \tag{36}$$

or

$$\langle\Psi_0[BB]^I\Psi_0\rangle = O(c^2)\delta_{I0}. \tag{37}$$

Therefore, for the two-phonon $I=0$ state there is a contribution to the inelastic scattering from the terms independent of quasi-particle operators which is of the same order as the contribution from the $\eta_{j'j}$ terms. This could have an important effect on both the magnitude and shape of the cross section for the excitation of the two-phonon state of angular momentum zero compared to angular momentum two and four, as is seen below.

Comparing Eq. (34) with Eq. (21), one finds that the cross section which results from the part of the interaction free of quasi-particle operators, $(d\sigma/d\Omega)_{\text{no q.p.}}$, is (with K a constant)

$$(d\sigma/d\Omega)_{\text{no q.p.}}{}^{2\text{ ph},\,I=0}=K(d\sigma/d\Omega)^{\text{el}}; \tag{38}$$

i.e., for this part of the interaction the cross section for the excitation of the $I=0$ two-phonon state is in phase with the elastic cross section. Although this result seems to be in agreement with the phase rule of Lemmer, de-Shalit, and Wall,[14] its origin and physical content seem to be quite different, as is obvious from the fact that it does not pertain to the excitation of the two-phonon states of angular momentum 2 and 4.

From the part of the interaction involving the operators η, one finds

$$\langle k'\Psi_0[BB]_M{}^I|\mathcal{V}|\Psi_0 k\rangle\cong\sum_{jj'}k_{j'j}(U_jU_{j'}-V_jV_{j'})\int d^3r_p\,dr\,r^2f_I(r_p,r)R_{n'l'}(r)R_{nl}(r)\varphi_{k'}{}^*(r_p)\varphi_k(r_p)Y^I(\Omega_p). \tag{39}$$

where

$$k_{j'j}=40\left(\frac{5}{4\pi}\right)^{1/2}\frac{N_\omega{}^2}{(2I+1)(2I+1)^{1/2}}(2j'+1)^{1/2}C_{0\frac{1}{2}\frac{1}{2}}{}^{2j'j}(2j+1)^{1/2}(2j'+1)^{1/2}$$
$$\times\sum_{j_1}W(Ij'2j_1;j2)C_{0\frac{1}{2}\frac{1}{2}}{}^{2jj_1}C_{0\frac{1}{2}\frac{1}{2}}{}^{2j'j_1}(U_jV_{j_1}+U_{j_1}V_j)(U_{j'}V_{j_1}+U_{j_1}V_{j'})$$
$$\times\langle j|r^2|j_1\rangle\langle j'|r^2|j_1\rangle/(E_j+E_{j_1}+\hbar\omega)(E_{j'}+E_{j_1}-\hbar\omega). \tag{40}$$

It should be pointed out that in deriving the expression (40) one just uses the relation (13) and the commutation rules (20) and (21); the precise form of the wave function and the coefficients c_{1234} do not enter into the calculation.

In order to study the qualitative features of this result, let us evaluate the cross section in the plane-wave Born approximation. In this approximation

$$(d\sigma/d\Omega)^{2\text{ph}}=K'\left[\sum k_{j'j}(U_jU_{j'}-V_jV_{j'})\right.$$
$$\left.\times\int R_{n'l'}(r)R_{nl}(r)r^2j_I(qr)dr\right]^2. \tag{41}$$

As one can see from Eqs. (31) and (41) the most important qualitative difference between the cross section for the one-phonon excitation compared to that for two phonons is the difference in the occupation number factors. For the one-phonon excitation, the factor $(U_jV_{j'}+V_jU_{j'})$ ensures that the states at the Fermi level contribute most to the process, while the factor $(U_jU_{j'}-V_jV_{j'})$ does not allow any contribution from these states for the two-phonon process. If the radial wave functions of the states which do contribute to the two-phonon process differ markedly from the states at the Fermi level, one can expect cross sections for the one-phonon and two-phonon processes to be out of phase.

For example, let us consider the Ni^{62} isotope, in which the neutrons are mainly filling the $f_{5/2}$ level. Presumably, the one-phonon cross section, to a large extent, is given by

$$(d\sigma/d\Omega)^{1\text{ph}}\cong K_1\left[\int R_{0f}{}^2(r)j_2(qr)r^2dr\right]^2, \tag{42}$$

while the cross section for the excitation of two phonons might be approximately

$$(d\sigma/d\Omega)^{2\text{ph}}\cong K_2\left[\int R_{1p}{}^2(r)j_I(qr)r^2dr\right]^2, \tag{43}$$

since the $p_{3/2}$ level is the other level which is being filled in this region. Since the one-node p radial wave function approximates the shape of the derivative of the nodeless f function, one finds (with these very rough assumptions)

$$\left(\frac{d\sigma}{d\Omega}\right)^{2\text{ph}}\cong K_2\left[\int\left(\frac{dR_{0f}}{dr}\right)^2j_I(qr)r^2dr\right]^2,$$

which, by comparison with Eq. (32), predicts that the one-phonon and two-phonon cross sections have the phase rule of reference 14.

However, even under these extreme and unrealistic conditions this phase rule cannot be expected to be universally valid, and the shape of the differential cross section will generally depend upon the levels which are filled. For α-particle scattering, since these projectiles are so strongly absorbed within the nucleus, the tails of the particle wave functions contribute most to the scattering. Therefore, one might expect in this case that the positions of the maxima and minima of the cross section for excitation to a particular state depend largely upon the nuclear radius and the angular momentum transfer, although the magnitude of the cross section might be quite sensitive to the details of the structure of the wave function. On the other hand, for excitation of collective states by particles which are not strongly absorbed, Eqs. (30) and (39) show that both the magnitude and shape of the theoretical cross sections will depend upon the construction of the collective states in terms of the single-particle states.

D. Excitation of the Two-Phonon States in the Second Born Approximation

In addition to the one-step process described in the previous section, the two-phonon states can be excited by a two-step process through the one-phonon state. The second Born approximation matrix element is

$$\langle k'\Psi_0[B^L B^L]_M{}^I|\mathcal{U}^{(2)}|\Psi_0 k\rangle=\sum_{k''M'}\frac{\langle k'\Psi_0[B^L B^L]_M{}^I|\mathcal{U}|B_{M'}{}^{L\dagger}\Psi_0 k''\rangle\langle k''\Psi_0 B_{M'}{}^L|\mathcal{U}|\Psi_0 k\rangle}{(\hbar^2k''^2/2\mu)+\hbar\omega-(\hbar^2k^2/2\mu)}. \quad (44)$$

The matrix element for excitation of the one-phonon state is given in Eq. (30). Using the arguments given at the beginning of Sec. III B, one can see that only the $A_{j'j}{}^L$ and $A_{j'j}{}^{L\dagger}$ parts of the interaction can lead to the excitation of the two-phonon states from the one-phonon state. Using the commutation rule (8) and condition (13), one easily finds that in the Sawada approximation the matrix element is (for a spinlessprojectile)

$$\langle k'\Psi_0[B^L B^L]_M{}^I|\mathcal{U}|B_{M'}{}^I\Psi_0 k'\rangle$$
$$=\frac{8\pi}{2L+1}\sum_{i'j}\int d^3r_p dr\, r^2\varphi_{k'}{}^*(r_p)\varphi_{k''}(r_p)f_L(r_p,r)R_{j'}(r)R_j(r)(-1)^{M-M'}Y_{M-M'}L\frac{(U_{j'}V_j+U_jV_{j'})}{(2L+1)^{1/2}}\langle j'\|Y^L\|j\rangle$$
$$\times(a_{j'j}-b_{j'j})C_{M',M-M'}{}^{LLI}. \quad (45)$$

For a δ-function interaction the matrix element (44) can be written approximately as

$$\langle k'\Psi_0[B^L B^L]_M{}^I|\mathcal{U}^{(2)}|\Psi_0 k\rangle$$
$$\cong\frac{g^2}{V}\frac{4\mu}{(2L+1)^{1/2}}ik\sum_{1234}\sum_{ll_\alpha l_\beta}[4\pi(2l_\beta+1)]^{1/2}$$
$$\times(2l_\alpha+1)(2l+1)^{1/2}(i)^{l_\alpha-l_\beta}(-1)^L C_{000}{}^{l_\alpha l L}C_{000}{}^{l_\beta L l}$$
$$\times W(Ll_\beta Ll_\alpha;lI)C_{0MM}{}^{l_\alpha l_\beta I}Y_M{}^{l_\beta *}(k',k)$$
$$\times\int drdr'\, r^2r'^2R_1(r)R_2(r)R_3(r')R_4(r')$$
$$\times j_{l_\alpha}(kr')j_{l_\beta}(k'r)j_l(kr_<)h_l{}^{(1)}(kr_>), \quad (46)$$

in which $r_>$ ($r_<$) stand for the larger of (smaller of) r and r' in the integrations. In the derivation of (46) (which is similar to the analogous derivation in the appendix of reference 14), the excitation energy $\hbar\omega$ has been neglected in the energy denominator of Eq. (44).

A comparison between expressions (46) and (39) reveals that the second Born approximation matrix element is essentially of the same magnitude as the first Born approximation for the excitation of the two-phonon state. This is not surprising, since the two-step process proceeds via the Sawada terms and thus in the second order is of the same magnitude as the one-step process to which the Sawada terms cannot contribute. Thus one has a close analogy between the nature of the Sawada approximation with the corrections to this approximation and that of the linearized collective interaction with the addition of nonlinear terms.

Although the evaluation of Eq. (46) is straightforward it is quite complicated. For this reason it has not been included in the rough numerical calculations described in the following part.

IV. INELASTIC SCATTERING FROM Ni^{58}

A. Alpha Particles

In this section the cross sections for the excitation of the one- and two-phonon states are calculated in the first Born approximation, using plane waves for the alpha particles and a δ-function interaction

$$v_{pi}=g\delta(\mathbf{r}_p-\mathbf{r}_i)=g_0(\hbar^3/m_\pi{}^2c)\delta(\mathbf{r}_p-\mathbf{r}_i) \quad (48)$$

between the alpha particle and the ith nuclear particle, where g is the interaction strength. For convenience, the dimensionless coupling constant g_0 is introduced in terms of the π-meson mass. Only the f- and p-state neutrons are explicitly considered and the wave functions are taken as simple harmonic oscillator functions, neglecting all energy splittings. This calculation is an attempt to further study the method described above and is not intended to be quantitatively accurate. Also, it should be noted that Ni^{58} is not a particularly favorable isotope. It contains only two neutrons outside of the neutron and proton 28 shells. These shells are not very strongly closed, so that both protons and neutrons are involved, and they occupy approximately the same shell-model levels. Since the short-range neutron-proton force is neglected in finding the nuclear wave functions, this could be quite important.

The parameters used to determine the states are the same as those used in reference 24, with a pairing force strength of 0.331 MeV and the single-particle states chosen as $p_{3/2}(0)$, $f_{5/2}(0.78)$, and $p_{1/2}(1.56)$. The very small contribution of the $g_{9/2}$ level to all of the processes discussed here is neglected. In each case the calculation is redone including the $f_{7/2}$ level of the 20–28 shell. This is not intended as an improvement (in fact, to be consistent one would have to at least include the $f_{7/2}$ protons also), but as a test of the sensitivity of the method to the choice of parameters.

TABLE I. Coefficients $d_{j'j}$ for 1.45-MeV one-phonon cross section.

j' \ j	$f_{7/2}$	$p_{3/2}$	$f_{5/2}$	$p_{1/2}$
$f_{7/2}$	0.146	0.553	0.0649	0
$p_{3/2}$	0.553	0.345	0.0683	0.123
$f_{5/2}$	0.0649	0.0683	0.172	0.0774
$p_{1/2}$	0	0.123	0.0774	0

From Eq. (31), the cross section for the excitation of one phonon is written as

$$(d\sigma/d\Omega)^{1\text{ph}}=K_1\sum d_{j'j}\mathcal{R}_{j'j}^{(2)}(qR_0), \tag{49}$$

in which K_1 is a constant, q is the momentum transfer, and the $d_{j'j}$ are given in Table I. The radial integrals $\mathcal{R}_{j'j}^{I}(qR_0)$, defined as

$$\mathcal{R}_{j'j}^{(I)}(qR_0)=\int_{R_0}^{\infty}dr\,r^2j_I(qr)R_{j'}(r)R_j(r), \tag{50}$$

are cut off at R_0, a radial parameter, to take into account the strong absorption. Since the radial wave functions in (49) are taken from a degenerate oscillator, there are only three radial integrals. In terms of them, the cross sections in the two cases are

$$(d\sigma/d\Omega)^{1\text{ph}}\ (\text{no } f_{7/2})=g_0^2\times4.14\times10^{-24}(0.591\mathcal{R}_{pp}^{(2)}+0.172\mathcal{R}_{ff}^{(2)}+0.291\mathcal{R}_{pf}^{(2)})^2\ \text{cm}^2, \tag{51a}$$

$$(d\sigma/d\Omega)^{1\text{ph}}\ (\text{with } f_{7/2})=g_0^2\times2.10\times10^{-24}(0.591\mathcal{R}_{pp}^{(2)}+0.172\mathcal{R}_{ff}^{(2)}+1.397\mathcal{R}_{pf}^{(2)})^2\ \text{cm}^2. \tag{51b}$$

The results are given in Fig. 1 for a cutoff radius of $R_0=1.6\times10^{-13}$ cm, which is reasonable. One can see that for a suitable choice of interaction strength g_0 either cross section is suitable, although there is a factor of five different in their magnitude for one choice of the constant. Of course, with a sharp cutoff and degenerate wave functions one almost eliminates the differences between the states insofar as the periodicity of the oscillations is concerned, which would not be true in an accurate calculation. With systematic studies, one probably could learn a good deal from the one-phonon cross sections.

Using the assumptions described above, the cross section for the excitation of the two-phonon state of spin I in the first Born approximation is

$$(d\sigma/d\Omega)_I^{2\text{ph}}=K_2^{(I)}\sum_{j'j}g_{j'j}^{I}\mathcal{R}_{j'j}^{(I)}(qR_0) \tag{52}$$

in which $K_2^{(I)}$ is a constant, and the $g_{j'j}^{I}$ are given in Tables II and III for the calculation without and with the $f_{7/2}$ level, respectively. For the case of Ni^{58} one has for the scattering from the 4+ and 2+ two-phonon levels without and with the $f_{7/2}$ level, respectively.

TABLE II. Coefficients $g_{j'j}^{(I)}$ for $I=2$ and 4 two-phonon cross sections with $f_{7/2}$ state.

j' \ j	$f_{7/2}$	$p_{3/2}$	$f_{5/2}$	$p_{1/2}$
		$g_{j'j}^{(4)}\times10^2$		
$f_{7/2}$	0.909	0.553	0.0208	0
$p_{3/2}$	0.782	0	−0.397	0
$f_{5/2}$	0.0226	−0.331	−0.200	0
$p_{1/2}$	0	0	0	0
		$g_{j'j}^{(2)}\times10^2$		
$f_{7/2}$	1.674	0.558	0.0070	0
$p_{3/2}$	0.724	−0.433	−0.0718	−1.07
$f_{5/2}$	0.0079	−0.0898	0.0769	−0.655
$p_{1/2}$	−0.698	−0.562	0	0

TABLE III. Coefficients $g_{j'j}^{(I)}$ for $I=2$ and 4 two-phonon cross sections without $f_{7/2}$ state.

j' \ j	$p_{3/2}$	$f_{5/2}$	$p_{1/2}$
	$g_{j'j}^{(4)}\times10^2$		
$p_{3/2}$	0	−0.387	0
$f_{5/2}$	−0.322	−0.229	0
$p_{1/2}$	0	0	0
	$g_{j'j}^{(2)}\times10^2$		
$p_{3/2}$	−0.147	−0.119	−1.068
$f_{5/2}$	0.0769	−0.0682	−0.655
$p_{1/2}$	−0.698	−0.562	0

$$(d\sigma/d\Omega)_{4+}^{2\text{ph}}\ (\text{no } f_{7/2})=g_0^2\times0.537\times10^{-21}\ \text{cm}^2(-0.00229\mathcal{R}_{ff}^{(4)}-0.00710\mathcal{R}_{pf}^{(4)})^2, \tag{53a}$$

$$(d\sigma/d\Omega)_{4+}^{2\text{ph}}\ (\text{with } f_{7/2})=g_0^2\times1.375\times10^{-22}\ \text{cm}^2(0.00753\mathcal{R}_{ff}^{(4)}+0.00606\mathcal{R}_{pf}^{(4)})^2, \tag{53b}$$

$$(d\sigma/d\Omega)_{2+}^{2\text{ph}}\ (\text{no } f_{7/2})=g_0^2\times0.533\times10^{-21}\ \text{cm}^2(-0.0191\mathcal{R}_{pp}^{(2)}-0.000682\mathcal{R}_{ff}^{(2)}-0.0102\mathcal{R}_{pf}^{(2)})^2, \tag{53c}$$

$$(d\sigma/d\Omega)_{2+}^{2\text{ph}}\ (\text{with } f_{7/2})=g_0^2\times1.36\times10^{-22}\ \text{cm}^2(-0.0220\mathcal{R}_{pp}^{(2)}+0.0177\mathcal{R}_{ff}^{(2)}-0.000955\mathcal{R}_{pf}^{(2)})^2. \tag{53d}$$

The cross sections are plotted in Figs. 2 and 3. One can see that the positions of the maxima and minima are in approximate agreement with experiment for the same value of the radial parameter $R_0=1.6\times10^{-13}$ cm as was used in the one-phonon calculation. The fact that this method predicts the correct phase relationships, which in this case are that the one-phonon and two-phonon 2+ states are in phase with each other and out of phase with the two-phonon 4+ state, indicates that the main factor in determining these rela-

tionships is the angular momentum transfer and not the number of phonons. This result depends upon the strong absorption with sharp cutoff, the use of degenerate harmonic oscillator states, and the neglect of the second Born term, but seems to be closer to the actual situation than the phase rule of reference 14. As has been pointed out by Glendenning[7] and others, whenever the reaction is concentrated at the surface the angular distribution is insensitive to the mechanism producing it. However, because of the possibility of coherent interference of the second Born terms, to which single-particle states contribute in a different manner than the first Born terms, the phase rule for the two-phonon states can provide a deeper dependence upon the reaction mechanism even for such strongly absorbing particles as alphas.

It is interesting to see that the one- and two-phonon 2+ states can be fit with approximately the same value of the coupling constant $g_0=27.4$. The 4+ state needs a force constant 14 times this value, but because of the neglect of the two-step process (as well as the other approximations) one should not consider this to be significant. Taking the value of $27.4(\hbar/M_\pi c)^3Mc^2$ for the magnitude of the δ-function interaction between α particles and nuclear particles, one can make an estimate of the optical model potential which would result for the elastic scattering of α particles from the nuclei. Assuming that the alpha-particle wave function

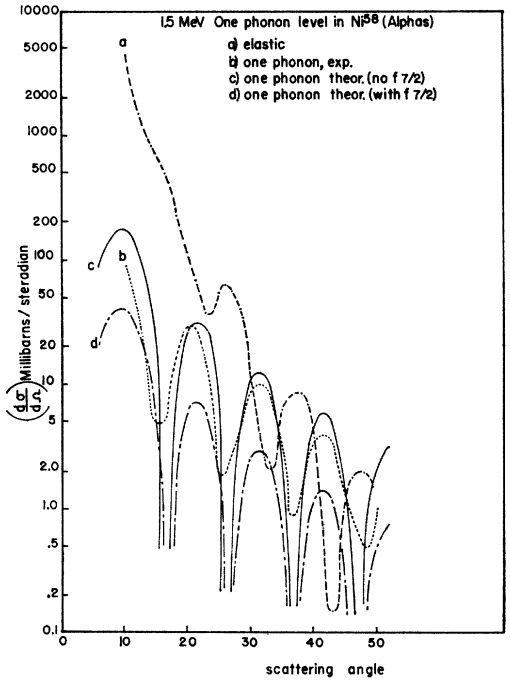

FIG. 1. The differential cross section for 45-MeV alpha-particle excitation of the 1.45-MeV one-phonon state in Ni^{58}. The experimental curves for Figs. 1–3 are taken from the results of Beurtey *et al.* (reference 9) and are consistent with those of Broek *et al.* (reference 9). See the text for a discussion of the two theoretical curves. The coupling constant used for the theoretical curves is $g_0=27.4$.

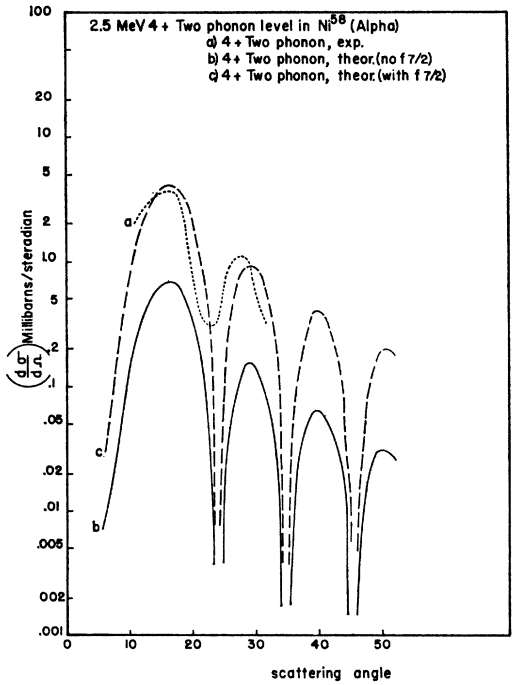

FIG. 2. The differential cross section for 45-MeV alpha-particle excitation of the 2.45-MeV 4+ two-phonon state in Ni^{58}. The magnitude of the interaction constant is $g_0=14\times41.1$.

and the optical potential are both constant within the nuclear volume V_N, one has

$$g_0=V_0\frac{V_N/A}{(\hbar/m_\pi c)^3m_\pi c^2}\cong1.44\times10^{-2}V_0, \qquad (54)$$

where V_0 is the depth of the uniform potential in MeV. The nuclear radius used to obtain Eq. (54) is $R=1.6\times10^{-13}\times1/3$ cm, the radius which results from the present work. This gives a value of $V_0=1900$ MeV.

The elaborate optical-model calculations by Igo[28] for α particles on nuclei give the real part of the optical potential, $1100\exp[-(r-4.53)/0.574]$ MeV for Ni, so that a value of 1100 MeV is obtained at the nuclear surface, which is consistent with the results of this work. On the other hand, Igo's results are quite insensitive to the real part of the optical potential, and one can see by comparison to other calculations[29] that the real part of the potential can be altered by a factor of 10 or even 100 in the optical-model calculations with rather minor changes in the imaginary part. In the present work, since we have used a sharp cutoff with wave functions having a small tail outside the nucleus and have neglected distortion, we find a coupling constant which is probably far too large. It would be extremely interesting to learn of the value of this interaction strength from an accurate calculation and to see if one force constant can systematically fit the data for various nuclei.

[28] G. Igo, Phys. Rev. **115**, 1665 (1959).
[29] G. Igo and R. M. Thaler, Phys. Rev. **106**, 126 (1957).

B. Electrons

The cross sections for excitation of the one- and two-phonon states in Ni^{58} are calculated in the first Born approximation using the simple nonrelativistic Coulomb interaction. Because of the well-known relationship

$$\int e^{iq\cdot r_p}\frac{1}{|\mathbf{r}_p-\mathbf{r}|}e^{-r_p\gamma}=\frac{4\pi}{q^2}e^{iq\cdot r}, \tag{55}$$

the plane-wave Born approximation scattering by the Coulomb interaction follows immediately from the calculation with the δ-function interaction. The relation between the matrix elements for the scattering of an alpha particle from momentum $\mathbf{k}$ to $\mathbf{k}'$ is

$$\langle k'|v_{\text{Coulomb}}|k\rangle = (4\pi e e_{\text{eff}}/q^2)(1/|k-k'|^2)\langle k'|v_{\delta\text{-}fcn}|k\rangle, \tag{56}$$

where e_{eff} is the effective charge of the nuclear particle.

Relationships analogous to Eqs. (49) and (52) are obtained, but with a factor of q^{-4} and in terms of radial integrals without cutoff. The result is

$$(d\sigma/d\Omega)^{1\text{ph}}\ (\text{no } f_{7/2}) = 3.56\times10^{-28}\ \text{cm}^2[e^{-q^2/4\nu}(0.156-0.0299q^2/\nu+0.00202q^4/\nu^2)]^2, \tag{57a}$$

$$(d\sigma/d\Omega)^{1\text{ph}}\ (\text{with } f_{7/2}) = 1.80\times10^{-28}\ \text{cm}^2[e^{-q^2/4\nu}(-0.0370+0.0223q^2/\nu-0.000116q^4/\nu^2)]^2, \tag{57b}$$

$$(d\sigma/d\Omega)_{4+}^{2\text{ph}}\ (\text{no } f_{7/2}) = 4.61\times10^{-26}\ \text{cm}^2[e^{-q^2/4\nu}(0.00108-0.000313q^2/\nu+0.0000131q^4/\nu^2)]^2, \tag{57c}$$

$$(d\sigma/d\Omega)_{4+}^{2\text{ph}}\ (\text{with } f_{7/2}) = 1.18\times10^{-26}\ \text{cm}^2[e^{-q^2/4\nu}(0.000746+0.0000284q^2/\nu-0.0000045q^4/\nu^2)]^2, \tag{57d}$$

$$(d\sigma/d\Omega)_{2+}^{2\text{ph}}\ (\text{no } f_{7/2}) = 4.58\times10^{-26}\ \text{cm}^2[e^{-q^2/4\nu}(-0.00345+0.000725q^2/\nu-0.0000585q^4/\nu^2)]^2, \tag{57e}$$

$$(d\sigma/d\Omega)_{2+}^{2\text{ph}}\ (\text{with } f_{7/2}) = 1.17\times10^{-26}\ \text{cm}^2[e^{-q^2/4\nu}(0.00106+0.000652q^2/\nu-0.0000684q^4/\nu^2)]^2. \tag{57f}$$

Using an effective charge of $1.4e$, which is consistent with the $B(E2)$,[24] the scattering cross sections for the 2+ one-phonon and two-phonon states are in reasonable agreement with experiment for the calculation without the $f_{7/2}$ level, while the 4+ two-phonon state is too small by two orders of magnitude (Figs. 4, 5, and 6). These results are quite similar to those for the α-particle scattering and suggest that the two-step process is especially important for the 4+ state.

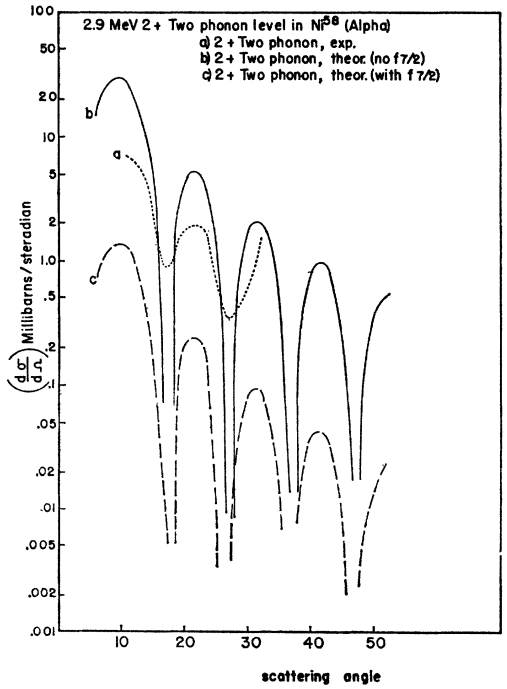

FIG. 3. The differential cross section for 45-MeV alpha-particle excitation of the 2.9-MeV 2+ two-phonon state in Ni^{58}. The magnitude of the coupling constant is $g_0=41.1$.

The most striking result is the strong dependence of both the shape and the magnitude of the cross sections upon choice of single-particle levels, which is borne out by the tremendous differences in the results when one includes the $f_{7/2}$ state. Since the cross sections result from a coherent sum of scatterings of the particles in the various levels, even small changes in the wave functions can shift the positions of the maxima and minima and generally produce important alterations in the results. For this reason it is clear that the information from the excitation of collective states by electrons might prove to be an excellent test of the single-particle structure of the collective wave functions. For quantitative calculations one must consider the magnetic moment scattering, purely relativistic effects, and the distortion of the electron wave functions, especially for large-angle scattering.

V. CONCLUSIONS

In conclusion, there are many similarities between this method of deriving the excitation of collective states by direct interaction between the scattered particle and the nuclear particles and that of the collective models. The dilute quasi-particle approximation leads to the first Born excitation of the one-phonon states and to excitation of the two-phonon states by a two-step process, in analogy to the collective Hamiltonian linear in the phonon operator. The correction terms, which occur both in the commutation rules and the operators and correspond to the interaction of quasi-particles with each other, give rise to a first Born excitation of the two-phonon states, in analogy to the nonlinear terms in the collective Hamiltonian.

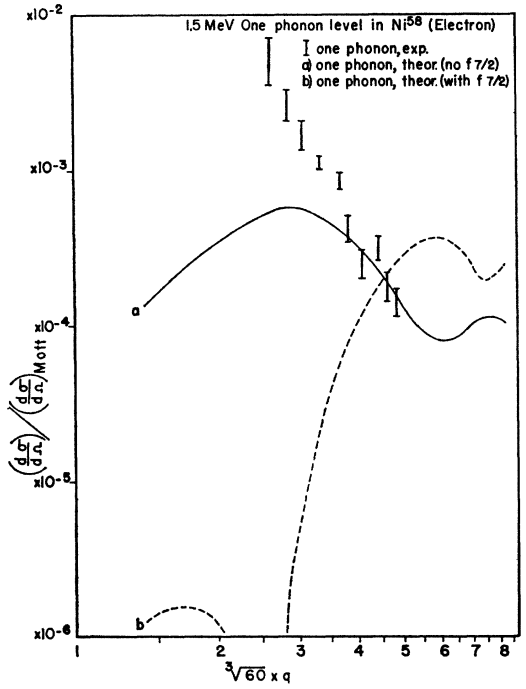

FIG. 4. The ratio of the differential cross section to Mott scattering for 183-MeV electron excitation of the 1.45-MeV one-phonon state in Ni^{58}. The experimental results of Figs. 4–6 are taken from Crannell *et al.* (reference 18). See the text for a discussion of the two theoretical curves. The effective charge of the neutron is taken as $e_{eff}=1.4e$.

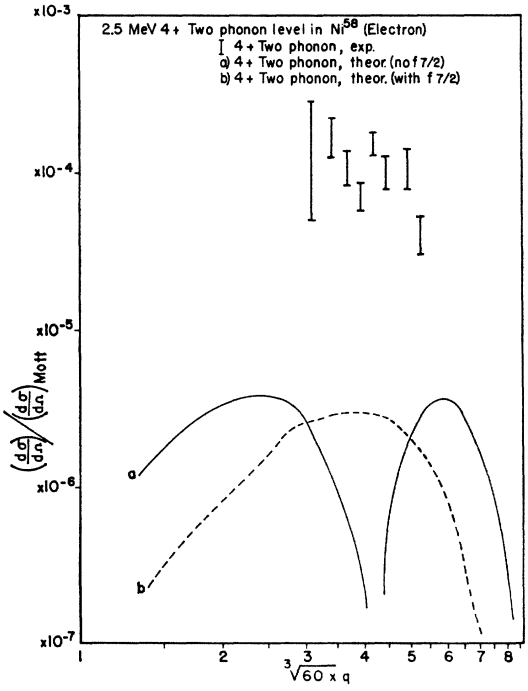

FIG. 5. The ratio of the differential cross section to Mott scattering for 183-MeV electron excitation of the 2.45-MeV 4+ two-phonon state in Ni^{58}. The effective charge of the neutron is taken as $e_{eff}=2.0e$.

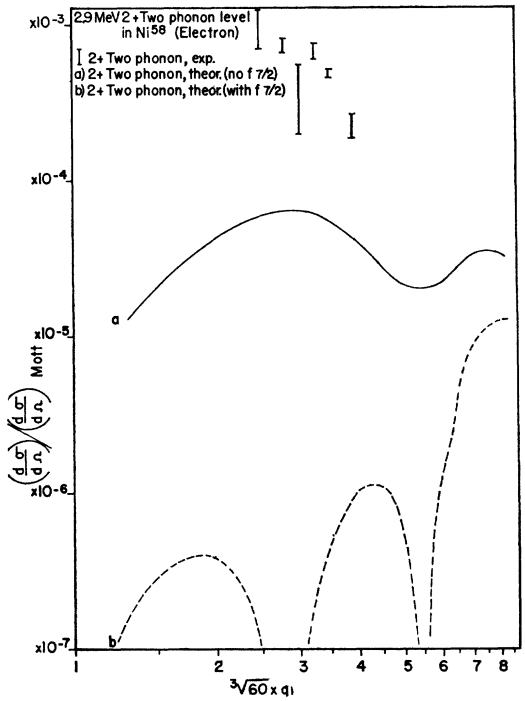

FIG. 6. The ratio of the differential cross section to Mott scattering for 183-MeV electron excitation of the 2.9-MeV 2+ two-phonon state in Ni^{58}. The effective charge of the neutron is taken as $e_{eff}=2.0e$.

In the first Born approximation, the main qualitative feature is that single-particle states which tend to be most important for the one-phonon process often contribute little to the two-phonon excitation. Although this will not lead to a specific phase rule between the one-phonon and two-phonon differential cross sections in general and, in fact, does not seem to be very significant in determining the relative phases for alpha scattering because of the strong absorption, in certain limits it results in a phase rule approximating that of Lemmer, de-Shalit, and Wall[14] for a surface interaction of alpha particles.

Because of the interference between the one-step and two-step process for the two-phonon excitation, one cannot expect a general phase rule to hold for the two-phonon excitation cross sections in comparison to the one-phonon ones. Since the two-step process takes place largely via the Sawada-type mechanism, which tends to emphasize different single-particle states than the quasi-particle scattering of the one-step process, the data on the phase as well as the magnitude of the two-phonon cross sections might involve many of the details of nuclear structure even for strongly absorbed particles.

The theoretical first Born differential cross sections for 44-MeV alpha-particle scattering in Ni^{58} agree with the experimental results as well as can be expected. The one-phonon and two-phonon 2+ states can be explained by the same coupling constant, while the 4+ two-phonon state is too small by two orders of magnitude,

presumably due to the neglect of the second Born terms. In the rough approximation of a constant alpha-particle wave function inside the nucleus, the interaction needed for the one-phonon process leads to an optical potential in agreement with the phenomenological calculation for elastic α scattering, but both results can be changed by one or two orders of magnitude. An important result of accurate and systematic calculations would be this interaction parameter.

The results for electron scattering indicate that one can obtain reasonable agreement between theory and experiment for the 2+ states with effective charges known from the $B(E2)$. The scattering from the 4+ two-phonon state in the first Born approximation is too small, just as with the alpha scattering. For electrons, the cross sections are extremely sensitive to the particle makeup of the states, which suggests the importance of further systematic studies—particularly for heavier isotopes where the method used in this work should be more nearly accurate.

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