Application of ξ-Limiting Process to Intermediate Bosons*

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(Received May 29, 1962)

The ξ-limiting process discussed in the preceding paper is used to study the electromagnetic properties of the intermediate boson $W^\pm$. Assuming that the limit $\xi \rightarrow 0$ exists, it is found that, by a rearrangement of the perturbation series, the radiative correction to the quadrupole moment of $W^\pm$ can be calculated and is proportional to $\alpha \ln\alpha$. Similar radiative corrections to the leptonic decay modes of $W^\pm$, μ decay, and the β decay of a "bare" nucleon are also discussed.

1. INTRODUCTION

In the preceding paper[1] a theory of charged vector meson, called ξ-limiting process, is discussed. For $\xi>0$, the theory is covariant and renormalizable; but the S matrix is not unitary. However, assuming that the limit $\xi\rightarrow 0$ exists, the limiting S matrix is shown to be unitary; therefore, it can be applied to physical problems.

In this paper, we apply this ξ-limiting process to the (as yet, hypothetical) intermediate bosons $W^\pm$ of the weak interactions.[2] Because of the absence of any direct strong interactions the $W^\pm$, if it exists, could serve as a test case for such a calculation.

For a nonzero value of ξ, the electromagnetic interactions of $W^\pm$ are renormalizable. Unlike a spin $\frac{1}{2}$ particle, both its charge and its magnetic moment require a renormalization. In terms of these renormalized quantities the quadrupole moment of $W^\pm$ can be calculated. The resulting power series expansion in the fine structure constant α for the quadrupole moment is finite if $\xi>0$, but becomes divergent as $\xi\rightarrow 0$. However, assuming that the limit $\xi\rightarrow 0$ exists for the entire sum of the power series, the summation over the most divergent terms (as $\xi\rightarrow 0$) in the series leads to a result for the quadrupole moment of $W^\pm$:

$$Q=-m_W^{-2}e[\kappa-(4\pi)^{-1}(\kappa+3)(\kappa-1)^2\alpha \times\ln(\alpha\kappa^2)+O(\alpha)], \quad (1)$$

where $(1+\kappa)$ = renormalized gyromagnetic ratio of W and m_W is the mass of W. The existence of terms like $[\alpha\ln(\alpha\kappa)^2]$ "explains" why the original power series expansion in α should be singular at $\xi=0$.

Similar considerations can be extended to the weak interactions of $W^\pm$. Radiative corrections to leptonic decay modes of $W^\pm$, μ-decay and the decay of a "bare" nucleon are also discussed.

It must be emphasized that, throughout this paper, the existence of a complete theory of $W^\pm$ in the limit $\xi\rightarrow 0$ is a *pure assumption.*

* Research supported in part by the Alfred P. Sloan Foundation.
[1] T. D. Lee and C. N. Yang, preceding paper [Phys. Rev. **128** 885 (1962)].
[2] See, for example, T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960). Throughout this paper, all unexplained notations are the same as those in references 1 and 2.

2. REVIEW OF THE FEYNMAN RULES

We review the Feynman rules for the electromagnetic interactions[1] of a charged vector meson in the ξ-limiting formalism (with a negative metric).

(i) It is convenient to represent the propagator of the $W^\pm$ in the interaction representation by a (4×4) matrix

$$S(p)=(-i)(p^2+m_W^2)^{-1} \times[1+(\xi p^2+m_W^2)^{-1}(1-\xi)p\tilde{p}], \quad (2)$$

where m_W is the mass of $W^\pm$ plus a negative infinitesimal quantity. p is a (4×1) column matrix whose matrix elements are $p_1, \cdots, p_4$ and $p^2=\tilde{p}p$. Throughout this paper, $\sim$ indicates the transpose of a matrix. The rows and the columns of the matrix determine, respectively, the final and the initial polarization states of W.

(ii) The three-point vertex for a $W^\pm$ with initial (incoming) momentum p and final (outgoing) momentum p' interacting with a photon is given by the (4×4) matrix

$$V_\lambda(p',p)=ie[(p+p')_\lambda+(\xi+\kappa)(\epsilon_\lambda\tilde{p}+p'\tilde{\epsilon}_\lambda) -(1+\kappa)(\epsilon_\lambda\tilde{p}'+p\tilde{\epsilon}_\lambda)], \quad (3)$$

where λ denotes the polarization of the photon, ϵ_λ is a (4×1) column matrix whose λth matrix element is 1 and other elements zero; e.g.,

$$\epsilon_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \epsilon_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \text{etc.} \quad (4)$$

(iii) The four-point vertex is given by the (4×4) matrix

$$U_{\lambda\mu}(p',p;k)=-ie^2[2\delta_{\lambda\mu}-(1-\xi)(\epsilon_\lambda\tilde{\epsilon}_\mu+\epsilon_\mu\tilde{\epsilon}_\lambda)], \quad (5)$$

where p, p' are the initial (incoming) and final (outgoing) momenta of $W^\pm$, respectively, λ and μ are the polarizations of the two photons, k is the incoming momentum of one of the photons (for definiteness, say, the one with polarization λ), and $(p'-p-k)$ is the incoming momentum of the other photon.

3. SOME IDENTITIES

We list first some simple identities satisfied by these matrices

(i) $$S^{-1}(p')-S^{-1}(p)=e^{-1}(p'-p)_\lambda V_\lambda(p',p), \quad (6)$$

where the inverse of $S(p)$ is

$$S^{-1}(p)=i[(p^2+m_W^2)-(1-\xi)p\tilde{p}]. \quad (7)$$

For a [(4×1) c-number column matrix] wave function $\varphi\perp p$ (i.e., $\tilde{\varphi}p=0$),

$$\tilde{\varphi}S^{-1}\varphi=0 \quad \text{at} \quad p^2+m_W^2=0. \quad (8)$$

On the other hand, if φ is proportional to p then

$$\tilde{\varphi}S^{-1}\varphi=0 \quad \text{at} \quad p^2+\xi^{-1}m_W^2=0. \quad (9)$$

Equations (8) and (9) give, respectively, the appropriate poles of $S(p)$ for its spin 1 and spin 0 parts.

(ii) $$\tilde{V}_\lambda(p',p)=-V_\lambda(-p,-p')=V_\lambda(p,p'), \quad (10)$$

which expresses the consequences of charge conjugation invariance and the space-time reversal invariance.

(iii) $$V_\lambda(p'+k,\,p+k)-V_\lambda(p',p)=-e^{-1}k_\mu U_{\mu\lambda}(p'+k,\,p;\,k)$$

and

$$\tilde{U}_{\lambda\mu}(p',p;k)=U_{\lambda\mu}(-p,-p';k)=U_{\lambda\mu}(p,p';-k)=U_{\mu\lambda}(p,p';p-p'+k). \quad (11)$$

Next, we consider $S'(p)$, $V_\lambda'(p,p')$ and $U_{\lambda\mu}'(p,p')$ which are defined to be, respectively, the *sum* of all diagrams that contribute to the propagation function of $W^\pm$, the three-point vertex function and the four-point vertex functions. Similar to the Ward's identity[3] (and related properties) for a spin $\frac{1}{2}$ field, there is the following generalization of (6)–(11):

(i) $$S'^{-1}(p')-S'^{-1}(p)=e^{-1}(p'-p)_\lambda V_\lambda'(p',p), \quad (12)$$

(ii) $$\tilde{V}_\lambda'(p',p)=-V_\lambda'(-p,-p')=V_\lambda'(p,p'), \quad (13)$$

(iii) $$V_\lambda'(p'+k,\,p+k)-V_\lambda'(p',p)=-e^{-1}k_\mu U_{\mu\lambda}'(p'+k,\,p;\,k), \quad (14)$$

(iv) $$\tilde{U}_{\lambda\mu}'(p',p;k)=U_{\lambda\mu}'(-p,-p';k)=U_{\lambda\mu}'(p,p';-k)=U_{\mu\lambda}'(p,p';p-p'+k). \quad (15)$$

In Appendix A we give a proof of these identities together with a discussion of some other well-known properties of charged vector mesons.

4. RENORMALIZATION ($\xi>0$)

Following the general method and notations developed by Dyson,[4] we find that the condition for primitive divergence (for $\xi>0$) is

$$E_W+E_\gamma\leq 4, \quad (16)$$

where E_W and E_γ are the number of external W lines and external photon lines, respectively. Because of gauge invariance and Furry's theorem, among graphs that satisfy (16) only those representing meson propagator, photon propagator, three-point vertex function, and meson-meson scattering are primitively divergent. The detailed program of renormalization for the present case ($\xi>0$) is similar to that of a charged scalar meson except for the difference in the spin variables. For example, the three-point vertex function $V_\lambda'(p',p)$ is now characterized by seven scalar functions $F_1,\cdots,F_7$ as follows:

$$\begin{aligned}V_\lambda'(p',p)=&\,p_\lambda F_1+p_\lambda'F_1^t+(\epsilon_\lambda\tilde{p})F_2+(p'\tilde{\epsilon}_\lambda)F_2^t\\&+(\epsilon_\lambda\tilde{p}')F_3+(p\tilde{\epsilon}_\lambda)F_3^t+(p\tilde{p}')[p_\lambda F_4+p_\lambda'F_4^t]\\&+(p'\tilde{p})[p_\lambda F_5+p_\lambda'F_5^t]+p_\lambda'(p\tilde{p})F_6\\&+p_\lambda(p'\tilde{p}')F_6^t+p_\lambda(p\tilde{p})F_7+p_\lambda'(p'\tilde{p}')F_7^t,\end{aligned} \quad (17)$$

where $F_1, F_2, \cdots, F_7$ are scalar functions; i.e.,

$$F_i=F_i(p^2,\tilde{p}'p,p'^2)$$

and $i=1,2,\cdots,7$. The F_i^t functions are related to F_i by

$$F_i^t=F_i(p'^2,\tilde{p}p',p^2). \quad (18)$$

Next, we represent the vertex functions, the meson-meson scattering function, and the derivative of photon propagator as sums over irreducible graphs. The existence of overlapping diagrams in such sums may generate complications. In this paper we do not enter any discussions on the overlapping problem but assume that such difficulties can be resolved (as would be the case if the electromagnetic interaction of a charged scalar meson is renormalizable). By a direct counting of momentum powers it can be seen that in the sum over irreducible graphs for the three point vertex function only F_1, F_2, F_3 [defined by (17)] are logarithmically divergent.

To remove these divergences it is necessary to add in the original Lagrangian a $\delta\xi$ and a $\delta\kappa$ term,

$$-ie\delta\kappa F_{\mu\nu}\varphi_\mu{}^\star\varphi_\nu-\delta\xi(\partial_\mu{}^\star\varphi_\mu{}^\star)(\partial_\nu\varphi_\nu), \quad (19)$$

where $F_{\mu\nu}$ and φ_μ are, respectively, the electromagnetic field tensor and the meson wave function. This gives rise to an additional three-point vertex

$$\delta V_\lambda(p',p)=ie[\delta\xi(\epsilon_\lambda\tilde{p}+p'\tilde{\epsilon}_\lambda)+\delta\kappa(k\tilde{\epsilon}_\lambda-\epsilon_\lambda\tilde{k})], \quad (20)$$

where $k=p'-p$. The values of ξ and κ in (2)–(5) are to be regarded as the renormalized constants. The divergences in F_1, F_2, and F_3 can then be removed by the use of $\delta\xi$, $\delta\kappa$, and the usual Z_1 factor. The renormalized three-point vertex function $V_{\lambda c}$ is defined by

$$V_{\lambda c}(p',p)=Z_1V_\lambda'(p',p) \quad (21)$$

and the boundary condition

$$\tilde{\varphi}V_{\lambda c}(p,p)\varphi=(2ie)p_\lambda, \quad (22)$$

where

$$\tilde{\varphi}\varphi=1$$

and

$$\tilde{\varphi}p=0. \quad (23)$$

[3] J. C. Ward, Proc. Phys. Soc. (London) **A64**, 54 (1951).
[4] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); A. Salam, *ibid.* **79**, 910 (1950); J. C. Ward, *ibid.* **84**, 897 (1951).

The renormalized meson propagator S_c can be obtained by using the generalized Ward's identity (12), which gives, after eliminating the Z_1 factor,

$$S_c^{-1}(p')-S_c^{-1}(p)=e^{-1}(p'-p)_\lambda V_{\lambda c}(p',p). \tag{24}$$

Similar to (8) and (9), S_c satisfies

$$\tilde{\varphi}S_c^{-1}(p)\varphi=0 \tag{25}$$

at

(i) $$p^2+m_W^2=0 \quad \text{if} \quad \tilde{\varphi}p=0$$

and

(ii) $$p^2+\xi^{-1}m_W^2=0 \quad \text{if} \quad \varphi=p.$$

The remaining divergences in the photon propagator D' and the meson-meson scattering can be removed by introducing in the original Lagrangian a (δe) term for the charge renormalization and two meson-meson scattering terms

$$(\lambda_1+\delta\lambda_1)(\varphi_\mu^\star\varphi_\mu^\star)(\varphi_\nu\varphi_\nu) + (\lambda_2+\delta\lambda_2)(\varphi_\mu^\star\varphi_\mu)(\varphi_\nu^\star\varphi_\nu), \tag{26}$$

where λ_i and $(\lambda_i+\delta\lambda_i)$ are, respectively, the renormalized and the unrenormalized coupling constants. The renormalized charge e is related to the unrenormalized charge $(e+\delta e)$ by

$$e=Z_3^{1/2}(e+\delta e), \tag{27}$$

where Z_3 is the usual ratio between the unrenormalized photon propagator D' and the renormalized one D_c.

To summarize, the renormalization of the present vector charged meson (for $\xi>0$) can be carried out by introducing renormalizations in e, m_W, ξ, κ and λ_1, λ_2. The magnetic moment of such a meson is

$$(2m_W)^{-1}e(1+\kappa), \tag{28}$$

where e, m_W, κ are all renormalized quantities.

5. QUADRUPOLE MOMENT AND THE LIMIT $\xi\to 0$

Let Q be the quadrupole moment which is defined to be the average value of $e(2z^2-x^2-y^2)$ for a W^+ at rest with $(\text{spin})_z=+1$. The radiative corrections of Q can be expressed as a power series in α,

$$Q=-\frac{e}{m_W^2}[\kappa+\sum_1^\infty A_n\alpha^n], \tag{29}$$

where $\alpha=$fine structure constant$\cong(137)^{-1}$ and A_n ($n=1, 2, \cdots$) are independent of α. If $\xi>0$, the A_n's are finite; but, as $\xi\to 0$, $A_n\to\infty$. The limit $\xi\to 0$ will be calculated by the following steps[5]:

(i) In (29), for each A_n we retain only the most singular part as $\xi\to 0$. (30)

(ii) Assume that the entire sum (29) does lead to a finite result in the limit $\xi\to 0$. (31)

[5] Similar considerations have been used in many-body problems. See, for example, Eq. (55) of T. D. Lee, K. Huang, C. N. Yang, Phys. Rev. **106**, 1135 (1957).

To carry out these two steps, we observe that A_n is a well-defined function of κ and ξ. As $\xi\to 0$ (proved in Appendix B),

$$A_1\to a_0\ln\xi$$

and

$$A_n\to a_{n-1}(\kappa^2/\xi^2)^{n-1} \quad \text{for} \quad n\geq 2, \tag{32}$$

where a_0 and a_{n-1} are *independent* of either ξ or α. Using (30), the quadrupole moment can be written as

$$Q=-(e\kappa/m_W^2)[1+\tfrac{1}{2}a_0\alpha\ln(\alpha\kappa^2)+\alpha f(x)], \tag{33}$$

where

$$x=(\alpha\kappa^2/\xi^2) \tag{34}$$

and $f(x)$ does not *explicitly* depend on either α or ξ. For small values of x,

$$f(x)=-\tfrac{1}{2}a_0\ln x+\sum_{m=1}^\infty a_m x^m. \tag{35}$$

When $\xi\to 0$, x becomes infinity. Applying (31), we demand that Q is finite as $\xi\to 0$. This is possible *only if*

$$\lim_{x\to\infty} f(x)=\text{finite}. \tag{36}$$

Therefore,

$$Q=-(e/m_W^2)[\kappa+\tfrac{1}{2}a_0\alpha\ln(\alpha\kappa^2)+O(\alpha)], \tag{37}$$

where $O(\alpha)=\alpha f(\infty)$. Explicit evaluation of a_0 gives [cf. Appendix B]

$$a_0=-(2\pi)^{-1}(\kappa+3)(\kappa-1)^2. \tag{38}$$

Similar considerations can be extended to other radiative corrections.

6. RADIATIVE CORRECTIONS TO LEPTONIC DECAYS OF $W^\pm$

The weak interactions between W and the other particles are described by[2]

$$\mathfrak{L}=\mathfrak{L}_{W-l}+\mathfrak{L}_{W-J}+\mathfrak{L}_{W-S},$$

where $\mathfrak{L}_{W-J}$ and $\mathfrak{L}_{W-S}$ describe, respectively, the interactions between W and the strongly interacting particles that conserve the strangeness S and violate S conservation. The term $\mathfrak{L}_{W-l}$ describes the interaction between W and the leptons

$$\mathfrak{L}_{W-l}=-ig_0\varphi_\lambda^\star[\psi_e^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_\nu + \psi_\mu^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_{\nu'}]+\text{conjugate terms}, \tag{39}$$

where g_0 is the unrenormalized coupling constant and $\dagger$ indicates Hermitian conjugation. Throughout this paper we assume[6]

$$\nu\neq\nu'.$$

[6] If $\nu=\nu'$, then arguments similar to that used in Sec. 5 would lead to a ratio of rates: $[\text{rate}(\mu\to e+\gamma)/\text{rate}(\mu\to e+\nu+\bar{\nu})] = (8\pi)^{-1}3\alpha[(\kappa-1)\ln(\alpha\kappa^2)]^2+O(\alpha)$, which does not agree with the experimental results. [See, for example, D. Bartlett, S. Devons, and A. M. Sachs. Phys. Rev. Letters **8**, 120 (1962); S. Frankel, J. Halpern, L. Holloway, W. Wales, M. Yearian, O. Chamberlain, A. Lemonick, and F. M. Pipkin, *ibid.* **8**, 23 (1962).]

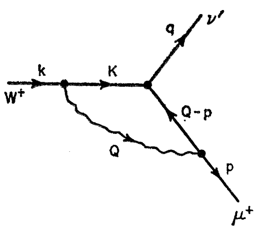

FIG. 1. A diagram for radiative correction to $W^+ \to \mu^+ + \nu'$.

For $\xi > 0$, the vertices for $W^+ \to e^+ + \nu$ and $W^+ \to \mu^+ + \nu'$ both require renormalizations. [In this paper, we consider only renormalization due to electromagnetic interactions.] For convenience, the *renormalized* $W-l$ coupling constant g may be defined in terms of the rate λ for W^+-decay into e^+ and ν

$$\lambda(W^+ \to e^+ + \nu) \equiv (6\pi)^{-1} m_W g^2. \qquad (40)$$

By using arguments similar to those in Sec. 4, it can be shown that in terms of g, all other physical quantities are finite if $\xi > 0$. The limit $\xi \to 0$ can then be carried out by applying (30) and (31) of the preceding section. Let $\Gamma_\lambda(q,p)$ denote the vertex function for

$$W^+ \to \mu^+ + \nu', \qquad (41)$$

where p and q are the (outgoing) 4-momenta of μ^+ and ν', respectively. The result (after taking the limit $\xi \to 0$) for Γ_λ is given by the following theorem:

Theorem 1: If in (41) both μ^+ and ν' are physical (hence, $p^2 = -m_\mu^2$ and $q^2 = 0$), then

$$\begin{aligned} g_1^{-1}\Gamma_\lambda(q,p) &= [u_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1+\gamma_5) u_\mu]\{1 + (16\pi m_W^2)^{-1}\alpha \\ &\quad \times \ln(\alpha\kappa^2)[(k^2 + m_W^2)(1+\tfrac{5}{6}\kappa) + m_\mu^2(1+\kappa)]\} \\ &\quad + i(16\pi m_W^2)^{-1} m_\mu \alpha \ln(\alpha\kappa^2)[u_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1-\gamma_5) u_\mu] \\ &\quad \times [k_\lambda(2+\tfrac{1}{3}\kappa) + 2p_\lambda(-1+\kappa)] + O(\alpha), \end{aligned} \qquad (42)$$

where u_μ, $u_{\nu'}$, are the free c-number spinors for μ and ν', m_μ is the mass of μ, k_λ is the incoming four-momentum of W^+, i.e.,

$$k_\lambda = p_\lambda + q_\lambda.$$

In (42), the constant g_1 is related to g by

$$\begin{aligned} (g/g_1)^2 = (1+v_e)^{-3} 2 v_e^2 \{ (3+v_e) + (m_e/m_W)^2 (8\pi)^{-1} \alpha \\ \times \ln(\alpha\kappa^2)[3(1+v_e) + \kappa(3-v_e)]\}, \end{aligned} \qquad (43)$$

where g is defined by (40), m_e is the mass of the electron, and v_e is the velocity of e^+ in the decay $W^+ \to e^+ + \nu$ observed in the rest system of W,

$$v_e = [1 + (m_e^2/m_W^2)]^{-1}[1 - (m_e^2/m_W^2)]. \qquad (44)$$

If one neglects (m_e^2/m_W^2) as compared to 1, then

$$g_1 = g.$$

Proof. To prove (42), we begin with $\xi > 0$ and calculate the renormalized vertex function Γ_λ as a power series in α. The diagram in Fig. 1 gives rise to a vertex function

$$\begin{aligned} (2\pi)^{-4}(e^2 g_0) \int d^4Q\, (Q^2+\delta^2)^{-1} [\psi_{\nu'}^\dagger \gamma_4 \gamma_\alpha (1+\gamma_5)] \\ \times [-i\gamma_\sigma (Q-p)_\sigma - m_\mu]^{-1} \gamma_\beta \psi_\mu \\ \times \Delta^2 (K^2 + \Delta^2)^{-1} [S(K) V_\beta(K,p)]_{\alpha\lambda}, \end{aligned} \qquad (45)$$

where d^4Q is real, $K = k - Q$, $S(K)$ and $V_\beta(K,p)$ are given by (2) and (3), respectively. The factor $\Delta^2(K^2+\Delta^2)^{-1}$ and the modified photon propagator $(Q^2+\delta^2)^{-1}$ are introduced to give the integral (45) a definite value. A direct computation shows that as $\xi \to 0$

$$\begin{aligned} (45) = Z g_0 [\psi_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1+\gamma_5) \psi_\mu] + (8\pi m_W^2)^{-1} \alpha \\ \times \ln\xi [g_0 \psi_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1+\gamma_5) \psi_\mu][k^2(1+\tfrac{5}{6}\kappa) \\ + m_\mu^2 (1+\kappa)] + i(8\pi m_W^2)^{-1} m_\mu \alpha \\ \times \ln\xi [g_0 \psi_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1-\gamma_5) \psi_\mu][k_\lambda (2+\tfrac{1}{3}\kappa) \\ + 2p_\lambda(-1+\kappa)] + O(\alpha), \end{aligned} \qquad (46)$$

where $O(\alpha)$ is proportional to α and remains finite as $\xi \to 0$, Z is a constant independent of k_λ, p_λ, and m_μ. In deriving (40), we use the properties that $p^2 + m_\mu^2 = 0$ and $(p-k)^2 = 0$. It is easy to see that Fig. 1 is the only diagram which gives a radiative correction to the momentum dependent part of the vertex function that is proportional to $\alpha \ln\xi$. Using the results obtained in Appendix B, it can be shown that the higher order radiative corrections are of the same form as (32). (For example, we may express in (33), instead of Q, the coefficients of $[\psi_{\nu'}^\dagger \gamma_4 \gamma_\lambda (1+\gamma_5) \psi_\mu] k^2$ or that of $[\psi_{\nu'}^\dagger \gamma_4 \gamma_\lambda \times (1+\gamma_5) \psi_\mu] p^2$, etc.) Similar to (36) and (37) the limit $\xi \to 0$ can be taken which yields the final result for Γ_λ given by (42). The ratio $(g/g_1)^2$ [Eq. (43)] can then be established by applying (42) (but changing μ, ν' to e, ν) to evaluate the rate $W^+ \to e^+ + \nu$.

We list in the following some consequences of Theorem 1:

(1) In the decay of $W^+ \to \mu^+ + \nu'$, let N_L and N_R be the number of left-handed μ^+ (i.e., helicity $= -\frac{1}{2}$) and that of right-handed μ^+ (i.e., helicity $= +\frac{1}{2}$), respectively. The ratio (N_L/N_R) is given by

$$\begin{aligned} (N_L/N_R) = \tfrac{1}{2}(m_\mu^2/m_W^2)\{1 - [4\pi(1+v_\mu)]^{-1} v_\mu (\kappa - 1)\alpha \\ \times \ln(\alpha\kappa^2) + O(\alpha)\}, \end{aligned} \qquad (47)$$

where v_μ is the velocity of μ^+ in the rest system of W,

$$v_\mu = [1 + (m_\mu^2/m_W^2)]^{-1}[1 - (m_\mu^2/m_W^2)]. \qquad (48)$$

(2) The branching ratio (summing over the helicities of e^+ and μ^+)

$$\begin{aligned} [\text{rate}(W^+ \to e^+ + \nu + \cdots)]^{-1} \\ \times [\text{rate}(W^+ \to \mu^+ + \nu' + \cdots)] \end{aligned} \qquad (49)$$

is given by

$$[(1+v_\mu)^3 v_e^2(3+v_e)]^{-1}[(1+v_e)^3 v_\mu^2(3+v_\mu)] \times\{1+(8\pi m_W{}^2)^{-1} m_\mu{}^2\alpha\ln(\alpha\kappa^2)(3+v_\mu)^{-1} \times[3+3v_\mu+\kappa(3-v_\mu)]-(8\pi m_W{}^2)^{-1}m_e{}^2\alpha \times\ln(\alpha\kappa^2)(3+v_e)^{-1}[3+3v_e+\kappa(3-v_e)]\} +O(\alpha)+O[(m_\mu{}^2/m_W{}^2)\alpha]. \quad (50)$$

In (49) the "$\cdots$" indicates possible presence of photons. In (50) the correction $O(\alpha)$ depends on $\ln(m_\mu/m_e)$ and is, therefore, important. However, it can be shown that the result of the infinite sum [similar to the $O(\alpha)$ term in (37)] leads only to a correction $O[(m_\mu{}^2/m_W{}^2)\alpha]$ in (50). Therefore, the important term $O(\alpha)$ can be calculated without summing over any infinite power series in α.

7. RADIATIVE CORRECTIONS TO μ DECAY

Similar considerations can be applied to μ decay. It turns out that the final result for the electron spectrum in μ decay including radiative corrections is quite simple. It essentially consists of only two parts which are (i) corrections[7] due to the nonlocal effects in μ decay induced by the presence of intermediate boson (without radiative corrections) and (ii) radiative corrections[8–10] but regarding the μ decay as due to a Fermi-type point interaction. Both effects (i) and (ii) have been calculated in the literature. For clarity, we state the final result of the electron spectrum in the form of a theorem:

Theorem 2. The electron spectrum in the decay of a completely polarized μ meson is given by

$$dN=x^2dxd(\cos\theta)[3\times2^6\times\pi^3]^{-1}m_\mu{}^5G_\mu{}^2 \times[1+\tfrac{3}{5}(m_\mu/m_W)^2]\{(3-2x)-\tfrac{1}{5}(m_\mu{}^2/m_W{}^2) \times[9-16x+5x^2]+(2\pi)^{-1}\alpha f(x)+\cos\theta[(1-2x) -\tfrac{1}{5}(m_\mu{}^2/m_W{}^2)(3-6x+5x^2)+(2\pi)^{-1}\alpha g(x)] +O[(m_\mu{}^2/m_W{}^2)\alpha\ln(\alpha\kappa^2)]\}, \quad (51)$$

where G_μ is related to the renormalized constant g by

$$G_\mu=m_W{}^{-2}2^{1/2}g^2\{1+O[\alpha\ln(\alpha\kappa^2)]\}, \quad (52)$$

$x=$ (electron momentum)/(maximum electron momentum), $\theta=$ angle between the momentum of electron and the spin of μ^-. The functions $f(x)$, $g(x)$ are given explicitly by Eqs. (2.4), (2.5), and (2.6) of reference 8.

It is important to notice that (51) is simply the sum of the above two effects (i) and (ii). In the spectrum (apart from the correction in G_μ) deviation from these two effects is only of the order of $(m_\mu{}^2/m_W{}^2)\alpha\ln(\alpha\kappa^2)$ which is smaller than either (i) or (ii). The radiative correction to the coupling constant G_μ is, however, of the order of $\alpha\ln(\alpha\kappa^2)$.

[7] T. D. Lee and C. N. Yang, Phys. Rev. **108**, 1611 (1957).
[8] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).
[9] S. M. Berman, Phys. Rev. **112**, 267 (1958).
[10] R. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. **101**, 868 (1956).

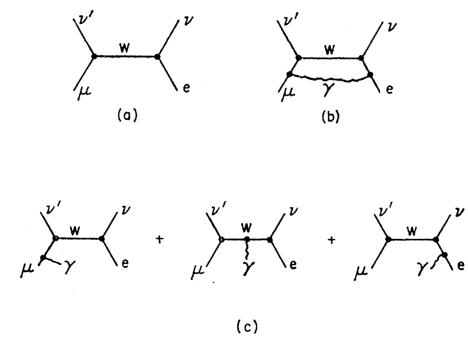

FIG. 2. Diagrams for μ decay.

Proof. To calculate radiative corrections we consider the skeleton graphs (a), (b), and (c) in Fig. 2.

In graph (a) we use the *renormalized* propagator and the *renormalized* vertex function Γ_λ. By using (42) it is seen that at zero momentum transfer and zero incoming momentum of the external line the radiative correction to Γ_λ is proportional to $\alpha\ln(\alpha\kappa^2)$ which contributes to $O[\alpha\ln(\alpha\kappa^2)]$ in (52). However, it is easy to see that the change of radiative corrections in Γ_λ at the physical momentum range of μ decay from that at zero momentum transfer and zero external momentum is of the order of $(m_\mu{}^2/m_W{}^2)\alpha\ln(\alpha\kappa^2)$. Identical conclusions also hold for the radiative corrections due to the renormalized propagator of W. Therefore, neglecting terms $O[\alpha\ln(\alpha\kappa^2)]$ in the coupling constant G_μ and $O[(m_\mu{}^2/m_W{}^2)\alpha\ln(\alpha\kappa^2)]$ in the relative magnitudes of the electron spectrum, the contribution of graph (a) to μ decay becomes identical with that given by the intermediate boson theory of μ decay in the absence of electromagnetic interactions.[7]

For $\xi>0$, graph (b) is completely finite. At $\xi\rightarrow0$, graph (b) can be separated into a sum of three terms

$$O(\alpha\ln\xi)+O[(m_\mu{}^2/m_W{}^2)\alpha\ln\xi]+O(\alpha), \quad (53)$$

where the singular term $O[\alpha\ln\xi]$ is independent of either the momentum transfer or the external momenta. By using the same arguments as that used in the previous section we find in the limit $\xi\rightarrow0$ the first and second terms in (53) together with their corresponding infinite sums contribute, respectively, to the terms $O[\alpha\ln(\alpha\kappa^2)]$ and $O[(m_\mu{}^2/m_W{}^2)\alpha\ln(\alpha\kappa^2)]$ in (52) and (51). The remaining term $O(\alpha)$ in (53) is completely finite (except for infrared divergence) and is identical with the result of radiative corrections in the Fermi theory with a point interaction for the μ decay. To be more specific, this term $O(\alpha)$ in (53) is identical with Eq. (7) of reference 10 except for the replacement of the ultraviolet cutoff in reference 10 by m_W. Theorem 2 is then proved by adding the effects of these graphs (a)–(c) and by using the results given in references 7 and 8.

A simple consequence of (51) is that the lifetime τ_μ

of the μ meson is given by

$$\tau_\mu^{-1}=(3\times 2^6\times\pi^3)^{-1}m_\mu^5G_\mu^2 \times\{1+(3/5)(m_\mu/m_W)^2-(2\pi)^{-1}\alpha(\pi^2-25/4) +O[(m_\mu/m_W)^2\alpha\ln(\alpha\kappa^2)]\}. \quad (54)$$

8. RADIATIVE CORRECTION TO THE β DECAY OF A "BARE" NUCLEON

By a "bare" nucleon we refer to a hypothetical particle without any strong interactions but with the same mass and electric charge as that of the physical nucleon. The weak-interaction Lagrangian $\mathcal{L}_{W-J}$ is assumed to be

$$\mathcal{L}_{W-J}=-ig_0\varphi_\lambda^\star[\psi_n^\dagger\gamma_4\gamma_\lambda(1+\gamma_5)\psi_p]+\text{H.c.}, \quad (55)$$

where g_0 is the same unrenormalized constant as that in (39).

Similar to Sec. 6, the renormalization of the vertex functions for $n\rightleftharpoons p+W^-$ and $p\rightleftharpoons n+W^+$ can be obtained. The final result is given by the following theorem:

Theorem 3. Let p_σ and n_σ be, respectively, the outgoing momentum of p and the incoming momentum of n in the vertex

$$n\rightarrow p+W^-. \quad (56)$$

For nucleon states with zero momentum transfer [i.e., $p^2=n^2=-(\text{nucleon mass})^2$ and $p_\sigma=n_\sigma$] the renormalized vertex function for (56) is given by

$$g_1\{1+(16\pi)^{-1}[1+(5/6)\kappa]\alpha\ln(\alpha\kappa^2)\}\{u_p^\dagger\gamma_4\gamma_\lambda(1+\gamma_5) \times u_n[1+(8\pi)^{-1}(m_N/m_W)^2(13/24)\kappa\alpha\ln(\alpha\kappa^2)] -u_p^\dagger\gamma_4\gamma_\lambda(1-\gamma_5)u_n(8\pi)^{-1}(m_N/m_W)^2(11/24)\kappa\alpha\ln(\alpha\kappa^2) +ip_\lambda u_p^\dagger\gamma_4u_n(8\pi)^{-1}(m_N/m_W)^2(1/12)\kappa\alpha\ln(\alpha\kappa^2)\}, \quad (57)$$

where m_N is the nucleon mass, the subscript λ indicates the polarization state of W and u_p, u_n are the free (c number) spinor solutions for p and n, respectively. In (57), g_1 is the same renormalized coupling constant used in (42). In the rest system of the nucleon, if $\lambda\neq 4$,

$$(57)=ig_2u_p^\dagger\sigma_\lambda u_n[1+(8\pi)^{-1}(m_N/m_W)^2\alpha\kappa\ln(\alpha\kappa^2)]; \quad (58)$$

if $\lambda=4$,

$$(57)=g_2u_p^\dagger u_n, \quad (59)$$

where σ_1, σ_2, σ_3 are the usual spin matrices and

$$g_2=g_1[1+(16\pi)^{-1}(1+\tfrac{5}{6}\kappa)\alpha\ln(\alpha\kappa^2)].$$

In a similar way, we can calculate the renormalized functions for either

$$n\rightleftharpoons p+e^-+\bar\nu$$

or

$$p\rightleftharpoons n+e^++\nu. \quad (60)$$

It is found that on keeping the correction term, which is $O[(m_N/m_W)^2\alpha\ln(\alpha\kappa^2)]$, but neglecting terms that are of the order of either α, or $[(m_N/m_W)^2\alpha]$, or $[(m_\mu/m_W)^2\alpha\ln(\alpha\kappa^2)]$, the renormalized vector and axial-vector β-decay coupling constants (for the decay of a "bare" nucleon) are given, respectively, by

$$G_V^2=G_\mu^2\{1+O(\alpha)+O[(m_N/m_W)^2\alpha] +O[(m_\mu/m_W)^2\alpha\ln(\alpha\kappa^2)]\} \quad (61)$$

and

$$G_A^2=G_\mu^2\{1+(4\pi)^{-1}(m_N/m_W)^2\alpha\kappa\ln(\alpha\kappa^2) +O(\alpha)+O[(m_N/m_W)^2\alpha] +O[(m_\mu/m_W)^2\alpha\ln(\alpha\kappa^2)]\}, \quad (62)$$

where G_μ is given by (54) [or (52)].

It is important to notice that both (61) and (62) are accurate to the order $(m_N/m_W)^2\alpha\ln(\alpha\kappa^2)$ and $\alpha\ln(\alpha\kappa^2)$; but due to apparently accidental cancellations such terms are absent in (61). Among the terms that are neglected, the $O[(m_N/m_W)^2\alpha]$ term is the most difficult one to be evaluated. Similar to the $O(\alpha)$ term in (37), it can only be calculated by summing up an infinite series. The $O[(m_\mu/m_W)^2\alpha\ln(\alpha\kappa^2)]$ term can be obtained relatively easily; but is found to be unimportant. The remaining $O(\alpha)$ term is identical with the radiative corrections already obtained in the literature[8–10] by assuming the usual Fermi theory of β decay provided that the ultraviolet cutoff is replaced by m_W. Consequently, the result is now completely finite. This is similar to the effect (ii) discussed in the preceding section. To be specific, the effect of the $O(\alpha)$ term in the electron spectrum of β decay is given explicitly by Eq. (4.1) of reference 8, except that the cutoff parameter λ is not replaced by m_W.

The comparison between the observed β-decay constant G_V and the μ-decay coupling has been discussed extensively in the literature.[8,11,12] According to Hendrie and Gerhart,[13] by using their recently observed value of G_V together with the radiative corrections calculated by Kinoshita and Sirlin[8] [i.e., without the assumption of an intermediate boson, choosing the ultraviolet cutoff $\lambda=m_N$, and neglecting further unknown structure effects of strong interactions], the calculated lifetime of the μ meson is found to be about 2.282×10^{-6} sec. The presence of an intermediate boson now changes this value to

$$[\tau_\mu]_{\text{th}}=[2.282\times10^{-6}\text{ sec}]\{1-\tfrac{3}{5}(m_\mu/m_W)^2 +(3\alpha/\pi)\ln(m_W/m_N)+O[(m_N/m_W)^2\alpha]\}. \quad (63)$$

If we ignore the $O[(m_N/m_W)^2\alpha]$ term[13a] and take, e.g., $m_W=5m_\mu$, then

$$(\tau_\mu)_{\text{th}}=2.218\times10^{-6}\text{ sec}, \quad (64)$$

[11] See, for example, R. P. Feynman, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1960), p. 499.

[12] The effects of intermediate boson on the ration between the coupling constant in μ decay and that in O^{14} decay have been discussed in the literature. See footnote 22 of reference 8; R. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961); S. Oneda and J. C. Pati, Phys. Rev. Letters **2**, 125 (1959).

[13] D. L. Hendrie and J. B. Gerhart, Phys. Rev. **121**, 846 (1961).

[13a] *Note added in proof.* It seems quite likely that the absence of $O[(m_N/m_W)^2\alpha\kappa\ln(\alpha\kappa^2)]$ in G_V [cf. (61)] implies that the $O[(m_N/m_W)^2\alpha]$ term in (63) is actually zero.

which is to be compared with the observed value[14]

$$(\tau_\mu)_{\text{exp}} = (2.211 \pm 0.003) \times 10^{-6} \text{ sec.} \qquad (65)$$

Because of the neglect of the $O[(m_N/m_W)^2\alpha]$ term[13a] and the unknown effects due to strong interactions, such a comparison is certainly not to be taken seriously.

ACKNOWLEDGMENTS

The author wishes to thank C. N. Yang and G. Feinberg for discussions. He also wishes to thank C.E.R.N. for its kind hospitality during the summer of 1961 when part of this work was done.

APPENDIX A

Let $G_p(N)$ denote any graph or part of a graph which connects a single W^+ line of incoming momentum p and final outgoing momentum p' with n photon lines of incoming momenta $k_1, k_2, \cdots k_N$ and polarizations $\Gamma_1, \Gamma_2, \cdots \Gamma_N$. In order to describe more explicitly the topological connection between the N photon lines and the W line we may also represent, e.g., a typical $G_p(N=4)$ graph in Fig. 3 by

$$G_p[(4)(3,2)(1)], \qquad (A1)$$

where each number i stands for the ith photon which has momentum k_i and polarization Γ_i. The order and the grouping of these numbers indicate, respectively, the consecutive order and the manner of the interaction between the photons and the W^+. For example, in (A1) the first set of parentheses (1) means that the W^+ first interacts with the photon k_1 through a three-point vertex; the next set of parentheses (3,2) indicates that the next interaction of the W^+ is with photons k_2 and k_3 through a four-point vertex, etc.

Each such graph (or part of a graph) $G_p(N)$ contributes a factor according to the Feynman rule. For example,

$$\begin{aligned} G_p[(4)(3,2)(1)] &= V_{\Gamma_4}(p', p'-k_4) S(p'-k_4) U_{\Gamma_3,\Gamma_2}(p'-k_4, p+k_1; k_3) \\ &\quad \times S(p+k_1) V_{\Gamma_1}(p+k_1, p), \qquad (A2) \end{aligned}$$

where

$$p' = p + \sum_{i=1}^{4} k_i. \qquad (A.3)$$

Throughout this discussion, both the graph and its corresponding factor are represented by $G_p(N)$.

Definition. For every given $G_p(N)$ we define $G_p{}^{q,\lambda}(N)$ to be a sum over certain graphs $G_p(N+1)$, each of which consists of the same N photons as in the given $G_p(N)$ plus another photon line of incoming momentum q and polarization λ

$$G_p{}^{q,\lambda}(N) \equiv \sum G_p(N+1), \qquad (A4)$$

[14] See, for example, *Proceedings of the 1960 Annual International Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1960), p. 878.

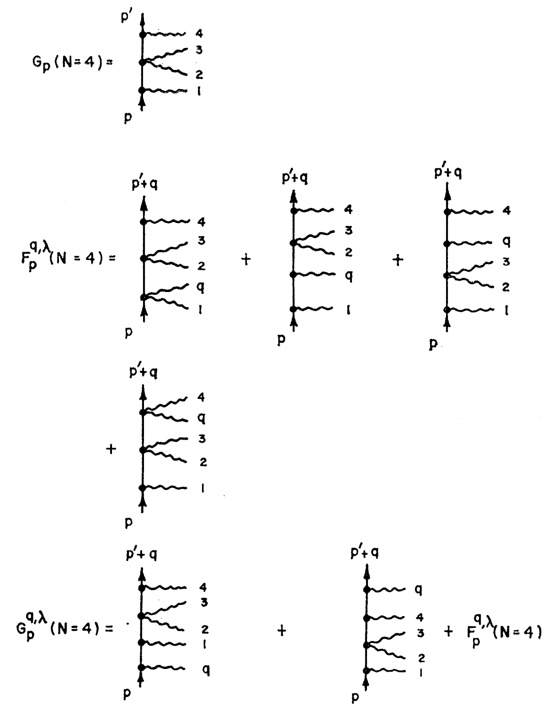

FIG. 3. An example of $G_p(N=4)$ and its associated graphs $F_p{}^{q,\lambda}$ and $G_p{}^{q,\lambda}$. [See (A1), (A15), and (A4) for their definitions.]

where each $G_p(N+1)$ in the sum satisfies the property that if the photon line q is simply erased then the remaining graph is topologically identical with the original graph $G_p(N)$. The sum (A4) extends over *all such* different $G_p(N+1)$.

An example of such sum is given in Fig. 3.

Theorem A1. For any given $G_p(N)$, the corresponding sum $G_p{}^{q,\lambda}(N)$ satisfies

$$\begin{aligned} \sum_{\lambda=1}^{4} q_\lambda G_p{}^{q,\lambda}(N) &= eS^{-1}(p'+q)S(p')G_p(N) \\ &\quad - eG_{p+q}(N)S(p+q)S^{-1}(p), \qquad (A5) \end{aligned}$$

where

$$p' = p + \sum_{i=1}^{N} k_i. \qquad (A6)$$

Proof. Assume that

$$\text{(A5) holds for all } G_p(N), \text{ where } N \le (n-1). \qquad (A7)$$

Any graph $G_p(n)$ must belong to either of the following two classes [cf. the notations used in (A1) and (A2)]:

(i) $$\begin{aligned} G_p(n) &= G_p[(n)\cdots] \\ &= V_{\Gamma_n}(p',p'')S(p'')G_p(n-1), \qquad (A8) \end{aligned}$$

where

$$p'' = p + \sum_{i=1}^{n-1} k_i = p' - k_n \qquad (A9)$$

and $G_p(n-1)$ represents the remaining part of $G_p(n)$

excluding the nth photon; or,

(ii) $$G_p(n)=G_p[(n, n-1)\cdots]$$
$$=U_{\Gamma_n,\Gamma_{n-1}}(p',p'';k_n)S(p'')G_p(n-2), \quad (A.10)$$

where

$$p''=p+\sum_{i=1}^{n-2} k_i=p'-k_n-k_{n-1}. \quad (A11)$$

As in the case of the $G_p(n-1)$ in (A8), the $G_p(n-2)$ in (A10) represents the remaining graph (excluding the last two photons, k_n and k_{n-1}). For case (i),

$$G_p{}^{q,\lambda}(n)=V_{\Gamma_n}(p'+q,\, p''+q)S(p''+q)G_p{}^{q,\lambda}(n-1)$$
$$+U_{\Gamma_n,\lambda}(p'+q,\, p'';k_n)S(p'')G_p(n-1)$$
$$+V_\lambda(p'+q,\, p')S(p')V_{\Gamma_n}(p',p'')S(p'')G_p(n-1). \quad (A12)$$

For case (ii),

$$G_p{}^{q,\lambda}(n)=U_{\Gamma_n,\Gamma_{n-1}}(p'+q,\, p''+q;k_n)S(p''+q)$$
$$\times G_p{}^{q,\lambda}(n-2)+V_\lambda(p'+q,\, p')S(p')$$
$$\times U_{\Gamma_n,\Gamma_{n-1}}(p',p'';k_n)S(p'')G_p(n-2). \quad (A13)$$

(A5) can then be verified directly for either (A12) or (A13) by using the assumption (A7) together with the identities (6) and (11). We observe further that (A5) is true for either case (i) and $N=1$, or case (ii) and $N=2$. Theorem A is then proved by induction.

Corollary. If in $G_p{}^{q,\lambda}(N)$ the external W lines are physical, then for the initial W state $S^{-1}(p)=0$ and for the final W state $S^{-1}(p'+q)=0$. Therefore,

$$\sum_{\lambda=1}^{4} q_\lambda G_p{}^{q,\lambda}(N)=0, \quad (A14)$$

which can be used to establish that the probability amplitude for emitting a longitudinal photon by a physical W is zero.

Definition. For every $G_p(N)$, we define

$$F_p{}^{q,\lambda}(N)\equiv G_p{}^{q,\lambda}(N)-G_{p+q}(N)S(p+q)V_\lambda(p+q,\, p)$$
$$-V_\lambda(p'+q,\, p')S(p')G_p(N), \quad (A15)$$

where

$$p'=p+\sum_{i=1}^{N} k_i.$$

An example of $F_p{}^{q,\lambda}(N)$ is given in Fig. 3. In general, $F_p{}^{q,\lambda}(N)$ can be obtained from $G_p{}^{q,\lambda}(N)$ by deleting the graphs $G_p[(q)\cdots]$ and $G_p[\cdots(q)]$. By using Theorem A1 and (6), it is easy to establish the following theorem:

Theorem A2. For any $G_p(N)$, the corresponding $F_p{}^{q,\lambda}(N)$ satisfies

$$\sum_{\lambda=1}^{4} q_\lambda F_p{}^{q,\lambda}(N)=e[G_p(N)-G_{p+q}(N)]. \quad (A16)$$

Definition. Let $\Lambda(p)$ be the sum over all proper self-energy diagrams of W:

$$S'^{-1}(p)=S^{-1}(p)-\Lambda(p).$$

Corollary. By regarding $\Lambda(p)=\sum G_p(N)$ and applying Theorem A2, the generalized Ward's identity (12) follows. Similarly, by applying Theorem A2 to the proper diagrams for V_λ' we establish (14).

Remarks. The identities (13) and (15) can be directly verified by using (10), (11), and

$$\tilde{S}(p)=S(p)=S(-p).$$

By using (13) and (15) the well-known Furry theorem can be easily proved.

APPENDIX B

To calculate the radiative correction for the quadrupole moment Q, we notice that $Q+(e\kappa/m_W{}^2)$ is given by

$$\text{the coefficient of } (iP_\lambda K\tilde{K}) \quad (A17)$$

in $V_\lambda'(P',P)$ for a physical W at the limit $K\to 0$, where

$$K=P'-P.$$

Therefore, by using (17) we find

$$Q+(e\kappa/m_W{}^2)=2iF_4 \quad (A18)$$

evaluated at $P'=P$ and $P^2+m_W{}^2=0$. By a straightforward counting of the degree of divergence of the relevant integrals at $\xi=0$, it can be easily established that as $\xi\to 0$ (32) describes the correct asymptotic power dependence on ξ for A_n. To prove that (32) is indeed correct, *without* any further factors such as $(\ln\xi)$, needs a much more detailed examination of the asymptotic behavior of the integrals.

Among the various groups for the three-point function $V_\lambda'(P',P)$ let us first consider a graph $\mathcal{G}_N$ which consists of I_W internal W lines, I_γ internal γ lines, N three-point vertices, and *no* four-point vertex. Therefore,

$$I_W=2I_\gamma=(N-1). \quad (A19)$$

The contribution of $\mathcal{G}_N$ to (A17) can be written as an integral over its

$$(I_W+I_\gamma-N+1)=\tfrac{1}{2}(N-1) \quad (A20)$$

independent internal momenta. From the definition (29), one sees that this integral forms a part of the coefficient A_n, where

$$n=\tfrac{1}{2}(N-1). \quad (A21)$$

The limit of this integral at $\xi\to 0$ depends on the corresponding asymptotic behavior of its integrand at large momenta $\sim\xi^{-1/2}m_W$.

The following properties are of importance:

(a) So far as the asymptotic behavior is concerned, the propagator $S(p)$ of every internal W line (with one possible single exception) can be replaced by

$$S_0(p)\equiv m_W^{-2}(p\tilde{p})\mathfrak{D}, \tag{A22}$$

where

$$\mathfrak{D}=-i[(p^2+m_W^2)^{-1}-(p^2+\xi^{-1}m_W^2)^{-1}]. \tag{A23}$$

(b) With this replacement, the propagator $-i(k^2)^{-1}$ of an internal photon line can be eliminated. Let $V_\lambda(p',p)$ and $V_\mu(q',q)$ be the two vertices connected by this photon line $[k=(p'-p)=(q'-q)]$. By using the identity

$$\tilde{p}'V_\lambda(p',p)p=ie\kappa[k^2p_\lambda-k_\lambda(k\cdot p)] +ie\xi[p^2p_\lambda'+(p')^2p_\lambda], \tag{A24}$$

we find that the product

$$[\tilde{p}'V_\lambda(p',p)p][\tilde{q}'V_\mu(q',q)q](k^2)^{-1}\delta_{\lambda\mu} \tag{A25}$$

becomes at $\xi=0$, simply

$$-e^2\kappa^2[k^2(p\cdot q)-(k\cdot p)(k\cdot q)] \tag{A26}$$

in which the factor $(k^2)^{-1}$ is completely canceled.

(c) The single possible exception mentioned in (a) refers to one of the two W lines that are connected to the external photon. Let p, p', and K be, respectively, the momenta of these two W lines and the external photon. By using (A22) and (A24), it is seen that

$$\lim_{\xi\to0} S_0(p')V_\lambda(p',p)S_0(p)$$

is proportional to $[K^2p_\lambda-K_\lambda(K\cdot p)]$. If we replace both propagators by S_0, the graph $\mathcal{G}_N$ can contribute to (A17) only through the second term on the right-hand side of (A24) which carries an extra factor ξ.

We, therefore, differentiate two cases:

Case I. All propagators of the internal W lines in $\mathcal{G}_N$ are replaced by S_0.

Case II. The propagator of one of the two W lines which are connected to the external photon is replaced by

$$D_W(p)=(-i)(p^2+m_W^2)^{-1}, \tag{A27}$$

but all other W propagators are replaced by S_0. In this case this particular W line is called an *exceptional* W line.

It is also useful to define the *exceptional* γ lines as those photon lines that are connected to either the external or the exceptional W lines and the *exceptional* vertices as those vertices that are in contact with either the exceptional γ line or the exceptional W line.

Let (I) and (II) represent, respectively the integrals for (A16) in Case I and Case II. Since $S(p)=S_0(p)+D_W(p)$, the sum (I)+(II) gives the complete contribution of the graph $\mathcal{G}_N$ as $\xi\to0$.

(d) By using (A17), (A20) (A21) and (A27) it can be verified by a direct counting that, at $\xi=0$ the integral (II) diverges like (momentum)^s, where

$$s=4[\tfrac{1}{2}(N-1)]+N-2I_\gamma-3-2=4(n-1), \tag{A28}$$

and the integral $[\xi^{-1}(\mathrm{I})]$ diverges like (momentum)$^{s+2}$. Furthermore, it follows from (A26) that each nonexceptional vertex gives a factor $(e\kappa)$ to the integral. Since the total number of exceptional vertices is limited by an upper limit *independent* of n and since the introduction of a small ξ gives a cutoff for the internal momentum at $\sim\xi^{-1/2}m_W$ we find, as $\xi\to0$

$$A_n\sim(\kappa^2/\xi^2)^{n-1} \tag{A29}$$

apart from possible further multiplicative factors such as $(\ln\xi)$. To establish the absence of such factors we have to exhibit more explicitly these integrals for $\mathcal{G}_N$.

Definition. It is convenient to define a reduced graph, called $\mathcal{G}_N'$, which is obtained by shrinking all the nonexceptional γ lines in $\mathcal{G}_N$ to zero length. The reduced graph $\mathcal{G}_N'$ contains only W lines (among these, at most one is exceptional), exceptional γ lines, exceptional three-point vertices, and *new* meson-meson scattering vertices.

For definiteness, we label the momenta carried by the l nonexceptional lines and m exceptional (W or γ) lines in $\mathcal{G}_N'$ by, respectively,

$$p_1,\ p_2,\ \cdots\ p_l \quad\text{and}\quad k_1,\ \cdots,\ k_m, \tag{A30}$$

where for Case I

$$l=I_W=2n,\qquad m\leq2, \tag{A31}$$

and for Case II

$$l=2n-1,\qquad m\leq4. \tag{A32}$$

The modified Feynman rules for $\mathcal{G}_N'$ are given in the following.

The vertex function for the *new* meson-meson scattering vertex is given by (A26).

The propagator for the jth nonexceptional W line is $\mathfrak{D}(p_j)$ [given by (A23)] which can also be represented by the following parametric representation:

$$\mathfrak{D}(p_j)=m_W^{-2}\xi\int_0^\infty \{\exp[-i\xi r_j]-\exp[-ir_j]\} \times\exp\left(-i\xi r_j\frac{p_j^2}{m_W^2}\right)dr_j, \tag{A33}$$

where $j=1,\ 2,\ \cdots,\ l$.

The propagator for the exception W line is

$$D_W(k_1)=-i(k_1^2+m_W^2)^{-1} =m_W^{-2}\xi\int_0^\infty \exp\left[-i\xi\left(\frac{k_1^2}{m_W^2}+1\right)r_{l+1}\right]dr_{l+1}. \tag{A34}$$

The propagator for the internal γ line (which is always exceptional)

$$D_\gamma(k_s)=-i(k_s^2)^{-1} =m_W^2\xi\int_0^\infty \exp\left(-i\xi\frac{k_s^2}{m_W^2}r_{l+s}\right)dr_{l+s}. \tag{A35}$$

In Case I the three-point vertex interacting with the external γ line is given by [cf. (A24)]

$$ie\xi[p^2p_\lambda'+(p')^2p_\lambda].$$

Otherwise, all three-point vertices in $\mathcal{G}_N'$ are given by $[V_\lambda(p',p)]_{\xi=0}$ multiplied by the appropriate $\tilde{p}'$ or p or both, depending on the number and the propagation directions of its neighboring nonexceptional W lines.

These modified Feynman rules for $\mathcal{G}_N'$ clearly gives the same result as the original Feynman rules for $\mathcal{G}_N$.

The graph $\mathcal{G}_N$ (or its reduced graph $\mathcal{G}_N'$) consists of n loops. Let $q_1, q_2, \cdots q_n$ be the n independent internal momenta carried by these loops. By using the above modified Feynman rules we find that, for the two cases (I) and (II), the graph $\mathcal{G}_N$ is given by

$$\xi\int[\mathcal{P}_a]\left[\prod_1^l \mathfrak{D}(p_i)\right]\left[\prod_1^m D_\gamma(k_j)\right]\prod_1^n d^4q_\alpha \quad (A36)$$

and

$$\int[\mathcal{P}_b]\left[\prod_{i=1}^l \mathfrak{D}(p_i)\right]\left[D_W(k_1)\prod_{j=2}^m D_\gamma(k_j)\right]\prod_1^n d^4q_\alpha, \quad (A37)$$

respectively. In the above integrals p_i, k_j are linear functions of the external W momenta P, P' and the independent internal momenta $q_1, \cdots, q_n$; $\mathcal{P}_a$, $\mathcal{P}_b$ are both homogeneous polynomials of these momenta of degree

$$1+4l-4n+2m \quad (A38)$$

[so that (the integrand$\times\prod d^4q_\alpha$)$\sim$(momentum)^1 in both (A36) and (A37)]. The coefficient (A17) of these two integrals (A37) and (A38) gives the desired A_n. It is important to notice that $\mathcal{P}_a$, $\mathcal{P}_b$, D_W, and D_γ do not depend on ξ.

To evaluate (A36) and (A37) we take advantage of the analogy between an electric circuit and the Feynman graph. Consider a circuit which has the same topological structure as $\mathcal{G}_N'$. Let P, P', K be the corresponding external currents of the circuit, $q_1, q_2, \cdots, q_n$ be the internal circulating currents of its n loops, and $r_1, r_2, \cdots, r_{l+m}$ be the resistances of the $l+m$ branches of the circuit. In order to maintain such a current distribution, the total electric power supplied by internal and external sources is a quadratic polynomial in q_i, which can be written as

$$\tilde{q}Rq+\tilde{S}q+\tilde{q}S+T, \quad (A39)$$

where q is an $(n\times1)$ column matrix whose elements are $q_1, q_2, \cdots$, the matrix R is an $(n\times n)$ symmetric matrix whose matrix elements are linear functions in $r_1, r_2, \cdots$ with coefficients ±1 or 0, and S is an $(n\times1)$ matrix whose elements depend linearly on both r_i and the external currents. The remaining function T depends linearly on r_i but quadratically on the external currents. The matrix R can be diagonalized by a real orthogonal matrix U

$$UR\tilde{U}=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}\equiv\Lambda, \quad (A40)$$

where $\lambda_\alpha(\alpha=1, 2, \cdots, n)$ are the eigenvalues of R. In terms of Λ the power (A39) becomes

$$\tilde{I}\Lambda I+Q, \quad (A41)$$

where

$$I=U(q+R^{-1}S)$$

and

$$Q=T-\tilde{S}R^{-1}S. \quad (A42)$$

In the absence of any internal power supply, the current distribution of the circuit is given by $I=0$ and its power supply$=Q$. By using (A33)–(A35) the integrals (A36) and (A37) can be written in the form

$$\int\prod_1^n d^4I_\alpha\int\prod_1^{l+m} dr_jf_j(r_j)\times(\mathcal{P}_a \text{ or } \mathcal{P}_b)$$
$$\times\exp[-im_W^{-2}\xi(\tilde{I}\Lambda I+Q)] \quad (A43)$$

multiplied by $\xi m_W^{-2(l+m)}$ or $m_W^{-2(l+m)}$, respectively, where if r_j=resistance of the nonexceptional line (i.e., $j\leq l$),

$$f_j=\exp(-i\xi r_j)-\exp(-ir_j); \quad (A44)$$

if r_j=resistance of the exceptional γ line,

$$f_j=1; \quad (A45)$$

and if r_j=resistance of the exceptional W line,

$$f_j=\exp(-i\xi r_j). \quad (A46)$$

To evaluate the coefficient of $(P_\lambda K\tilde{K})$ for (A36) and (A37) we use (A43) and the integral

$$\int(k^2)^n\exp(-i\lambda k^2)d^4k=(i)^n\frac{\partial^n}{\partial\lambda^n}\frac{-i\pi^2}{\lambda^2}. \quad (A47)$$

The coefficient A_n, is, then found to be of the form [*valid* for both (A36) and (A37)]

$$\xi^{-2(n-1)}\int\left(\prod_{\alpha=1}^n\lambda_\alpha^{N_\alpha}\right)^{-1}F\left(\prod_1^{l+m}f_jdr_j\right)\exp(-i\xi\mathcal{Q}), \quad (A48)$$

where the integrations are from $r_j=0$ to ∞ and F, $\mathcal{Q}$, λ_α are independent of ξ. To establish (32) we need to prove that the integral in (A48) exists in the limit $\xi\to0$ for $n\geq2$. (For clarity, we regard the external currents, the mass of W and κ as pure numerical constants in the subsequent discussion.)

We list the following simple properties:

(i) F is a homogeneous function of r_j of degree 0 and F is bounded,

$$|F|\leq\text{constant}. \quad (A49)$$

(ii) λ_α is a homogeneous function of r_j of degree 1. It is clear that for any electric circuit

$$\sum_1^n \lambda_\alpha \geq \sum_1^{l+m} r_j. \tag{A50}$$

(iii) $\mathcal{Q}$ is a homogeneous function of r_j of degree 1. If $\sum_1^n \lambda_\alpha$ is bounded then $\mathcal{Q}$ is also bounded.

(iv)
$$N_\alpha \geq 2. \tag{A51}$$

(v)
$$\sum_1^n N_\alpha = 2l+m-1, \tag{A52}$$

which is a consequence of (A38) and (A47).

(vi) For $\xi>0$, the integrand, in (A48) multiplied by $\prod_1^{l+m} r_j$, is $\sim O(r_j)$ as $r_j \to 0$. Therefore, the integral (A48) exists if $\mathcal{G}_N$ is a skeleton graph. Otherwise, the convergence of the integration is insured for $\xi>0$ only if we add to $\mathcal{G}_N$ other graphs that are necessary for the renormalization purpose. In the following we assume that this is done and that the integral (A48) does exist provided $\xi>0$. The following theorem establishes the existence of (A48) in the limit $\xi\to 0$.

Theorem B. If the integral (A48) exists for any $\xi>0$, then for $n\geq 2$ the integral

$$A_n^0 \equiv \int \left[\prod_{\alpha=1}^n \lambda_\alpha^{N_\alpha}\right]^{-1} F\left[\prod_1^{l+m} f_j^0 dr_j\right] \tag{A53}$$

also exists, where the integration is from $r_j=0$ to ∞. The function f_j^0 is given by

$$f_j^0 = 1-\exp(-ir_j) \tag{A54}$$

for a nonexceptional line (i.e., $j\leq l$) and by

$$f_j^0 = 1 \tag{A55}$$

for an exceptional (W or γ) line (i.e., $j>l$).

Proof. Define

$$\mathfrak{F} \equiv \left(\prod_{\alpha=1}^n \lambda_\alpha^{N_\alpha}\right)^{-1} F \prod_{j=1}^{l+m} f_j^0 \tag{A56}$$

and

$$\mathcal{I}(R) \equiv \int_\Omega \mathfrak{F} \prod_1^{l+m} dr_j, \tag{A57}$$

where Ω is the region in which

$$r_j \geq 0,$$

but

$$R \leq \sum_{\alpha=1}^n \lambda_\alpha \leq 2R. \tag{A58}$$

The existence of (A48) for any $\xi>0$ implies that $\mathcal{I}(R)$ exists for all finite R. To prove Theorem B it is only necessary to prove that $\lim_{R\to\infty}\mathcal{I}(R)=0$.

Let Ω_0 be a subdomain of Ω, in which all λ_j are uniformly large; say

$$\lambda_\alpha \geq \epsilon_\alpha R,$$

where ϵ_α is independent of R. By using (A49)–(A52), we find

$$\int_{\Omega_0} \mathfrak{F} \prod_1^{l+m} dr_j \sim R^{-l+1} \tag{A59}$$

which $\to 0$ as $R\to\infty$ provided $n\geq 2$ [cf. (A31) and (A32)]. Next, we consider another subdomain Ω_1 of Ω. In Ω_1 all λ_α, except one, are uniformly large. Without loss of generality we may choose

$$\lambda_1 \leq \epsilon R, \qquad \lambda_{\alpha\neq 1} \geq \epsilon' R, \tag{A60}$$

and

$$\epsilon' \gg \epsilon,$$

where ϵ' and ϵ are both independent of R. By using properties of electric circuits it can be readily shown that the extreme case

$$\lambda_1 \ll \lambda_{\alpha\neq 1}$$

can happen only if one of the loops in the circuit $\mathcal{G}_N'$ is developing a short circuit. Furthermore,

$$\textstyle\sum_s r_s = \lambda_1 \times \text{constant}. \tag{A61}$$

In (A61), as well as in the following, we use s (or r_s) to represent the various branches (or their corresponding resistances) in that short circuit. [The constant in (A61) is 1 if the short circuit coincides with one of the original loops chosen for the assignment of q in (A39).]

We fix all r_s in the short circuit and integrate over the remaining $r_{j\neq s}$

$$\tau \equiv \int_{\Omega_1} [\prod_{\alpha\neq 1} \lambda_\alpha^{N_\alpha}]^{-1} \prod_{j\neq s} f_j^0 dr_j, \tag{A62}$$

$$\langle F\rangle \equiv \tau^{-1} \int_{\Omega_1} [\prod_{\alpha\neq 1} \lambda_\alpha^{N_\alpha}]^{-1} F \prod_{j\neq s} f_j^0 dr_j. \tag{A63}$$

For large R,

$$\tau \sim \text{constant} \times R^p, \tag{A64}$$

$$p = (l-l_1)+(m-m_1) - \sum_{\alpha=2}^n N_\alpha, \tag{A65}$$

and

$$\langle F\rangle \sim O[(r_s/R)^\beta], \tag{A66}$$

where l_1 and m_1 are the number of exceptional lines and that of nonexceptional lines in the short circuit, respectively, and [in order that (A49) holds]

$$\beta \geq 0. \tag{A67}$$

To show that the remaining integration in the integral

$$\int_{\Omega_1} \mathfrak{F} \prod_1^{l+m} dr_j = \int (\lambda_1)^{-N_1} \tau \langle F\rangle \prod_s f_s^0 dr_s \tag{A68}$$

does become zero as R approaches infinity we first establish the following lemma.

Lemma.

$$(\beta-p)\geq 1. \qquad \text{(A69)}$$

Proof. For any large but finite R (A68) exists. Consequently

$$\beta+2l_1+m_1-N_1\geq 1 \qquad \text{(A70)}$$

in order to have the convergence at small r_s. Combining (A70) with (A65) and (A52), we obtain the inequality

$$(\beta-p)\geq(l-l_1), \qquad \text{(A71)}$$

which proves the lemma if $(l-l_1)\neq 0$.

The special case

$$l-l_1=0 \qquad \text{(A72)}$$

means that excluding the short-circuit loop the remaining graph consists of only exceptional lines. Since the number of exceptional lines is limited by (A31) and (A32), it can be shown that (A72) is possible only if the remaining graph consists of one single (exceptional) photon line; i.e.,

$$(m-m_1)=1.$$

Therefore, $p=1-\sum_{\alpha\neq 2}N_\alpha$. The lemma follows by using (A51) and (A67).

To perform the integration (A68) we separate Ω_1 into two regions,

$$\Omega_a:\quad 0\leq\lambda_1\leq A$$

and

$$\Omega_b:\quad A\leq\lambda_1\leq\epsilon_1 R, \qquad \text{(A73)}$$

where A is a constant independent of R.

Integration over Ω_a in (A68) gives

$$\int_{\Omega_a}\mathfrak{F}\prod_{j=1}^{l+m}dr_j\sim R^{-\beta+p}\leq R^{-1}. \qquad \text{(A74)}$$

For the region Ω_b, we change the variables to θ and x_s

$$\lambda_1=\theta R \qquad \text{(A75)}$$

and

$$r_s=x_s\lambda_1, \qquad \text{(A76)}$$

where

$$\textstyle\sum_s x_s=\text{constant} \qquad \text{(A77)}$$

which is the same constant in (A61). We keep λ_1 fixed and integrate first over all the x_s in (A68). By using (A52) and the inequality $|f_j^0|\leq 2$, we find

$$\left|\int_{\Omega_b}\mathfrak{F}\prod_1^{l+m}dr_j\right|<\text{constant}\times R^{-l+1}\times\int_{(A/R)}^{\epsilon}\theta^L d\theta, \qquad \text{(A78)}$$

where the power L is given by

$$L=-N_1+\beta+l_1+m_1-1 \qquad \text{(A79)}$$

which is also equal to [by using (A52) and (A65)]

$$(\beta-p-l). \qquad \text{(A80)}$$

Therefore, $|\int_{\Omega_b}\mathfrak{F}\prod_1^{l+m}dr_j|$ is less than R^{-l+1}, or $R^{-l+1}(\ln R)$, or $R^{-\beta+p}$ depending on whether $(L+1)$ is >0, or $=0$, or <0. In either one of these cases,

$$\lim_{R\to\infty}\int_{\Omega_b}\mathfrak{F}\prod_1^{l+m}dr_j=0. \qquad \text{(A81)}$$

Therefore, we establish

$$\int_{\Omega_1}\mathfrak{F}\prod_1^{l+m}dr_j\to 0\text{ as,}\quad R\to\infty.$$

In a similar way we can prove

$$\lim_{R\to\infty}\int_{\Omega_2}\mathfrak{F}\prod_1^{l+m}dr_j=0,$$

where Ω_2 is another subregion of Ω in which, instead of (A60), two of the λ_α are small, etc. Theorem B is then proved. Similar proof can be constructed for $\mathcal{G}_N$ which contains also four-point vertices. Therefore, (32) is established. The coefficient a_n in (32) can be obtained by explicitly calculating integrals such as (A53). The result for a_0 is given by (38).