Coulomb Energy of He^3 and Charge Distribution of Nucleon*

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The effect of finite nucleon size on the Coulomb energy of He^3 has been investigated. The experimental value of the difference between the binding energies of H^3 and He^3 is 0.764 MeV, while the calculated Coulomb energy is approximately equal to or greater than 1.0 MeV if the nuclear force has no repulsive core. We show that, if the finite size of nucleon is taken into consideration, the Coulomb energy of He^3 is reduced by about 15–20%. The effect of finite charge distribution is determined mainly by the mean square radius. If there is a hard core (with radius D), the calculated Coulomb energy (assuming point nucleons) is already smaller, with the values 0.8–0.9 MeV for $D=0.2\times10^{-13}$ cm, $\sim$0.7 MeV for $D=0.6\times10^{-13}$ cm. The reduction of Coulomb energy due to the finite size is about 8% and 3%, respectively, for two different models. The Coulomb potential between extended unpolarized nucleons is given in closed form for exponential and Yukawa charge distributions.

1. INTRODUCTION

THE recent electron scattering experiments at Stanford University[1] have established the finite size of the proton. If a finite charge distribution is assumed for the proton, the Coulomb energy (referred to as CE) arising from the small interproton distance in the He^3 nucleus is reduced. We shall estimate this effect. Since the actual wave function for He^3 is not known, the best thing we can do at present may be to consider several simplified wave functions, and see semiquantitatively how large the effect of the finite size is.

The experimental value of the difference between binding energies of H^3 and He^3 is 0.764 MeV,[2] which is mainly due to the CE of He^3 if the same p-p and n-n nuclear forces are assumed. On the other hand, the calculated CE of He^3 (assuming point nucleon) is approximately equal to or greater than 1.0 MeV,[3,4] if the nuclear force has no repulsive core. In Sec. 2 it is shown that the theoretical CE is reduced by about 15–20% by taking into consideration possible charge distributions. The reduction is roughly proportional to the square of the mean-square radius of the proton-charge distribution. The mean-square radius of the neutron-charge distribution is assumed zero, consistent with experiment.

If the nuclear force has a repulsive core (with radius D) the calculated CE (assuming point nucleon) is already smaller, with the values[5] of 0.8–0.9 MeV for $D=0.2\times10^{-13}$ cm, $\sim$0.7 MeV for $D=0.6\times10^{-13}$ cm. The reduction of CE due to the finite size is expected to be smaller for larger values of D, because the hard core prevents the nucleons from approaching each other closely. The evaluation with two simple wave functions for He^3 yields reductions of $\sim$8% and 3%, respectively (Sec. 3). The general formula of the Coulomb potential between two particles with extended charge distributions is given in the Appendix. If the charge distribution is of the Yukawa or exponential type (or their combination), the Coulomb potential is expressed by elementary functions.

The difference of the energies of two mirror nuclei (other than H^3 and He^3) can be considered similarly, but the effect of the finite size may be smaller.

2. NO HARD-CORE CASE

We shall assume a simple wave function for He^3: the spatial part is given by

$$\psi=N^{-\frac{1}{2}}e^{-\mu(r_{12}+r_{23}+r_{31})}, \tag{2.1}$$

with a doublet spin function which is antisymmetric with respect to the exchange of two protons. μ is an adjustable parameter which is determined by the usual variational method. ψ in (2.1) is very simple, but is a fairly good wave function. (See reference 4.)

As the two-body nuclear force we assume (1) an exponential central potential or (2) a Yukawa central potential. The potential depths and ranges are adjusted so as to fit the following low-energy data of the two-body system:

Binding energy of deuteron	=	2.226 MeV,
triplet scattering length	=	0.5378×10^{-12} cm,
singlet scattering length	=	-2.369×10^{-12} cm,
singlet effective range	=	2.7×10^{-13} cm.

The binding energy of H^3 thus calculated is 10.26 MeV for the exponential potential, and 12.49 MeV for the Yukawa potential, while the experimental value is 8.49 MeV. The adjusted value of μ is 0.479 (in 10^{13} cm^{-1}) and 0.605, respectively.

The electron-scattering experiment at low energies tells us the mean-square radius of the charge distribution which is about 0.8×10^{-13} cm for protons and

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[1] For example, see R. Hofstadter, F. Bumiller, and M. R. Yearian, Revs. Modern Phys. **30**, 482 (1958).
[2] C. W. Li, Ward Whaling, W. A. Fowler, and C. C. Lauritsen, Phys. Rev. **83**, 512 (1951).
[3] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952).
[4] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 204.
[5] T. Ohmura (formerly Kikuta), M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) **15**, 222 (1956); *ibid.* **17**, 326 (1957); T. Ohmura *ibid.* **21**, 34 (1959).

0.0×10^{-13} cm for neutrons. We shall consider (1) the Yukawa distribution and (2) the exponential distribution for the protons only, with the value 0.8×10^{-13} cm for the mean-square radius. The neutron is assumed completely neutral.

$$\rho_Y(r)=\frac{e\beta_1^2}{4\pi}\frac{e^{-\beta_1 r}}{r},\quad \langle r^2\rangle=6/\beta_1^2 \tag{2.2}$$

$$\rho_E(r)=\frac{e\beta_2^3}{8\pi}e^{-\beta_2 r},\quad \langle r^2\rangle=12/\beta_2^2. \tag{2.3}$$

The experimental data up to a few hundred MeV is well reproduced by an exponential charge distribution, but is not accurately fitted by the Yukawa distribution. We shall consider both of them just for the purpose of comparison. The recent higher energy experiment can give more detailed information on the shape of the charge distribution. Hofstadter and Herman[6] have proposed the following form:

$$\rho_p(r)=\frac{e}{4\pi}\left(\frac{0.24\delta(r)}{r^2}+\frac{0.28\beta^2e^{-\beta r}}{r}+\frac{0.60\gamma^2e^{-\gamma r}}{r}\right)$$

for the proton, and

$$\rho_n(r)=\frac{e}{4\pi}\left(\frac{0.64\delta(r)}{r^2}+\frac{0.28\beta^2e^{-\beta r}}{r}-\frac{0.60\gamma^2e^{-\gamma r}}{r}\right) \tag{2.4}$$

for the neutron, with $\beta=2.162\times10^{13}$ cm^{-1} and $\gamma=3.162\times10^{13}$ cm^{-1}. The mean-square radius of ρ_p and ρ_n is 0.85×10^{-13} cm and zero, respectively. The charge distribution (2.2) or (2.3) modifies the Coulomb potential e^2/r between two protons. In order to get the modified Coulomb potential, it seems enough to assume that the proton charge is always distributed in a spherically symmetric way around the center of the proton as if it were alone, because the polarization of the charge is estimated to be very small. Thus we find the Coulomb potential between two protons as follows:

$$V_Y(r)=\frac{e^2}{r}(1-e^{-\beta_1 r}-\tfrac{1}{2}\beta_1 re^{-\beta_1 r})\quad\text{for}\quad \rho_Y, \tag{2.5}$$

$$V_E(r)=\frac{e^2}{r}\left[1-\left(1+\frac{11\beta_2 r}{16}+\frac{3(\beta_2 r)^2}{16}+\frac{(\beta_2 r)^3}{48}\right)e^{-\beta_2 r}\right]\quad\text{for}\quad \rho_E. \tag{2.6}$$

To construct the Coulomb potential between two protons with the charge distribution (2.4), we need to know the potential between $\rho_1=(e\beta^2/4\pi)(e^{-\beta r}/r)$ and $\rho_2=(e\gamma^2/4\pi)(e^{-\gamma r}/r)$, which is given by

$$V_{12}(r)=\frac{e^2}{r}\left(1+\frac{\beta^2e^{-\gamma r}}{\gamma^2-\beta^2}+\frac{\gamma^2e^{-\beta r}}{\beta^2-\gamma^2}\right). \tag{2.7}$$

If we take the limit $\gamma\rightarrow\beta=\beta_1$, (2.7) is reduced to (2.5). The Coulomb potential V_1 between the proton with the distribution ρ_1 and the other point proton is obtained by taking the limit, $\gamma\rightarrow+\infty$.

$$V_1(r)=\frac{e^2}{r}(1-e^{-\beta r}). \tag{2.8}$$

The Coulomb potential between two protons, or between proton and neutron, is expressed in terms of V_Y, V_{12}, and V_1. The general formula for the Coulomb potential between arbitrary charge distributions $\rho(r)$ and $\rho'(r)$ is given in Appendix I. The Coulomb energy of He^3 is the expectation value of this modified Coulomb potential with the wave function (2.1). The calculation can be carried out analytically (see Appendix II). The results are given in Table I.

TABLE I. Coulomb energy of He^3 in MeV.

Charge distribution	Nuclear force: Exponential	Yukawa
Point nucleon	0.986	1.246
Yukawa (proton)	0.839	0.996
Exponential (proton)	0.833	0.985
Hofstadter-Herman	0.799	0.929
Experimental	0.764	

Let us first consider the case of an exponential potential. The CE is about 1.0 MeV for point nucleons. The more elaborate calculation of Pease and Feshbach also gives almost the same value. They assume a Yukawa shape for the central and the tensor potentials (without repulsive cores) which are adjusted by the low energy two-body data (scattering lengths, effective ranges, quadrupole moment of the deuteron) and the binding energy of the triton. The CE value is reduced by 15–16% for the Yukawa and the exponential charge distributions, and by 19% for Hofstadter and Herman's distribution.

The mean-square radius would be the main factor involved in the reduction of CE. The reason is the following. The CE can be expanded in terms of the moments of the charge distribution. If high-momentum components are not strongly represented in the wave function, the effect of the finite size is mainly determined by the first term (the second moment) only.

ΔE in Table II is the difference between the CE of point protons and that of extended protons with various mean-square radii. (The Yukawa distribution is assumed for Table II.) ΔE is proportional to $\langle r^2\rangle$ for small value of $\langle r^2\rangle$. The reduction of the CE in the case of Hofstadter and Herman's distribution is somewhat

[6] R. Hofstadter and R. Herman, Phys. Rev. Letters **6**, 293 (1961). Also see L. N. Hand, D. G. Miller, and R. Wilson, *ibid.* **8**, 110 (1962); and S. Goto, Progr. Theoret. Phys. (Kyoto) **27**, 273 (1962).

larger than in the Yukawa or exponential distribution. This is reasonable because the former has a mean-square radius of 0.85×10^{-13} cm in contrast with 0.80×10^{-13} cm in the latter.

The numerical values in Table I for the Yukawa potential are somewhat larger, and the discrepancy between the calculated values and the experimental value is also larger. The Yukawa potential, however, gives too large a binding energy for He^3 (12.49 MeV is the value obtained by the variational method; the true value will be larger by 1 MeV or so), so that the extension of the wave function is too small and gives too large a value of the CE. For this reason the values for the Yukawa potential are somewhat unrealistic. They are included in Table I just for the purpose of comparison.

3. INCLUSION OF THE HARD CORES IN NUCLEAR FORCES

If the nuclear force has a repulsive core (with radius D), the CE is already smaller even if a point nucleon is assumed, since the repulsive core prevents the nucleons from approaching each other too closely. To estimate the effect of the finite size we shall take the following example.

The exponential central potential with a hard core is assumed for the nuclear force. The radius of the hard core, D, is varied from $D=0$ to $D=0.6\times10^{-13}$ cm. The depths and ranges of the potential are adjusted so as to fit the low-energy data of two-nucleon system as given in Sec. 2. The wave function of He^3 is

$$\psi=\prod_{\text{cyclic}}\left(e^{-\mu(r_{ij}-D)}-e^{-\nu(r_{ij}-D)}\right)\quad\text{for all}\quad r_{ij}>D \tag{3.1}$$
$$=0\quad\text{otherwise,}$$

where r_{ij} is the distance between ith and jth nucleons, and μ and ν are adjustable parameters which are determined by the standard variational method. (For details see reference 5.) Only the proton is assumed to have a finite size with the Yukawa distribution (2.2). The Coulomb potential is thus given by (2.5). The Coulomb energies are summarized in Table III. The CE is reduced about by 15% for $D=0$. The reduction becomes smaller and smaller as D increases. The CE of He^3 as well as the binding energy of H^3 seems to fit the experimental data if the hard-core radius is taken to be $\sim0.2\times10^{-13}$ cm.

TABLE II. The reduction of the Coulomb energy of He^3 as a function of the mean-square radius of the charge distribution.

$\langle r^2\rangle$ in 10^{-26} cm²	E in MeV
ϵ $(0\leq\epsilon\ll1)$	0.483ϵ
0.25	0.075
0.64	0.147
1.00	0.196

TABLE III. The Coulomb energy of He^3 for the point and extended protons. The Yukawa charge distribution (2.2) with a mean-square radius of 0.8×10^{-13} cm is assumed for extended protons.

Hard core D in 10^{-13} cm	Binding energy of H^3 in MeV	Coulomb energy of He^3 in MeV	
		Point	Extended
0.0	10.3	0.99	0.84 (15%)
0.2	8.8	0.84	0.77 (8%)
0.6	5.8	0.69	0.66 (3%)
Experimental	8.5	0.76	

4. DISCUSSION AND CONCLUSION

The discussion in Secs. 2 and 3 has been based on oversimplified models.

(*i*) *Wave function.* Since we have used the simple trial function, (2.1) and (3.1), some errors will be included in the results obtained. It is estimated that the trial function is, however, fairly good[5] so far as the central potential is assumed: The true binding energy will not be larger by more than 1 MeV than that obtained by the variational method with the trial function (2.1) and (3.1). The calculated CE will be larger than the true value by only several percent.

(*ii*) *Nuclear forces.* The largest error will come from our omission of tensor forces. The inclusion of the tensor potential in the two-body force may decrease the Coulomb energy. This fact can be understood by comparing the calculated values of Pease and Feshbach with Table I of the present paper. Using the Yukawa well for both central and tensor parts, Pease and Feshbach obtained 1.01–1.04 MeV for the CE, while Table I gives 1.245 MeV for the Yukawa well. The reason may be as follows. The tensor potential mixes D states (and P states) in He^3. These states are pushed away by the centrifugal forces, so that the CE arising from these mixed parts of the wave function decreases considerably. There is also another reason, that the inclusion of tensor force reduces the binding energy so that the nucleus has a larger extension.

Thus, the numerical values given in Table I, II, and III may have only an approximate meaning. However, we shall follow a different logic. We shall first take a standard value for the CE of He^3. Pease and Feshbach adjusted six parameters for the nuclear forces (without repulsive core) so as to fit the deuteron data, the low-energy scattering data, and the binding energy of H^3. They get ~1 MeV as the CE of He^3. Therefore, we may believe that the figure, 0.986 in Table I, can be taken as a standard value. Table I shows that the CE will be reduced to 0.80–0.84 MeV from ~1.0 MeV if the finite size of nucleon is taken into account. There seems a possibility that the CE is reduced to the experimental value even if the hard core is not assumed. The values listed in the column "point" in Table III may be considered as standard values for the following reason. The values of μ (which determines the damping

TABLE IV. The ratio of V_Y in (2.5) or V_E in (2.6) with the usual Coulomb potential e^2/r. The unit of length is the mean-square radius of the charge distribution.

r in $\langle r^2\rangle^{\frac{1}{2}}$	$(r/e^2)V_Y$	$(r/e^2)V_E$
0.0	0.000	0.000
0.2	0.237	0.213
0.4	0.441	0.408
0.6	0.601	0.571
0.8	0.721	0.700
1.0	0.808	0.797
1.2	0.869	0.865
1.4	0.905	0.913
1.6	0.937	0.945
1.8	0.958	0.966
2.0	0.973	0.979

of the wave function in the outer region) are 0.479, 0.462, 0.457 and 0.450×10^{13} cm^{-1} for $D=0.0$, 0.2, 0.4, and 0.6×10^{-13} cm, respectively. The constancy of μ suggests that the diminution of the CE as a function of D is not due to the behavior of the wave function at large distances, which is mainly determined by the binding energy, but is due to the vanishing of the wave function at short distances. Thus, Table III may show the effect of the finite size with reasonable accuracy. When D is large, the effect mainly comes from the outer region of the charge distribution. Therefore, the effect is not always determined by the mean-square radius only. However, a different charge distribution, such as the exponential type, will give similar results, because both give rather similar (modified) Coulomb potentials. (See Table IV.)

Tables I and III will supplement the result obtained in the reference 5 which treats the effect of hard cores on the binding energies of H^3 and He^3.

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APPENDIX I. COULOMB POTENTIAL BETWEEN EXTENDED CHARGED PARTICLES

The spherically symmetrical charge distributions $\rho_1(r)$ and $\rho_2(r)$ are assumed not to be polarized when they approach each other. [$4\pi\int_0^\infty \rho(r)r^2dr=$total charge.] The potential $V(r)$ is given by

$$V(r)=\frac{16\pi^2}{r}\int_0^r s^2\rho_1(s)ds\int_0^{r-s}t^2\rho_2(t)dt+\frac{4\pi^2}{r}\int_0^\infty s\rho_1(s)ds$$
$$\times\int_{|r-s|}^{|r+s|}[2(st+sr+tr)-(s^2+t^2+r^2)]t\rho_2(t)dt$$
$$+16\pi^2\int_0^\infty s^2\rho_1(s)ds\int_{r+s}^\infty t\rho_2(t)dt$$
$$+16\pi^2\int_r^\infty s\rho_1(s)ds\int_0^{s-r}t^2\rho_2(t)dt.$$

The last two terms are equal when $\rho_1=\rho_2$. The second integral is often simplified by the transformation, $(s,t)\rightarrow(x,y)$, $x=s+t$, $y=s-t$. If $\rho_2(t)$ is the point-charge distribution, the second and third terms vanish. The potential between the point charged particle (charge e) and the extended charged particle with the distribution ρ is given by

$$V(r)=\frac{4\pi e}{r}\int_0^r s^2\rho(s)ds+4\pi e\int_r^\infty s\rho(s)ds.$$

APPENDIX II. EXPLICIT FORMULAS FOR COULOMB ENERGY

In the course of the Coulomb energy calculation the following formulas are needed. Although some of them are already given in reference 4, these are included for completeness. ψ is defined in (2.1) with $N=7/(2^{10}\mu^6)$. $d\tau\equiv r_{12}r_{23}r_{31}dr_{12}dr_{23}dr_{31}$. The domain of integration is restricted by the trigonometric inequality: $x\equiv2\mu/\beta$.

$$\int\frac{1}{\beta r_{12}}e^{-\beta r_{12}}\psi^2d\tau=\frac{8x^3}{7(2x+1)^2}\left[1+\frac{2x}{1+2x}+\frac{1}{2}\left(\frac{2x}{1+2x}\right)^2\right],$$

$$\int e^{-\beta r_{12}}\psi^2d\tau=\frac{1}{7}\left(\frac{2x}{1+2x}\right)^3\left[2+3\left(\frac{2x}{1+2x}\right)+2\left(\frac{2x}{1+2x}\right)^2\right],$$

$$\int\beta r_{12}e^{-\beta r_{12}}\psi^2d\tau=\frac{16}{7}\left\{\frac{2}{1+x}\left(\frac{x}{1+x}\right)^2-\frac{1}{4(1+2x)}\left(\frac{2x}{1+2x}\right)^2\left[\left(\frac{x}{1+x}\right)^3\left(\frac{2x}{1+2x}\right)+2\left(\frac{x}{1+x}\right)^2\left(\frac{2x}{1+2x}\right)^2\right.\right.$$
$$\left.\left.+\frac{5}{2}\left(\frac{x}{1+x}\right)\left(\frac{2x}{1+2x}\right)^3+2+\frac{3}{2}\left(\frac{2x}{1+2x}\right)\right]\right\},$$

$$\int(\beta r_{12})^2e^{-\beta r_{12}}\psi^2d\tau=\frac{96}{7}\left\{\frac{x^2}{1+x}-\frac{x^2(1+4x)}{(1+2x)^5}-\frac{x^6}{(1+x)(1+2x)^4}\left[\frac{1}{(1+x)^3}+\frac{4}{(1+x)^2(1+2x)}\right.\right.$$
$$\left.\left.+\frac{10}{(1+x)(1+2x)^2}+\frac{20}{(1+2x)^3}\right]\right\}.$$

The calculation of the CE with the wave function (3.1) can be carried out by the formulas given in reference 5, when a charge distribution of Yukawa type is assumed. The factor $e^{-\beta_i D}$ in the formula for potential energy[7] should be deleted. If an exponential charge distribution is assumed, the calculation becomes much more complicated.

[7] T. Ohmura, M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) 17, 329 (1957), line 5.

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Distorted-Waves Theory of Double-Excitation by Inelastic Scattering*

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It is shown how the distorted-waves theory of inelastic scattering can give rise to angular distributions with "anomalous phase," such as observed in recent experiments with alpha particles. These occur when the scattering takes place through second-order effects; excitation of a two-phonon vibrational state is studied explicitly. There is an important cancellation between the amplitude for simultaneous excitation and part of that for successive excitation, without which the anomalous phase would not be observed. These conclusions are contrasted with the predictions of a theory using plane-wave Born approximation. Further study of the cancellation is suggested as a sensitive test of the optical model.

I. INTRODUCTION

The inelastic scattering of strongly-absorbed projectiles is known[1,2] to show a clearly-defined phase-rule relationship among the oscillatory angular distributions for exciting different states of a given target. According to this rule, angular distributions corresponding to odd values of the angular momentum transfer L have their maxima and minima out of phase with those of angular distributions corresponding to even L. Also the odd-L patterns are in phase with the elastic angular distributions. The conditions under which these rules should be reliable are not very restrictive,[3,4] and are fulfilled very well by (α,α') reactions at energies of about 40 MeV. The use of such reactions in the excitation of the lowest 2^+ and 3^- levels has provided many experimental verifications of the phase rule.

It was especially interesting, therefore, when several (α,α') excitations of known 4^+ levels of even nuclei were found[5,6] to be in phase with the 3^- excitations of the same nuclei, and out of phase with the 2^+ excitations. The 4^+ levels in question are believed to be part of the two-phonon triplet of quadrupole vibrational states of these nuclei, and it was suggested that two-phonon excitation somehow reverses the phase rule.

Indeed Blair[7] and Drozdov[8] applied a formula of the adiabatic theory, for the scattering from a "black" ellipsoid, and discovered that in this model the part of the scattering amplitude which is second order in the deformation does give agreement with experiment. The

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[1] J. S. Blair, Phys. Rev. 115, 928 (1959).
[2] E. Rost and N. Austern, Phys. Rev. 120, 1375 (1960).
[3] N. Austern, Ann. Phys. (New York) 15, 299 (1961).
[4] E. Rost, Ph.D. thesis, University of Pittsburgh, 1961; and to be published.
[5] R. Beurtey, P. Catillon, R. Chaminade, M. Crut, H. Faraggi, A. Papineau, J. Saudinos, and J. Thirion, Comp. rend. 252, 1756 (1961).
[6] H. Broek, J. L. Yntema, and B. Zeidman, Phys. Rev. 126, 1514 (1962).
[7] J. S. Blair (private communication).
[8] S. I. Drozdov, J. Exptl. Theoret. Phys. (USSR) 38, 499 (1960) [translation: Soviet Phys.—JETP 11, 362 (1960)].