rotation interaction, respectively, the Σ appearing in the exponent being 0 or 1 according to whether the Σ term in question is the Σ^+ or Σ^- term.

At a glance it is clear that formula (1) and the first members of the right side of (2), which then come directly from the spin-orbit interaction, are formally identical; thus, upon the Λ-type splitting of the $^3\Pi$ term in Hund's case (b) the spin-spin interaction again supplies exactly the same expressions as the perturbations transmitted by the spin-orbit interaction. Consequently, it is without doubt that the coefficient of the first term of the formula (2) is correctly: $\beta=\epsilon+\frac{1}{2}C_0$, that is, the Λ splittings observed experimentally can be traced back to the combined effect of these two interactions. Obviously, the same is true of the Λ-type splitting of the $^4\Pi$ terms in Hund's case (b) whose explicit expressions the author, by taking into account the spin-orbit interaction, has formerly given also.[10]

Summarizing, it can be concluded that in the multiplet splittings of the Σ and Π terms of higher multiplicity than the doublet, as well as in the Λ-type doubling of the Π terms in the first approximation, the spin-spin and the spin-orbit interactions give expressions of the same form, and the splittings observed experimentally can be attributed to the combined effect of the two interactions.

[10] I. Kovács, Can. J. Phys. **36**, 329 (1958).

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Harmonics in the Scattering of Light by Free Electrons

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The presence of harmonics in the scattering of light by free electrons is pointed out. The cross section for the second harmonic is proportional to the energy flux and to the square of the wavelength of the incident light. Thus experiments in which intense monochromatic microwave beams are scattered by plasmas are the most promising for the detection of the effect.

INTRODUCTION

THE author and Punhani[1,2] have recently pointed out that classical electrodynamics predicts the presence of harmonics in the radiation scattered by *free* electrons. The reason for this is simple to understand if we write the equations of motion of the electron in a plane light wave field taking account not only of its electric vector,[3] $\mathbf{E}=\mathbf{E}_0\cos\omega$, but also of the magnetic vector, $\mathbf{H}=\mathbf{H}_0\cos\omega$. Since the velocity, $\mathbf{v}$, of the electron is proportional to $\sin\omega$, the magnetic force, $e\mathbf{v}\times\mathbf{H}$, is proportional to $\sin\omega\cos\omega$, i.e., to $\sin 2\omega$. One can therefore at once see that the electron will oscillate not only with frequency k_0 but also with $2k_0$, which, in turn, will give rise to scattered electromagnetic waves of half the incident wavelength. The following calculation shows that if intense monochromatic beams of the same order as those obtained with ruby lasers are available in the microwave region, the cross section for the second harmonic will be comparable with that for the principal mode (the Thomson cross section), but the effect is too small in the optical region to be measurable with the present devices [see Eqs. (14)–(18)].

[1] Vachaspati and Sudarshan L. Punhani, Proc. Nat. Inst. Sci (India) (to be published).

[2] Vachaspati, Proc. Nat. Inst. Sci. (India) (to be published).

[3] $\omega=k_0z_0-\mathbf{k}\cdot\mathbf{z}$; $(z_0,\mathbf{z})$ are the time-space coordinates of the electron; $|\mathbf{H}_0|=|\mathbf{E}_0|=E_0$.

ELECTRON EQUATIONS OF MOTION

It has been shown in reference 2 that the relativistic electron equations of motion,

$$m\dot{v}_\mu=ev^\alpha f_{\alpha\mu}\cos\omega,\quad \mu,\alpha=0,1,2,3, \tag{1}$$

where

$$(f_{01},f_{02},f_{03})=\mathbf{E}_0,\quad (f_{23},f_{31},f_{12})=\mathbf{H}_0,$$
$$v_\mu=dz_\mu/d\tau,\quad \dot{v}_\mu=dv_\mu/d\tau,\quad d\tau=dz_0[1-(d\mathbf{z}/dz_0)^2]^{1/2},$$

(m= electron mass, speed of light = 1), can be solved exactly. If we use a coordinate system in which the electron has no translatory motion, the origin is taken at the mean position around which it oscillates, and the clocks are so adjusted that the zero of the observer's time, z_0, coincides with the zero of the electron proper time, τ, the solution of (1) is

$$k_0z_0=k_0'\tau-\tfrac{1}{8}q'\sin(2k_0'\tau), \tag{2a}$$

$$k_0\mathbf{z}=-\mathbf{e}_o(eE_o/mk_0')\cos(k_0'\tau)-\mathbf{n}_o\tfrac{1}{8}q'\sin(2k_0'\tau), \tag{2b}$$

where

$$\mathbf{e}_o=\mathbf{E}_o/E_o,\quad \mathbf{n}_o=\mathbf{k}/k_0,\quad k_0'=k_0(1+\tfrac{1}{2}q)^{1/2},$$
$$q'=(k_0/k_0')^2q,\quad q=e^2E_o^2/m^2k_0^2. \tag{3}$$

The occurrence of k_0' in (2a) and (2b) does not imply that the frequency of the observed radiation will change. The reason is that the frequency which an observer measures is the one that is coupled to his own time, z_0, and not the one that is coupled to the proper time of the electron. If we take the time average Eq. (2a), we find that $k_0'\tau$ equals $k_0 z_0$. This shows that the observed frequency does not change from k_0 to k_0' [see also (7)].

RADIATION FIELD OF THE ELECTRON

The expressions (2a,b) can be used to find the retarded electric and magnetic radiation fields which the electron produces:

$$\mathbf{E}^{\text{scatt}}=\mathbf{H}^{\text{scatt}}\times\mathbf{n}, \qquad \mathbf{H}^{\text{scatt}}=(e/r)(\mathbf{n}\times\mathbf{M}),$$
$$r=|\mathbf{x}-\mathbf{z}|\approx|\mathbf{x}|, \qquad \mathbf{n}=(\mathbf{x}-\mathbf{z})/r\approx\mathbf{x}/r, \tag{4}$$

where the approximate equalities hold for large distances from electron;

$$\mathbf{M}=(v_0-\mathbf{n}\cdot\mathbf{v})^{-3}[(\dot{v}_0-\mathbf{n}\cdot\dot{\mathbf{v}})\mathbf{v}-(v_0-\mathbf{n}\cdot\mathbf{v})\dot{\mathbf{v}}], \tag{5}$$

where $x_\mu=(x_0,\mathbf{x})$ is the point of observation. These expressions should be evaluated at that instant, $\tau=\tau_{\text{ret}}$, at which

$$x_0-z_0=r. \tag{6}$$

The retarded time, τ_{ret}, can be readily found. Using (2a) in (6) we get

$$k_0'\tau_{\text{ret}}-(q'/8)\sin(2k_0'\tau_{\text{ret}})=k_0(x_0-r),$$

from which

$$k_0'\tau_{\text{ret}}=\psi+\tfrac{1}{2}\phi, \tag{7}$$

where

$$\psi=k_0(x_0-r),$$

and

$$\phi=\lim_{n\to\infty}\phi_n;$$

here the sequence $\phi_1, \phi_2, \phi_3, \cdots$ is defined by means of the relation

$$\phi_n=(q'/4)\sin(2\psi+\phi_{n-1}), \quad n=1, 2, \cdots,$$

with $\phi_0=0$. In the lowest order which is sufficient for our purpose, this gives

$$\phi=\phi_1=(q'/4)\sin 2\psi.$$

When (2a,b) are inserted in (5), τ_{ret} is replaced by means of (7), and the result is expanded in powers of E_0, we get after a rather lengthy calculation[4]

$$\mathbf{M}=\mathbf{M}^{(1)}\cos\psi+\mathbf{M}^{(2)}\sin 2\psi+\mathbf{M}^{(3)}\cos 3\psi, \tag{8}$$

where

$$\mathbf{M}^{(a)}=\mathbf{e}_0A^{(a)}+\mathbf{n}_0B^{(a)}, \quad a=1, 2, 3,$$
$$A^{(1)}=-(eE_0/m)[1-\tfrac{1}{2}q(\tfrac{5}{8}-3\cos^2\alpha)], \qquad B^{(1)}=-\tfrac{3}{8}(eE_0/m)q\cos\alpha,$$
$$A^{(2)}=-\tfrac{3}{2}k_0q\cos\alpha, \qquad B^{(2)}=-\tfrac{1}{2}k_0q,$$
$$A^{(3)}=-(eE_0/2m)q(9/8-\cos\theta-3\cos^2\alpha), \quad B^{(3)}=\tfrac{5}{8}(eE_0/m)q\cos\alpha,$$
$$\cos\alpha=(\mathbf{n}\cdot\mathbf{e}_0), \quad \cos\theta=(\mathbf{n}\cdot\mathbf{n}_0).$$

CROSS SECTION

The time average of the square of the magnetic field in (4) can be easily obtained. We find that up to terms of order E_0^4

$$\langle\mathbf{H}^{\text{scatt}^2}\rangle_{\text{av}}=(e^2/2r^2)[(\mathbf{n}\times\mathbf{M}^{(1)})^2+(\mathbf{n}\times\mathbf{M}^{(2)})^2].$$

The explicit expressions for $(\mathbf{n}\times\mathbf{M}^{(1)})^2$ and $(\mathbf{n}\times\mathbf{M}^{(2)})^2$ are

$$(\mathbf{n}\times\mathbf{M}^{(1)})^2=(e^2E_0^2/m^2)[\sin^2\alpha-(q/8)(5\sin^2\alpha+6\cos\theta\cos^2\alpha-6\sin^2 2\alpha),$$

and

$$(\mathbf{n}\times\mathbf{M}^{(2)})^2=\tfrac{1}{16}(e^2E_0^2/m^2)q[9\sin^2 2\alpha-24\cos^2\alpha\cos\theta+4\sin^2\theta].$$

The differential cross section, $d\sigma$, for the scattering of radiation into a solid angle $d\Omega$ is given by

$$d\sigma/d\Omega=(e^2/E_0^2)(\mathbf{n}\times\mathbf{M})^2,$$

and can be written as the sum of two cross sections, $d\sigma^{(1)}$ and $d\sigma^{(2)}$, which describe, respectively, the scattering into waves of the same frequency as the original radiation and into waves of twice the original frequency[5]:

$$d\sigma/d\Omega=d\sigma^{(1)}/d\Omega+d\sigma^{(2)}/d\Omega;$$
$$(d\sigma^{(1)}/d\Omega)_{\text{polarized light}}=(e^2/m)^2[\sin^2\alpha-\tfrac{1}{8}q(5\sin^2\alpha+6\cos\theta\cos^2\alpha-6\sin^2 2\alpha)], \tag{9}$$

and

$$(d\sigma^{(2)}/d\Omega)_{\text{polarized light}}=\tfrac{1}{16}(e^2/m)^2q[9\sin^2 2\alpha-24\cos^2\alpha\cos\theta+4\sin^2\theta]. \tag{10}$$

[4] It should be noticed that the arguments of the sine and the cosines involve $\psi=k_0(x_0-r)$ and not τ_{ret}. This results in somewhat different expressions for $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(3)}$ than those given in reference 1.

[5] The term proportional to q is somewhat different in (9) from that given in the Eq. (26b) of reference 1. Consequently, the expressions (11) and (13) below are also different from the previous results. The reason is that the condition (6) given above was not properly taken into account in reference 1.

If we have unpolarized incident light, we can take the average over the polarization angle α. Then

$$(d\sigma^{(1)}/d\Omega)_{\text{unpolarized light}}=d\sigma_T/d\Omega-\tfrac{1}{16}(e^2/m)^2q[18\cos^4\theta-6\cos^3\theta-7\cos^2\theta+6\cos\theta-1], \tag{11}$$

and

$$(d\sigma^{(2)}/d\Omega)_{\text{unpolarized light}}=\tfrac{1}{32}(e^2/m)^2q\sin^2\theta[27\cos^2\theta-24\cos\theta+17], \tag{12}$$

where

$$d\sigma_T/d\Omega=\tfrac{1}{2}(e^2/m)^2(1+\cos^2\theta)$$

is the well-known Thomson cross section and q is defined in (3).

The total cross sections obtained by integrating the above expressions over the solid angle are

$$\sigma^{(1)}=[1-(7/80)q]\sigma_T, \tag{13}$$

$$\sigma^{(2)}=(7/10)q\sigma_T, \tag{14}$$

$$\sigma_T=(8\pi/3)(e^2/m)^2.$$

If we introduce the speed of light, c, and the incident energy flux,

$$I_0=cE_0^2/(8\pi),$$

into our expressions, we can write

$$q=QI_0\lambda^2, \tag{15}$$

where

$$Q=2e^2/\pi m^2c^5, \tag{16}$$

and $\lambda=2\pi/k_0$ is the wavelength of the incident light. Numerically

$$Q=0.7\times10^{-10}\ \text{W}^{-1},$$

and therefore

$$\sigma^{(2)}=0.5\times10^{-17}I_0\lambda^2\sigma_T\ \text{cm}^2.$$

[The incident intensity I_0 is in ergs/(cm^2 sec); the wavelength λ is in cm.]

EXPERIMENTAL DETECTION AND CONCLUDING REMARKS

The monochromatic light beam produced by means of pulsed ruby optical masers, when focused, has been reported[6] to exhibit electric fields of the order of 10^5 V/cm. Since the electron mass is 1/2 MeV, we have 10^5 V$=mc^2/(5e)$ in cgs units; hence

$$QI_0=\frac{2e^2}{\pi m^2c^5}\left[\frac{c}{8\pi}(mc^2/5e)^2\right]=\frac{1}{100\pi^2}\ \text{cm}^{-2}=10^{-3}\ \text{cm}^{-2} \tag{17}$$

[see (16)]. Therefore

$$QI_0\lambda^2=0.5\times10^{-11},\quad (\lambda=6943\ \text{Å}\approx7\times10^{-5}\ \text{cm}). \tag{18}$$

The intensity of the second harmonic is thus 10^{-11} times that of the principal mode. However, this weak intensity is due essentially to the factor λ^2 which is very small in the optical region. If experiments are done with microwaves, the second harmonic will be much more significant. For waves of the same intensity as above but 1-cm wavelength, the second harmonic cross section will be 1/1000 of the Thomson cross section, which should be possible to detect experimentally.

The following points about the results obtained above may be noted:

(1) The double-frequency cross section, $\sigma^{(2)}$, is the more important the longer the incident wavelength is [see (14) and (15)].

(2) The second harmonic differential cross section for unpolarized light, (12), is maximum at about 120° angle of scattering.

(3) Radiation reaction effects will be unimportant to consider in $d\sigma^{(2)}/d\Omega$, because they are significant only for short-wave phenomena. For the same reason, quantum corrections are expected to be negligible.

(4) The same-frequency cross sections, $d\sigma^{(1)}/d\Omega$ and $\sigma^{(1)}$, are different from the Thomson cross sections, if the incident radiation is intense [Eqs. (9), (11), and (13)].

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[6] P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, Phys. Rev. Letters **7**, 118 (1961). T. H. Maiman, Nature **187**, 493 (1960). R. J. Collins, D. F. Nelson, A. L. Schawlow, W. Bond, C. G. B. Garrett, and W. Kaiser, Phys. Rev. Letters **5**, 303 (1960).