Scattering and Production Amplitudes with Unstable Particles*

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An S-matrix theory is developed for a system of strongly interacting particles in which unstable particles are included. The particular system considered is $\pi+N \leftrightarrow \pi+N$, $\pi+N \leftrightarrow \rho+N$, and $\rho+N \leftrightarrow \rho+N$. Only the one-pion-exchange interaction is included. The relationship to the strip approximation and the application to the higher resonances in pion-nucleon scattering are discussed. Complex singularities are evaluated, and their relationship to the generalized unitarity condition for the pion-pion system is stressed. An extended ND^{-1} method is used to develop a system of nonsingular, uncoupled, Fredholm integral equations, from which the transition amplitudes can be evaluated.

I. INTRODUCTION

IN the past few years experiments have revealed the existence of a large number of resonances in elementary particle reactions. Many physicists have found attractive the idea of considering these resonances as unstable particles, decaying via strong interactions, but to be treated on an equal basis with the stable particles. This poses the challenge to the S-matrix theory of strong interactions[1] of dealing with processes in which unstable particles occur as intermediate or external particles. Forces arising from the exchange of unstable particles have been the subject of intensive investigation; i.e., the exchange of a ρ meson ($J=I=1$ pion-pion resonance) in pion-pion scattering. In this paper we are concerned with processes such as $\pi+N \rightarrow \rho+N$, where the unstable particle occurs as an external particle, and with the effect of this process on elastic pion-nucleon scattering. One cannot, of course, insert the concept of an unstable external particle directly into an S-matrix theory, because in such a theory transitions are defined only between asymptotic states. Therefore, we consider processes such as $\pi+N \rightarrow \pi+\pi+N$, using the existence of the pion-pion resonance to reduce the complexity of the three-body state.

A specific reason for interest in such a program is the fact that recent theoretical work indicates that important effects are to be expected in elastic scattering in the energy region where the cross section in a competing inelastic channel is rising rapidly; for example, at the "threshold" for the production of an unstable particle or resonance state.[2–6] In pion-nucleon scattering, there are strong indications that the higher resonances may be understood in terms of the $J=1$, $I=1$ pion-pion resonance[7] (ρ-meson) in the pion production channel.[2,8] Thus, it is important that the Chew-Mandelstam program[1] for the calculation of low-energy scattering processes on the basis of analyticity and unitarity be extended to include some effects of inelastic channels. One such attempt is the "strip approximation" proposed by Chew and Frautschi,[9] based on the work of Mandelstam[10] and Cutkosky.[11] These authors noted that those portions of the double-spectral functions nearest to the physical scattering region could be expressed in terms of equations involving only two-body scattering processes. The approximation of including only these inelastic effects corresponds to the peripheral-collision model for high-energy collisions.[12] In both cases the longest range part of the interaction is assumed to dominate. This assumption becomes especially plausible if one attempts to calculate only high partial waves.

The strip approximation was applied by Ball and Frazer[2,13] to the calculation of inelastic effects on pion-nucleon scattering partial waves of angular momentum $l \geqslant 2$ in the energy range of the higher resonances. This amounted to calculating the diagram in Fig. 1 with the pions in a $J=1$, $I=1$ resonant state, and imposing the unitarity condition only in the pion-

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[1] For reviews of this theory, see Geoffrey F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961); and *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, Edinburgh, 1961).

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[10] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); Phys. Rev. **115**, 1741 and 1752 (1959).

[11] R. E. Cutkosky, Phys. Rev. Letters **4**, 624 (1960); J. Math. Phys. **1**, 429 (1960).

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[13] J. S. Ball and W. Frazer (to be published).

nucleon channel. Qualitative agreement was obtained in the assignment of quantum numbers, as well as rough agreement with the observed positions of the peaks. Similarly, indications were found that the peak in K^{-}-p scattering at about 1 BeV/c lab momentum is associated with the K^* production threshold.[14] Detailed quantitative calculations proved impossible because the inelastic effects calculated from the strip approximation diagram of Fig. 1 exceeded the unitarity limit. On the other hand, in reference 4, the unitarity condition was imposed on the transition amplitudes in all channels at the outset, while no detailed assumptions were made about the interactions. It is the purpose of the present paper to outline a method in which it is possible to calculate those inelastic effects which result from the longest range part of the interaction and at the same time preserve the unitarity condition.[15]

In order to make our program tractable it seems necessary to avoid the full complexity of the three-body states. We propose to do this by considering only that part of the reaction $\pi+N\rightarrow\pi+\pi+N$ in which two of the final particles are produced in a resonant state; i.e., $\pi+N\rightarrow\rho+N$ and $\pi+N\rightarrow\pi+N^*$. As a first approximation we shall consider in addition to the pion-nucleon channel only the ρ-nucleon channel, since the interaction leading to the ρ-nucleon state is the longest range interaction leading to a final state of one nucleon and two pions (see Fig. 2). In this respect we are remaining within the spirit of the strip approximation, although we must go beyond it in order to satisfy the requirements of unitarity. We shall do this by imposing requirements of analyticity and unitarity on the partial-wave amplitudes for the three processes $\pi+N\rightarrow\pi+N$, $\pi+N\rightarrow\rho+N$, and $\rho+N\rightarrow\rho+N$, solving the resulting equations by an extended ND^{-1} method.[16,17] The only interaction (left-hand cut) which

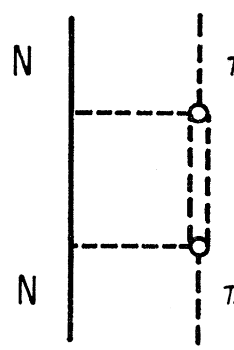

FIG. 1. A strip-approximation diagram for pion-nucleon scattering.

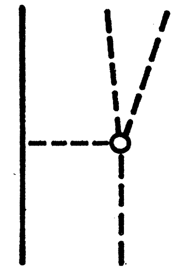

FIG. 2. One-pion-exchange interaction.

we shall include is that shown in Fig. 2. It is easily seen that the lowest order term in the solution of such a set of equations is just the strip-approximation diagram shown in Fig. 1 and calculated in references 2 and 13. The qualitative success of that calculation encourages us to believe that the present program may provide a quantitative explanation of the higher resonances in pion-nucleon scattering. However, we do not expect this program to be meaningful for the low partial waves in pion-nucleon scattering, nor for the production processes except perhaps for those events in which the two final pions are observed to be in resonance, because the longest range part of the interaction cannot be assumed to be dominant in these cases.

Let us now consider this program in more detail. In Sec. II we discuss the expressions to be used for the calculation of the discontinuities across the physical branch cuts. The well-known fact that production amplitudes have complex singularities[18] implies that the discontinuity across the physical cut is not simply related to the imaginary part. Therefore, the unitarity condition does not give us the discontinuity, and an appropriate modification proposed by Blankenbecler[17] is presented. We discuss the relationship of these equations to the unitarity condition. We then discuss the form these equations take when the two-pion system is in a resonant state.

In Sec. III we discuss the simplification of the discontinuity equations by projection of partial waves and by factorization of the sharp dependence of the amplitudes on the energy of the two-pion system. We introduce the one-pion-exchange interaction in Sec. IV, and describe its analytic properties. In Sec. V we formulate the integral equations satisfied by the transition amplitudes, and in Sec. VI we use the ND^{-1} method to reduce these equations to Fredholm form. The anomalous thresholds and complex singularities are dealt with by analytic continuation in the mass of the two-pion system.[19] Considerable care is required to insure that the complex singularities do not introduce violations of the generalized unitarity condition for the pion-pion system.

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II. DISCONTINUITIES ACROSS THE PHYSICAL BRANCH CUTS

We discuss in this paper an extension of the Chew-Mandelstam program[1] for the calculation of two-particle scattering amplitudes to processes involving unstable particles. An unstable particle is described here as a two-particle resonance, since all the particle states in an S-matrix theory must be stable. It is, therefore, necessary to discuss three-particle production and scattering amplitudes. To be specific, we consider the $\pi+N$ and $\pi+\pi+N$ states, and assume that the two pions are in a resonant state. For simplicity we take zero angular momentum for the resonance, and call it a ρ meson. We also neglect isotopic spin and nucleon spin.

For the amplitude T_{11} for the pion-nucleon elastic scattering process $\pi+N\rightarrow\pi+N$, we use the standard variable s, the square of the center-of-mass energy, and t, the invariant momentum transfer. For the production process $\pi+N\rightarrow\pi+\pi+N$, whose amplitude will be designated T_{21}, we will be concerned with the dependence on the variables s, t, and ω where ω is the square of the energy of the two final pions in their center-of-mass frame. In general, the production amplitude T_{21} depends on two more variables, but our restriction to processes in which the two pions are produced as a ρ meson corresponds precisely to neglect of this dependence. Similarly, the amplitude T_{22} for the process $\pi+\pi+N\rightarrow\pi+\pi+N$ will be considered as a function of s, t, ω plus an additional variable ω' corresponding to the energy squared of the incoming pair of pions in their center-of-mass system. According to time reversal and space reflection invariance the amplitude T_{12} for the inverse process $\pi+\pi+N\rightarrow\pi+N$ is equal to T_{21} while T_{22} is symmetric in ω and ω'.

We assume that as a function of the square of the center-of-mass energy of any pair of particles, the transition amplitudes T_{ij} which we are considering are analytic functions except for branch points in the region of physical energies, and poles associated with single-particle intermediate states. The physical values of these amplitudes are obtained by approaching each of the branch cuts from the *upper* half of the corresponding complex plane. In general, the location of each cut depends on the other variables of the transition amplitudes; for some range of values of these variables the cut extends below the physical threshold and into the complex plane. In this section we are concerned with the calculation of discontinuities across the cuts in the s and ω variables above the physical thresholds. We use the equations for these discontinuities obtained by Blankenbecler[17] from the L.S.Z. formalism[20] but emphasize their connection to the physical unitarity condition in the channel where s is the square of the total energy.

The unitarity condition in the s channel is, for all variables having physical values $[t<0;\omega,\omega'>4\mu^2, s>(M+\mu)^2]$,

$$\operatorname{Im}T_{11}(s_+,t)=\sum[T_{11}(s_+,t')T_{11}(s_-,t'')+T_{21}(s_+,t',\omega_+'')T_{21}(s_-,t'',\omega_-'')],\tag{2.1a}$$

$$\operatorname{Im}T_{21}(s_+,t,\omega_+)=\sum[T_{21}(s_+,t',\omega_+)T_{11}(s_-,t'')+T_{22}(s_+,t',\omega_+,\omega_+'')T_{21}(s_-,t'',\omega_-'')],\tag{2.1b}$$

$$\operatorname{Im}T_{22}(s_+,t,\omega_{1+},\omega_{2+})=\sum[T_{21}(s_+,t',\omega_{1+})T_{21}(s_-,t'',\omega_{2-})+T_{22}(s_+,t',\omega_{1+},\omega_+'')T_{22}(s_-,t'',\omega_{2-},\omega_-'')],\tag{2.1c}$$

where the primes designate intermediate variables, and the summation symbol $\sum$ represents the phase-space integrals over the 4-momenta q_i of the pions and nucleon in the intermediate states with total 4-momenta P,

$$\sum=\frac{1}{2}\int\Pi_i\left[\frac{d^4q_i}{(2\pi)^4}2\pi\delta(q_i^2-m_i^2)\theta(\epsilon_i)\right]\times(2\pi)^4\delta^4(\sum_i q_i-P).\tag{2.2}$$

The subscript $+(-)$ on the variables in Eqs. (2.1) indicates that the corresponding variable is taken above (below) the cut. In the right-hand side of Eqs. (2.1) we have assumed that the transition amplitudes satisfy a generalized Schwarz reflection principle

$$T_{ij}^*(x,y,\cdots)=T_{ij}(x^*,y^*,\cdots),\tag{2.3}$$

which is obviously satisfied by any perturbation graph.

The left-hand sides of Eqs. (2.1) can also be written

$$\begin{aligned}2i\operatorname{Im}T_{11}(s_+,t)&=T_{11}(s_+,t)-T_{11}(s_-,t),\\2i\operatorname{Im}T_{21}(s_+,t,\omega_+)&=T_{21}(s_+,t,\omega_+)-T_{21}(s_-,t,\omega_-),\\2i\operatorname{Im}T_{22}(s_+,t,\omega_{1+},\omega_{2+})&=T_{22}(s_+,t,\omega_{1+},\omega_{2+})\\&\quad-T_{22}(s_-,t,\omega_{1-},\omega_{2-}).\end{aligned}\tag{2.4}$$

These discontinuities given us by unitarity are not in a useful form for dispersion-theoretic calculations. We would like to know the discontinuities across the physical cuts in s when the other variables are held fixed. Consider, for example, T_{21}. The discontinuity we want to calculate is the following:

$$\begin{aligned}T_{21}(s_+,t,\omega_+)-T_{21}(s_-,t,\omega_+)&=2i\operatorname{Im}T_{21}(s_+,t,\omega_+)\\&\quad-[T_{21}(s_-,t,\omega_+)-T_{21}(s_-,t,\omega_-)].\end{aligned}\tag{2.5}$$

The term in brackets on the right-hand side of Eq. (2.5) is the discontinuity of T_{21} in ω for fixed values of

[20] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 425 (1955).

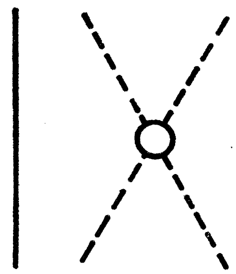

FIG. 3. Disconnected diagram for $\pi+\pi+N\to\pi+\pi+N$.

s and t. Retaining only two-pion intermediate states, we have for $\omega\geq 4\mu^2$

$$T_{21}(s,t,\omega_+)-T_{21}(s,t,\omega_-)=2ie^{i\delta(\omega)}\sin\delta(\omega)T_{21}(s,t,\omega_-),\quad (2.6)$$

where $\delta(\omega)$ is the S-wave pion-pion scattering phase shift. Equation (2.6) is a generalized form of the unitarity condition discussed by Mandelstam,[19] Cutkosky,[11] and Blankenbecler.[17]

Now let T_{22}^D be the contribution to the amplitude T_{22} due to a disconnected process in which the two pions scatter without interacting with the nucleon (see Fig. 3):

$$T_{22}^D=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')f(\omega),\quad (2.7)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final nucleon 3-momenta, and $f(\omega)$ is the S-wave pion-pion scattering amplitude,

$$f(\omega)=16\pi[\omega/(\omega-4\mu^2)]^{\frac{1}{2}}e^{i\delta(\omega)}\sin\delta(\omega).\quad (2.8)$$

Then Eq. (2.6) can be rewritten in the form

$$T_{21}(s,t,\omega_+)-T_{21}(s,t,\omega_-)=2i\sum T_{22}^D(s,t',\omega_+\omega_+'')T_{21}(s,t'',\omega_-'').\quad (2.9)$$

Substituting this expression in Eq. (2.5), we arrive at a relation for the discontinuity of T_{21} in s for fixed t and ω:

$$T_{21}(s_+,t,\omega)-T_{21}(s_-,t,\omega)=2i\sum T_{21}(s_+,t',\omega)T_{11}(s_-,t'')+T_{22}^C(s_+,t',\omega,\omega_+'')T_{21}(s_-,t'',\omega_-''),\quad (2.10)$$

where $T_{22}^C=T_{22}-T_{22}^D$ is the contribution to the amplitude T_{22} due to all the connected scattering processes. Since both sides of Eq. (2.10) are analytic functions of ω, this equation is valid for all values of ω. Equation (2.10) also follows directly from the generalized form of the unitarity condition in the s channel. The important thing to note is that any two of Eqs. (2.1b), (2.6), and (2.10), together with our assumption about analyticity, imply the validity of the remaining equation. The procedure that we adopt here is to use the discontinuity Eqs. (2.6) and (2.10). We emphasize that if *both* these equations are satisfied, then the unitarity condition in the s channel, Eq. (2.1b), is automatically satisfied also. We see in Sec. V that when complex singularities arise, considerable care is required to preserve the discontinuity equation in the ω channel.

Similar considerations apply to T_{22}. Here the discontinuity condition for the ω channel takes the form

$$T_{22}(s,t,\omega_+,\omega')-T_{22}(s,t,\omega_-,\omega')=2ie^{i\delta(\omega)}\sin\delta(\omega)T_{22}(s,t,\omega_-,\omega'),\quad (2.11)$$

which we assume to be valid by analytic continuation for all s and ω'. A similar expression exists for the discontinuity in ω'. With the help of these equations we are able, after some manipulation, to convert Eq. (2.1c) to the form

$$T_{22}^C(s_+,t,\omega_1,\omega_2)-T_{22}^C(s_-,t,\omega_1,\omega_2)=2i\sum\{T_{21}(s_+,t',\omega_1)T_{21}(s_-,t'',\omega_2)+T_{22}^C(s_+,t',\omega_1,\omega_+'')T_{22}^C(s_-,t'',\omega_2,\omega_-'')\}.\quad (2.12)$$

It is well known that the discontinuity conditions in ω and ω' can be satisfied quite easily by introducing the functions:

$$M_{21}(s,t,\omega)\equiv T_{21}(s,t,\omega)/f(\omega),\qquad M_{22}(s,t,\omega,\omega')\equiv T_{22}^C(s,t,\omega,\omega')/[f(\omega)f(\omega')].\quad (2.13)$$

It can readily be verified, using Eqs. (2.6) and (2.11), that M_{21} and M_{22} have no discontinuities in ω and ω' for ω, $\omega'>4\mu^2$. Our procedure will be to impose this condition on the equations we derive. It is possible to define other functions that have no discontinuities in ω and ω' by multiplying the right-hand side of Eq. (2.13) by any function analytic in the neighborhood of $\omega>4\mu^2$. However, we shall see later that Eq. (2.13) turns out to be the most useful separation in our scheme. We note that this is a particular form of the final-state interaction theorem of Watson.[21]

To complete our discussion, we should also include the unitarity condition in the t channel. However, the approximation we make here is to neglect the branch points of the transition amplitudes in the t variable, and keep only the pole term in T_{21} due to the exchange of a single π meson (see Fig. 2). This corresponds to the physical approximation of keeping only the longest range part of the force that produces the interaction in the s channel. We have

$$T_{21}=gf(\omega)/(t-\mu^2),\quad (2.14)$$

where g is the $\pi-N$ coupling constant. Note that G parity forbids a single-pion exchange in T_{11} and T_{22}. In our first approximation we attempted to neglect other interactions, but it turns out that it is then impossible to satisfy simultaneously the analyticity requirements and the unitarity conditions on the transition amplitudes. As we shall see later on, it is necessary to include at least the pole contribution from the single-particle states in Figs. 4 and 5 given by

$$T_{21}=gT_{11}/(\sigma-M^2),$$

[21] K. Watson, Phys. Rev. **95**, 228 (1954).

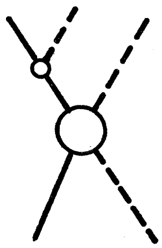

FIG. 4. Additional interaction necessary for consistent formulation of integral equations for T_{21}.

and

$$T_{22}(s,t,\omega',\omega)=\frac{gT_{21}(s,t,\omega')}{\sigma-M^2}+\frac{gT_{21}(s,t,\omega)}{\sigma'-M^2}, \qquad (2.15)$$

where σ is the energy square of one of the two pions and the nucleon in the $\pi+\pi+N$ state in their center-of-mass system.

III. PARTIAL-WAVE AMPLITUDES

The discontinuity equations, Eqs. (2.1a), (2.10), and (2.12), are greatly simplified if we expand our transition amplitudes T_{ij} in partial-wave amplitudes $T_{ij}{}^l$

$$T_{ij}=\sum_{l=0}^{\infty}(2l+1)T_{ij}{}^lP_l(\cos\theta), \qquad (3.1)$$

where θ is the nucleon scattering angle in the center-of-mass system.[22]

In terms of the partial amplitudes $M_{ij}{}^l$ defined by Eqs. (2.13) and (3.1), the discontinuity equations become, for $s\geq(M+\mu)^2$,

$$\begin{aligned}
[M_{11}{}^l(s_+)-M_{11}{}^l(s_-)]/2i&=M_{11}{}^l(s_+)\rho_1(s_+)M_{11}{}^l(s_-)+\textstyle\sum' M_{21}{}^l(s_+,\omega'')|f(\omega'')|^2M_{21}{}^l(s_-,\omega''),\\
[M_{21}{}^l(s_+,\omega)-M_{21}{}^l(s_-,\omega)]/2i&=M_{21}{}^l(s_+,\omega)\rho_1(s_+)M_{11}{}^l(s_-)+\textstyle\sum' M_{22}{}^l(s_+,\omega,\omega'')|f(\omega'')|^2M_{21}{}^l(s_-,\omega''),\\
[M_{22}{}^l(s_+,\omega,\omega')-M_{22}{}^l(s_-,\omega,\omega')]/2i&=M_{21}{}^l(s_+,\omega)\rho_1(s_+)M_{21}{}^l(s_-,\omega')+\textstyle\sum' M_{22}{}^l(s_+,\omega,\omega'')|f(\omega'')|^2M_{22}{}^l(s_-,\omega',\omega''),
\end{aligned} \qquad (3.2)$$

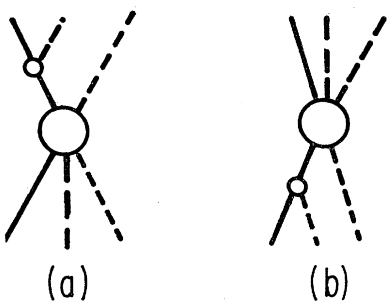

FIG. 5. Additional interactions necessary for consistent formulation of integral equations for T_{22}.

where the "summation" $\sum'$ now implies only an integration over ω'':

$$\textstyle\sum'=\theta(s-(M+2\mu)^2)\displaystyle\int_{4\mu^2}^{(s^{\frac{1}{2}}-M)^2}d\omega''\,\rho_2(s,\omega''), \qquad (3.3)$$

and

$$\rho_1(s)=\frac{1}{8\pi}\frac{q_1(s)}{\sqrt{s}}, \qquad (3.4)$$

$$\rho_2(s,\omega)=\frac{1}{4\pi^2}\frac{q_2(s,\omega)}{\sqrt{s}}\rho(\omega). \qquad (3.5)$$

In Eqs. (3.4) and (3.5) we have defined

$$\begin{aligned}
q_1(s)&=\frac{\{[s-(M+\mu)^2][s-(M-\mu)^2]\}^{\frac{1}{2}}}{2\sqrt{s}},\\
q_2(s,\omega)&=\frac{\{[s-(M+\sqrt{\omega})^2][s-(M-\sqrt{\omega})^2]\}^{\frac{1}{2}}}{2\sqrt{s}},
\end{aligned} \qquad (3.6)$$

and

$$\rho(\omega)=\frac{1}{16\pi}\left[\frac{\omega-4\mu^2}{\omega}\right]^{\frac{1}{2}}. \qquad (3.7)$$

We now use the fact that $|f(\omega)|$ is large only in the neighborhood of the resonance energy $\omega=m_\rho{}^2$ to simplify further Eqs. (3.2). Since the amplitudes M^l do not have singularities in ω on the physical cut, we may expand them in a Taylor series about $\omega=m_\rho{}^2$ *under the integral* in Eq. (3.2). If the resonance is narrow enough, a good approximation is to keep only the first term, and we then obtain for $s\geq(M+\mu)^2$

$$\begin{aligned}
[M_{11}{}^l(s_+)-M_{11}{}^l(s_-)]/2i&=M_{11}{}^l(s_+)\rho_1(s_+)M_{11}{}^l(s_-)+M_{21}{}^l(s_+,m_\rho{}^2)\rho_2(s_+)\theta[s-(M+2\mu)^2]M_{21}{}^l(s_-,m_\rho{}^2),\\
[M_{21}{}^l(s_+,\omega)-M_{21}{}^l(s_-,\omega)]/2i&=M_{21}{}^l(s_+,\omega)\rho_1(s_+)M_{11}{}^l(s_-)\\
&\qquad+M_{22}{}^l(s_+,\omega,m_\rho{}^2)\rho_2(s_+)\theta[s-(M+2\mu)^2]M_{21}{}^l(s_-,m_\rho{}^2),\\
[M_{22}{}^l(s_+,\omega,\omega')-M_{22}{}^l(s_-,\omega,\omega')]/2i&=M_{21}{}^l(s_+,\omega)\rho_1(s_+)M_{21}{}^l(s_-,\omega')\\
&\qquad+M_{22}{}^l(s_+,\omega,m_\rho{}^2)\rho_2(s_+)\theta[s-(M+2\mu)^2]M_{22}{}^l(s_-,\omega',m_\rho{}^2),
\end{aligned} \qquad (3.8)$$

where

$$\rho_2(s)=\int_{4\mu^2}^{(s^{\frac{1}{2}}-M)^2}d\omega'\rho_2(s,\omega')|f(\omega')|^2. \qquad (3.9)$$

Equation (3.8) is identical in form to the partial-wave unitarity condition for stable two-particle channels. The properties of the unstable particle are

[22] An elegant discussion of the partial-wave expansion of T_{21} and T_{22}, including spins, has been given by L. F. Cook, Jr., and B. W. Lee (to be published).

entirely contained in the generalized phase space integral, Eq. (3.9).

IV. THE INTERACTION SINGULARITIES

The interaction which is considered in this paper is the one-pion-exchange diagram in Fig. 2. This is, of course, a pole in the momentum transfer variable with residue $gf(\omega)$, where g is the pion-nucleon coupling constant, $f(\omega)$ is the $\pi-\pi$ scattering amplitude, and ω is the mass squared of the $\pi-\pi$ final state. Taking the S-wave projection of this pole, we find the contribution to T_{21} is $f(\omega)B(s,\omega)$, where

$$B(s,\omega)=\frac{\alpha(s,\omega)}{\pi}\ln\left[\frac{\beta(s,\omega)+\alpha(s,\omega)}{-\beta(s,\omega)+\alpha(s,\omega)}\right], \tag{4.1}$$

where

$$\beta(s,\omega)=\pi g\left[\frac{s^2-s(2M^2+\omega-\mu^2)+(M^2-\mu^2)(M^2-\omega)}{s}\right]^{-1}, \tag{4.2}$$

and

$$\alpha(s,\omega)=\pi sg\{[s-(M+\mu)^2][s-(M-\mu)^2]\times[s-(M-\sqrt{\omega})^2][s-(M+\sqrt{\omega})^2]\}^{-\frac{1}{2}}. \tag{4.3}$$

It is easily seen from Eq. (4.1) that $B(s,\omega)$ is singular at the zeros of the numerator or denominator of the argument of the log function. These zeroes are at $s=0$, and at $s=s_{\pm}$, given by

$$s_{\pm}(\omega)=M^2+\omega/2\pm[(\sqrt{\omega})/2\mu]\times[(4M^2-\mu^2)(4\mu^2-\omega)]^{\frac{1}{2}}. \tag{4.4}$$

If we let ω have a small positive imaginary part, i.e., $\omega\rightarrow\omega+i\delta$, where $\delta>0$, and keep $\omega\leq2\mu^2[1+(\mu/2M)]$, we can express the analytic properties of $B(s,\omega)$ in a convenient form by the representation

$$B(s,\omega+i\delta)=\frac{1}{\pi}\int_{\Gamma_+}\frac{\alpha(s',\omega+i\delta)}{s'-s}ds', \tag{4.5}$$

where the contour of integration Γ_+ and the cuts which determine the required branches of $\alpha(s,\omega)$ are shown in Fig. 6. As we increase ω the point s_+ moves to the right and reaches the physical threshold $s=(M+\mu)^2$ at $\omega=2\mu^2(1+\mu/2M)$. For $2\mu^2(1+\mu/2M)\leq\omega\leq4\mu^2$ the contour circles around the point $(M+\mu)^2$ and s_+ moves to the left below the cut in $\alpha(s,\omega)$ as shown in Fig. 6(b).

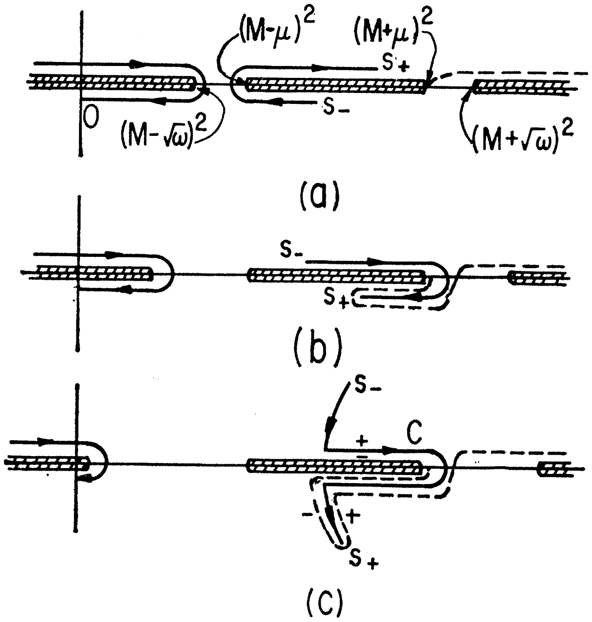

FIG. 6. Singularities of the one-pion-exchange interaction. The cross-hatched lines are the cuts of the function $\alpha(s,\omega)$; the solid lines denote the contour Γ; and the dashed line indicates the necessary deformation of the path of integration along the physical cuts of M_{21} and M_{22}. In Fig. 6(a) the situation for $\omega<2\mu^2(1+\mu/2M)$ is shown; in 6(b), for $2\mu^2(1+\mu/2M)<\omega<4\mu^2$, and 6(c), for $\omega>4\mu^2$. In Fig. 6(c) the contour C is shown, as well as the sense in which we have defined the discontinuity across it.

For $\omega\geq4\mu^2$, s_+ and s_- become complex, as is shown in Fig. 6(c). Nevertheless, it can be seen from Eq. (4.5) that $B(s,\omega)$ remains a real analytic function of s for fixed real ω. To show this one uses the relation

$$\alpha^*(s^*,\omega)=-\alpha(s,\omega), \tag{4.6}$$

which follows from our choice of branch cuts for $\alpha(s,\omega)$.

V. FORMULATION OF THE INTEGRAL EQUATIONS

In this section we derive the integral equations satisfied by the amplitudes $M_{ij}{}^l$. For simplicity we consider only the S-wave amplitudes $M_{ij}{}^0$. In the following we drop the superscript. If there were no anomalous thresholds or complex singularities present, the discontinuity equations, Eqs. (3.8), together with the interaction term, Eq. (4.1), would imply

$$M_{11}(s)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{11}(s_+')\rho_1(s_+')M_{11}(s_-')}{s'-s}+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',m_\rho^2)\rho_2(s_+')M_{21}(s_-',m_\rho^2)}{s'-s}, \tag{5.1a}$$

$$M_{21}(s,\omega)=B(s,\omega)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',\omega)\rho_1(s_+')M_{11}(s_-')}{s'-s}+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{M_{22}(s_+',\omega,m_\rho^2)\rho_2(s_+')M_{21}(s_-',m_\rho^2)}{s'-s}, \tag{5.1b}$$

$$M_{22}(s,\omega',\omega)=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',\omega')\rho_2(s_+')M_{21}(s_-',\omega)}{s'-s}+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{M_{22}(s_+',\omega,m_\rho^2)\rho_2(s_+')M_{22}(s_-',\omega,m_\rho^2)}{s'-s}, \tag{5.1c}$$

where $B(s,\omega)$ is the one-pion-exchange interaction [see Fig. 2 and Eq. (4.1)]. The phase-space factor ρ_2 has been defined in Eq. (3.9).

It is, however, well known that production amplitudes have complex singularities. Therefore, we should write down Eqs. (5.1) for ω, $\omega'<2\mu^2[1+(\mu/2m)]$, the value for which an anomalous threshold develops, and then continue analytically in ω and ω' to the value of physical interest, m_ρ^2.[19] However, we find that a straightforward application of this procedure leads to a contradiction of the discontinuity equation for the ω channel, Eq. (2.6). We showed in Sec. II that the factorization of the pion-pion scattering amplitude which was used to define the amplitudes M_{ij} in Eq. (2.13) should remove the physical cuts in ω and ω' from the M_{ij}. Therefore, if one performs the analytic continuation of Eqs. (5.1) by giving ω and ω' small imaginary parts, it should make no difference whether these imaginary parts are positive or negative. This is not the case. If positive imaginary parts are chosen, the physical cut beginning at $(M+\mu)^2$ must be deformed into the lower half-plane, whereas for negative imaginary parts it must be deformed into the upper half-plane. The discontinuity in ω can be calculated, and is found to be nonzero.

The meaning of this result can be seen most clearly by considering the simple perturbation graph discussed in Appendix A. It is found there that the discontinuity in ω produced by the continuation is identical to the term in the unitarity equation for the ω process which results from cutting the perturbation diagram in the manner shown in Fig. 7(c). Thus unitarity in ω cannot be satisfied unless the interaction shown in Fig. 8 is included. The natural generalization is that we must include the interactions shown in Figs. 4 and 5 in order to formulate a consistent set of equations for the M_{ij}. These diagrams are, of course, to be interpreted in the dispersion-theoretic sense: amplitudes which appear as residues of poles are to be evaluated on the mass shell.

Evaluating the diagram in Fig. 4 and projecting the outgoing pions into an S state we find, as described in Appendix A, the following contribution to $M_{21}(s,\omega)$:

$$\frac{M_{11}(s)}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)} ds'\frac{\rho_1(s_+')\alpha(s',\omega)}{s'-s}. \tag{5.2}$$

We cannot simply add this term to Eq. (5.1b) as it stands, because the physical cuts which it possesses are already included in the integrals in Eq. (5.1b). We want only the contribution of the left-hand singularities in s, which is given by

$$\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)} ds'\frac{\rho_1(s_+')\alpha(s',\omega)M_{11}(s')}{s'-s}. \tag{5.3}$$

Similarly, the diagrams of Figs. 5(a) and 5(b) suggest that the following terms be added to the equation for $M_{22}(s,\omega',\omega)$:

$$\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)} ds'\frac{\rho_1(s_+')M_{21}(s',\omega')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f(\omega')\rho(\omega')}\int_{s_+(\omega')}^{s_-(\omega')} ds'\frac{\rho_1(s_+')\alpha(s',\omega')M_{21}(s',\omega)}{s'-s}. \tag{5.4}$$

The remainder of this section is devoted to the analytic continuation of the integral equations for the M_{ij}. The additional terms we have introduced will result in equations which have no discontinuities in ω or ω' for physical values.

As we perform the continuation in ω and ω' our integral equations require modification when the contour Γ, along which $B(s,\omega)$ is singular, crosses the physical cut. Since this happens only in the vicinity of the point $s=(M+\mu)^2$, the integrals beginning at $s=(M+2\mu)^2$ are unaffected. Thus we see that Eq. (5.1a) is unchanged; i.e., there are no anomalous thresholds or complex singularities in the $\pi+N\rightarrow\pi+N$ channel, as is well known. Let us then proceed to the analytic continuation of the M_{21} equation. Our method is an extension of techniques used by Oehme[23] and by Blankenbecler, Goldberger, MacDowell, and Treiman[24] in discussions of anomalous thresholds. An anomalous threshold occurs when a singularity, such as the end of our contour Γ, moves from the second sheet through the physical cut and onto the physical sheet. The contour of integration of the physical cut is deformed to avoid the approaching singularity, and an anomalous threshold results. See Fig. 6 for an illustration of the contour deformation required when ω is given a positive imaginary part, as we shall assume throughout the following discussion.

Adding the term given in Eq. (5.3) to Eq. (5.1b) we have the following equation to continue in ω:

$$M_{21}(s,\omega)=\frac{1}{\pi}\int_{\Gamma(\omega)} ds'\frac{\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)} ds'\frac{\rho_1(s')\alpha(s',\omega)M_{11}(s')}{s'-s}$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{M_{21}(s_+',\omega)\rho_1(s_+')M_{11}(s_-')}{s'-s}+\cdots, \tag{5.5}$$

[23] R. Oehme, Z. Physik **162**, 426 (1961).

[24] R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

where we have used the dots to indicate the integral beginning at $s=(M+2\mu)^2$, which is unaffected by the continuation. In order to facilitate deformation of the contour from the physical cut into the lower half-plane, we use the relation

$$M_{21}^{\mathrm{II}}(s'-i\epsilon)=M_{21}(s'+i\epsilon), \tag{5.6}$$

where $M_{21}^{\mathrm{II}}(s)$ is the continuation of $M_{21}(s)$ through the cut in the inteval $(M+\mu)^2<s<(M+2\mu)^2$ onto the second sheet. The function $M_{21}^{\mathrm{II}}(s)$ can be found by manipulation of the unitarity relation in this interval,

$$\begin{aligned}M_{j1}(s+i\epsilon)-M_{j1}(s-i\epsilon)\\ =2i\rho_1(s+i\epsilon)M_{11}(s-i\epsilon)M_{j1}(s+i\epsilon),\end{aligned} \tag{5.7}$$

where $j=1$, 2, into the form

$$M_{j1}(s+i\epsilon)=\frac{M_{j1}(s-i\epsilon)}{1+2i\rho_1(s-i\epsilon)M_{11}(s-i\epsilon)}, \tag{5.8}$$

from which it is apparent that

$$M_{j1}^{\mathrm{II}}(s)=\frac{M_{j1}(s)}{1+2i\rho_1(s)M_{11}(s)}. \tag{5.9}$$

We have defined the branch cuts of $\rho_1(s)$ to be

$$s\geq(M+\mu)^2 \quad\text{and}\quad s\leq(M-\mu)^2,$$

from which it follows that

$$\rho_1^*(s^*)=-\rho_1(s). \tag{5.10}$$

Thus we can express the last term in Eq. (5.5) in the form

$$-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}^{\mathrm{II}}(s_-',\omega)\rho_1(s_-')M_{11}(s_-')}{s'-s}. \tag{5.11}$$

If we now continue to $\mathrm{Re}\omega>4\mu^2$, we find

$$\begin{aligned}M_{21}(s,\omega)=&\frac{1}{\pi}\int_{\Gamma(\omega)}ds'\frac{\alpha(s',\omega)}{s'-s}-\frac{1}{\pi}\int_{s_+(\omega)}^{(M+\mu)^2}ds'\frac{[\mathrm{disc}M_{21}^{\mathrm{II}}(s',\omega)]\rho_1(s_+')M_{11}(s')}{s'-s}\\ &-\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{(M+\mu)^2}ds'\frac{\rho_1(s')M_{11}^{\mathrm{II}}(s')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f(\omega)\rho(\omega)}\int_{(M+\mu)^2}^{s_-(\omega)}ds'\frac{\rho_1(s')M_{11}(s')\alpha(s',\omega)}{s'-s}+\cdots.\end{aligned} \tag{5.12}$$

To simplify the notation we have written ω, whereas $\omega+i\epsilon$ is intended throughout. Moreover, we have used the symbol "disc" to indicate the discontinuity of a function across a cut. The sense of the discontinuity is shown by the plus and minus signs in Fig. 6c, where we also define a contour $C(\omega)$. Taking the discontinuity of M_{21} across that portion of C which lies in the lower half-plane, and remembering that $\mathrm{disc}M_{11}=0$, we find

$$\mathrm{disc}M_{21}(s,\omega)=2i\alpha(s,\omega)+2i[\mathrm{disc}M_{21}^{\mathrm{II}}(s,\omega)]\rho_1(s)M_{11}(s)+\frac{2i\rho_1(s)M_{11}^{\mathrm{II}}(s)\alpha(s,\omega)}{f(\omega)\rho(\omega)}, \tag{5.13}$$

or

$$\mathrm{disc}M_{21}^{\mathrm{II}}(s,\omega)=2i\alpha(s,\omega)\left[1+\frac{\rho_1(s)M_{11}^{\mathrm{II}}(s)}{f(\omega)\rho(\omega)}\right]. \tag{5.14}$$

Combining Eqs. (5.12) and (5.14) we find that

$$\begin{aligned}M_{21}(s,\omega)=B(s,\omega)&-\frac{1}{\pi f^*(\omega)\rho(\omega)}\int_{s_+(\omega)}^{(M+\mu)^2}ds'\frac{\rho_1(s')M_{11}(s')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f(\omega)\rho(\omega)}\int_{(M+\mu)^2}^{s_-(\omega)}ds'\frac{\rho_1(s')M_{11}(s')\alpha(s',\omega)}{s'-s}\\ &+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')M_{21}(s_+',\omega)M_{11}(s_-')}{s'-s}+\cdots,\end{aligned} \tag{5.15}$$

where we have made use of the relation

$$\frac{1}{1+2i\rho_1M_{11}}=1-2i\rho_1M_{11}^{\mathrm{II}}, \tag{5.16}$$

and the unitarity condition for $\pi-\pi$ scattering in the form

$$\frac{1}{f(\omega+i\epsilon)}-\frac{1}{f(\omega-i\epsilon)}=-2i\rho(\omega). \tag{5.17}$$

We have defined the cuts of $\rho(\omega)$ to lie in the region $0\leq\omega\leq4\mu^2$.

We can obtain a further simplification of Eq. (5.15) by restricting our attention to $M_{21}(s,m_\rho^2)$, since

$f(m_\rho^2)\rho(m_\rho^2)=i$. We then obtain

$$M_{21}(s,m_\rho^2)=B(s,m_\rho^2)+\frac{i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\rho_1(s')M_{11}(s')\alpha(s',m_\rho^2)}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')M_{21}(s_+',m_\rho^2)M_{11}(s_-')}{s'-s}$$
$$+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{\rho_2(s_+')M_{21}(s_+',m_\rho^2)M_{21}(s_-',m_\rho^2)}{s'-s}. \quad (5.18)$$

Now if one carries out the continuation described above, but with $\mathrm{Im}\,\omega<0$, one finds exactly the same result. Thus the present formulation is consistent with the discontinuity equation for the ω channel. Moreover, one can verify from Eq. (5.15) that $M_{21}(s,\omega)$ is a real analytic function of s for physical values of ω. This reality would not hold if we had simply continued Eq. (5.1b) without adding the contribution of Eq. (5.3). As we shall see in the next section, the reality property greatly simplifies the task of solving the integral equations.

Finally, we must perform the continuation in ω and ω' of the following equation for $M_{22}(s,\omega',\omega)$:

$$M_{22}(s,\omega',\omega)=\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)}ds'\frac{\rho_1(s')M_{21}(s',\omega')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f(\omega')\rho(\omega')}\int_{s_+(\omega')}^{s_-(\omega')}ds'\frac{\rho_1(s')M_{21}(s',\omega)\alpha(s',\omega')}{s'-s}$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',\omega')\rho_1(s_+')M_{21}(s_-',\omega)}{s'-s}+\cdots. \quad (5.19)$$

Let us first continue in ω', holding ω fixed at a value less than $2\mu^2(1+\mu/2M)$. Again we give ω' a small positive imaginary part. Proceeding as above we find for $\mathrm{Re}\,\omega'>4\mu^2$

$$M_{22}(s,\omega',\omega)=-\frac{1}{\pi f^*(\omega')\rho(\omega')}\int_{s_+(\omega')}^{(M+\mu)^2}ds'\frac{\rho_1(s')M_{21}(s',\omega)\alpha(s',\omega')}{s'-s}+\frac{1}{\pi f(\omega')\rho(\omega')}\int_{(M+\mu)^2}^{s_-(\omega')}ds'\frac{\rho_1(s')M_{21}(s',\omega)\alpha(s',\omega')}{s'-s}$$
$$-\frac{1}{\pi f(\omega)\rho(\omega)}\int_{C(\omega)}ds'\frac{\rho_1(s')M_{21}(s',\omega')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',\omega')\rho_1(s_+')M_{21}(s_-',\omega)}{s'-s}+\cdots. \quad (5.20)$$

For $\omega'=m_\rho^2$ this equation simplifies to

$$M_{22}(s,m_\rho^2,\omega)=\frac{i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\rho_1(s')M_{21}(s',\omega)\alpha(s',m_\rho^2)}{s'-s}-\frac{1}{\pi f(\omega)\rho(\omega)}\int_{C(\omega)}ds'\frac{\rho_1(s')M_{21}(s',m_\rho^2)\alpha(s',\omega)}{s'-s}$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',m_\rho^2)\rho_1(s_+')M_{21}(s_-',\omega)}{s'-s}+\cdots. \quad (5.21)$$

If we now continue in ω we obtain

$$M_{22}(s,m_\rho^2,m_\rho^2)=\frac{2i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\rho_1(s')\alpha(s',m_\rho^2)}{s'-s}\{M_{21}(s_+',m_\rho^2)-i\alpha(s',m_\rho^2)[1+i\rho_1(s')M_{11}(s')]\}$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{M_{21}(s_+',m_\rho^2)\rho_1(s_+')M_{21}(s_-',m_\rho^2)}{s'-s}+\cdots. \quad (5.22)$$

The subscripts "+" in the integrals over the contour C designate that the contour is to be approached from the side marked "+" in Fig. 6(c). The derivation of Eq. (5.22) is straightforward, as long as one is careful to specify the limit to be used in defining M_{21} on $C(m_\rho^2)$.

VI. GENERALIZED N/D METHOD

In this section we show that a generalization of the N/D method reduces the rather formidable integral equations derived in Section V to a remarkably simple form, which appears to be quite amenable to numerical calculation. The N/D method, which was originated by Chew and Mandelstam,[1] has been generalized by Bjorken and Nauenberg[16] to multichannel reactions, and by Blankenbecler[17] to multiparticle reactions. Since we have reduced our problem to coupled integral equations in one variable by treating the pions as a resonant system, we need not use Blankenbecler's approach in its full generality. Nevertheless, we show that it is possible to obtain the dependence of the amplitudes M_{ij} on the energies of the pion pairs, ω and ω', as well as the dependence on s.

We define the relationship of the functions n_{ij} and d_{ij} to the amplitudes M_{ij} by the following equations:

$$n_{11}(s)=M_{11}(s)[1+d_{11}(s)]+M_{21}(s,m_\rho^2)d_{21}(s,m_\rho^2), \quad (6.1a)$$

$$n_{12}(s,\omega)=M_{11}(s)d_{12}(s,\omega)+M_{21}(s,\omega)+M_{21}(s,m_\rho^2)d_{22}(s,m_\rho^2,\omega), \quad (6.1b)$$

$$n_{21}(s,\omega)=M_{21}(s,\omega)[1+d_{11}(s)]+M_{22}(s,\omega,m_\rho^2)d_{21}(s,m_\rho^2), \quad (6.1c)$$

$$n_{22}(s,\omega',\omega)=M_{21}(s,\omega')d_{12}(s,\omega)+M_{22}(s,\omega',m_\rho^2)\times d_{22}(s,m_\rho^2,\omega)+M_{22}(s,\omega',\omega). \quad (6.1d)$$

We use small n's and d's to emphasize the fact that these quantities differ from the usual N's and D's in that the δ_{ij} term is absent in d_{ij}; i.e., if we ignored anomalous thresholds and complex singularities we would write

$$d_{11}(s)\equiv-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{ds'}{s'-s}\rho_1(s_+')n_{11}(s'), \quad (6.2a)$$

$$d_{12}(s,\omega)\equiv-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{ds'}{s'-s}\rho_1(s_+')n_{12}(s',\omega), \quad (6.2b)$$

$$d_{21}(s,m_\rho^2)\equiv-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}\frac{ds'}{s'-s}\rho_2(s_+')n_{21}(s',m_\rho^2), \quad (6.2c)$$

$$d_{22}(s,m_\rho^2,\omega)\equiv-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}\frac{ds'}{s'-s}\rho_2(s_+')n_{22}(s',m_\rho^2,\omega), \quad (6.2d)$$

and

$$n_{11}(s)=\frac{1}{\pi}\int_{\Gamma(m_\rho^2)}\frac{ds'}{s'-s}\alpha(s',m_\rho^2)d_{21}(s',m_\rho^2), \quad (6.3a)$$

$$n_{12}(s,\omega)=B(s,\omega)+\frac{1}{\pi}\int_{\Gamma(m_\rho^2)}\frac{ds'}{s'-s}\alpha(s',m_\rho^2)d_{22}(s',m_\rho^2,\omega), \quad (6.3b)$$

$$n_{21}(s,\omega)=B(s,\omega)+\frac{1}{\pi}\int_{\Gamma(\omega)}\frac{ds'}{s'-s}\alpha(s',\omega)d_{11}(s'), \quad (6.3c)$$

$$n_{22}(s,\omega',\omega)=\frac{1}{\pi}\int_{\Gamma(\omega')}\frac{ds'}{s'-s}\alpha(s',\omega')d_{12}(s',\omega). \quad (6.3d)$$

Equations (6.2) can be regarded as the definition of d_{ij}. Then Eqs. (6.3) follow from Eqs. (6.1), (6.2), and the equations for the discontinuities of M_{ij}, Eqs. (3.8). These sets of equations have the required property that in the weak-coupling limit $M_{11}=M_{22}=0$, whereas $M_{21}(s,\omega)=B(s,\omega)$.

We could now combine Eqs. (6.2) and (6.3) to form uncoupled linear integral equations for either the n_{ij} or the d_{ij}. Before doing this, however, we must undertake the principal task of this section, namely, the modification of Eqs. (6.2) and (6.3) to include the complex singularities discussed in the previous section. Again we cannot simply make an analytic continuation in ω and ω' of these equations because the result, in contradiction to the unitarity condition for the ω process, depends upon whether ω and ω' are given positive or negative imaginary parts. We must include the additional interaction found necessary in the previous section. In order to do this we formulate n and d equations directly from the integral equations obtained for the M_{ij}, Eqs. (5.1), (5.15), and (5.20).

We shall proceed in a manner which may seem rather arbitrary, but the final result will be easy to justify.

In order to keep the reader's attention as long as possible, we first specialize to the case $\omega=\omega'=m_\rho^2$, since the equations are simpler for this case. The general case will be treated in Appendix B. Let us also abbreviate the notation by writing $n_{12}(s)$ for $n_{12}(s,m_\rho^2)$, and similarly for the rest of the n_{ij} and d_{ij} and also for the M_{ij}. Finally, we write C and Γ for $C(m_\rho^2)$ and $\Gamma(m_\rho^2)$, the contours defined in Fig. 6; $B(s)$ for $B(s,m_\rho^2)$, and $\alpha(s)$ for $\alpha(s,m_\rho^2)$.

First we note that Eqs. (6.2c, d) for d_{21} and d_{22} remain unchanged, since they involve integrals beginning at $s=(M+2\mu)^2$ and thus never interact with the contour Γ. For d_{11} let us adopt the ansatz

$$d_{11}(s)=\frac{1}{2\pi i}\int_C ds'\frac{\mathrm{disc}\,d_{11}(s')}{s'-s}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')n_{11}(s')}{s'-s}, \quad (6.4)$$

where the sense of the discontinuity across the contour C is shown in Fig. 6, and where $\mathrm{disc}d_{11}(s)$ is to be determined. Now let us derive an equation for n_{11} by calculating its singularities from Eq. (6.1a). Using Eq. (5.18) to obtain the discontinuity of M_{21} across C,

$$\mathrm{disc}M_{21}(s)=2i\alpha(s)[1+i\rho_1(s)M_{11}(s)], \tag{6.5}$$

we obtain, suppressing the argument s,

$$\mathrm{disc}n_{11}=M_{11}\,\mathrm{disc}d_{11}+2i\alpha(1+i\rho_1M_{11})d_{21}. \tag{6.6}$$

In order that M_{11} not appear in $\mathrm{disc}n_{11}$, we must choose

$$\mathrm{disc}d_{11}=2\alpha\rho_1d_{21}. \tag{6.7}$$

Then n_{11} satisfies Eq. (6.3a). Similarly, we find that

$$\mathrm{disc}n_{12}=M_{11}\,\mathrm{disc}d_{12}+2i\alpha(1+i\rho_1M_{11})(1+d_{22}). \tag{6.8}$$

As before, to eliminate the explicit dependence of n_{12} on M_{11}, we are led to the choice:

$$\mathrm{disc}d_{12}=2\alpha\rho_1(1+d_{22}), \tag{6.9}$$

which leaves Eq. (6.3b) unchanged.

In order to calculate the discontinuities of n_{21} and n_{22} we need the following relation, derived from Eq. (5.22):

$$\mathrm{disc}M_{22}=-4\rho_1\alpha[M_{21}-i\alpha(1+i\rho_1M_{11})]. \tag{6.10}$$

It is then straightforward to show that, after some cancellations,

$$\begin{aligned}\mathrm{disc}n_{21}&=2i\alpha(1+d_{11})-2\alpha\rho_1[M_{11}(1+d_{11})+M_{21}d_{21}]\\&=2i\alpha(1+d_{11})-2\alpha\rho_1n_{11}.\end{aligned} \tag{6.11}$$

Since the first term is the result one obtains by using Eq. (6.3c), we are forced to add an extra term to this equation. Finally, the result for n_{22} is

$$\mathrm{disc}n_{22}=2i\alpha d_{12}-2\rho_1\alpha n_{12}. \tag{6.12}$$

Let us now gather together the n and d equations which we have derived for the case $\omega=\omega'=m_\rho^2$:

$$d_{11}(s)=-\frac{i}{\pi}\int_C ds'\frac{\rho_1(s')\alpha(s')d_{21}(s')}{s'-s}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')n_{11}(s')}{s'-s}, \tag{6.13a}$$

$$d_{12}(s)=-\frac{1}{\pi}\int_C ds'\frac{\rho_1(s')\alpha(s')[1+d_{22}(s')]}{s'-s}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')n_{12}(s')}{s'-s}, \tag{6.13b}$$

$$d_{21}(s)=-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{\rho_2(s_+')n_{21}(s')}{s'-s}, \tag{6.13c}$$

$$d_{22}(s)=-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{\rho_2(s_+')n_{22}(s')}{s'-s}, \tag{6.13d}$$

$$n_{11}(s)=\frac{1}{\pi}\int_\Gamma ds'\frac{\alpha(s')d_{21}(s')}{s'-s}, \tag{6.14a}$$

$$n_{12}(s)=B(s)+\frac{1}{\pi}\int_\Gamma ds'\frac{\alpha(s')d_{22}(s')}{s'-s}, \tag{6.14b}$$

$$n_{21}(s)=B(s)+\frac{1}{\pi}\int_\Gamma ds'\frac{\alpha(s')d_{11}(s')}{s'-s}+\frac{i}{\pi}\int_C ds'\frac{\rho_1(s')\alpha(s')n_{11}(s')}{s'-s}, \tag{6.14c}$$

$$n_{22}(s)=\frac{1}{\pi}\int_\Gamma ds'\frac{\alpha(s')d_{12}(s')}{s'-s}+\frac{i}{\pi}\int_C ds'\frac{\rho_1(s')\alpha(s')n_{12}(s')}{s'-s}. \tag{6.14d}$$

One can now verify that the M_{ij} given by Eqs. (6.1) and the above equations have the correct discontinuities, as derived in Sec. V.

The patient reader who has followed us to this point may wonder what has been gained by converting the three formidable equations for the M_{ij} into eight apparently equally formidable equations for the n_{ij} and the d_{ij}. The answer is found in the relative simplicity of the integral equations satisfied by the n_{ij}, which we now formulate. First consider Eq. (6.14a) for $n_{11}(s)$. Substituting for d_{21} according to Eq. (6.13c), we obtain by partial fractions

$$n_{11}(s)=\int_{(M+2\mu)^2}^{\infty}ds'\rho_2(s_+')K(s,s')n_{21}(s'), \tag{6.15}$$

where we define the kernel $K(s,s')$ by

$$K(s,s')\equiv[B(s)-B(s')]/\pi(s-s'). \tag{6.16}$$

Similarly, one obtains

$$n_{12}(s)=B(s)+\int_{(M+2\mu)^2}^{\infty}ds'\rho_2(s_+')K(s,s')n_{22}(s'). \tag{6.17}$$

The equation for n_{21} is more complicated, but by straightforward substitution one can express n_{21} in terms of n_{11} and d_{21}. Then substituting for these quantities according to Eqs. (6.15) and (6.13c) we find the following uncoupled linear integral equation for n_{21}:

$$n_{21}(s)=B(s)+\int_{(M+2\mu)^2}^{\infty}ds'\rho_2(s_+')L(s,s')n_{21}(s'), \tag{6.18}$$

where the kernel $L(s,s')$ is

$$L(s,s')=\int_{M(+\mu)^2}^{\infty} ds''\rho_1(s_+'')K(s,s'')K(s',s'')$$
$$+\frac{i}{\pi}\int_C ds''\rho_1(s'')\alpha(s'')\left[\frac{K(s_+'',s')}{s''-s}+\frac{K(s_-'',s)}{s''-s'}\right], \quad (6.19)$$

which can also be written in the form

$$L(s,s')=\int_{(M+\mu)^2}^{\infty} ds''\rho_1(s_+'')K(s,s'')K(s',s'')$$
$$+\frac{i}{\pi}\int_C ds''\rho_1(s'')\alpha(s'')\left[\frac{K(s_+'',s')}{s''-s}+\frac{K(s_+'',s)}{s''-s'}-\frac{2i\alpha(s'')}{\pi(s''-s)(s''-s')}\right]. \quad (6.20)$$

From the latter form it can easily be shown, with the help of Eqs. (4.6) and (5.10) that $L(s,s')$ is real for real s and s'. This would not be true if we had not included the additional interactions discussed in the previous section.

The reality of this kernel greatly increases the prospect of solving Eq. (6.18) by numerical methods. Moreover, we note that Eq. (6.18) is a nonsingular Fredholm equation of the second kind, since the complex singularities appear only in the calculation of the kernel.

Similarly, we find for n_{22} the following Fredholm integral equation:

$$n_{22}(s)=\bar{B}(s)+\int_{(M+2\mu)^2}^{\infty} ds'\rho_2(s_+')L(s,s')n_{22}(s'), \quad (6.21)$$

where we define

$$\bar{B}(s)=\int_{(M+\mu)^2}^{\infty} ds'\rho_1(s')K(s,s')B(s')$$
$$+\frac{i}{\pi}\int_C ds'\frac{\alpha(s')\rho_1(s')}{s'-s}[2B(s_+')-B(s)-2i\alpha(s')]. \quad (6.22)$$

Note that the integral equation satisfied by n_{22} differs from that satisfied by n_{21} only in the inhomogeneous term.

The procedure to be followed in solving for the amplitudes M_{ij} is then the following: First calculate the inhomogeneous term $\bar{B}$ and the kernel L, and solve the Fredholm equations for n_{21} and n_{22}. This is a routine numerical problem. Then calculate n_{11}, n_{12}, and the four d_{ij} by quadratures, using Eqs. (6.15), (6.17), and (6.13). Finally, one calculates M_{ij} from the matrix equation

$$M=n(I+d)^{-1}. \quad (6.23)$$

This procedure appears to be quite tractable with the help of existing computers. The important point is that the integral equations to be solved are not a great deal more complicated than those appearing in single-channel problems with normal thresholds. The additional complication of the complex singularities manifests itself only in quadratures which must be done to calculate kernels and to calculate the amplitudes from the functions n_{21} and n_{22}.

The generalization required to calculate the dependence of M_{ij} on ω and ω' is discussed in Appendix B.

VII. COMPARISON WITH THE STRIP APPROXIMATION

Let us now examine the relationship of the solution of the equations we have derived to the strip approximation.[9] If we solve these equations formally by iteration, we find that the lowest order contribution to $\mathrm{Im}M_{11}(s)$ is

$$\mathrm{Im}M_{11}(s)=\rho_2(s)B^2(s,m_\rho^2). \quad (7.1)$$

We wish to compare this formula with the strip approximation result for our spinless pion-nucleon model.

The strip approximation calculation proceeds along the lines first outlined by Mandelstam. In the energy region in which no more than $2\pi+N$ intermediate states contribute, it amounts to calculation of the contribution to $\mathrm{Im}M_{11}(s)$ arising from the diagram in Fig. 1. The projection of partial waves is discussed in reference 13. The result for the amplitude M_{11} calculated from the diagram of Fig. 1 is

$$\mathrm{Im}M_{11}(s)=\frac{1}{4\pi^2\sqrt{s}}\int_{4\mu^2}^{(s^{\frac{1}{2}}-M)^2} d\omega\rho(\omega)|f(\omega)|^2 \times q_2(s,\omega)[B(s,\omega)]^2, \quad (7.2)$$

where the pion-pion interaction has been assumed to be entirely S wave, and where $f(\omega)$ is the pion-pion scattering amplitude in this state.

As before, we use the fact that $|f(\omega)|$ is large only for $\omega\approx m_\rho^2$, and evaluate $B(s,\omega)$ at $\omega=m_\rho^2$. We then obtain the same result as that found in Eq. (7.1) for the lowest order interative solution of our equations.

VIII. SUMMARY

We have developed in this paper a procedure in which unstable particles can be included on a similar footing with stable particles in an S-matrix theory of strong interactions. We treated the unstable particle as a resonance in the scattering of its decay products. To be specific, we described how the effect of the ρ meson can be included in the treatment of the pion-nucleon interaction by the methods of dispersion theory. We ignored, however, the complication of spin and isotopic spin in order to keep the discussion as simple as possible while emphasizing the new features associated with the treatment of unstable particles. The essential new point is the appearance of the mass of the unstable particle as an additional variable in the transition amplitudes. In treating the unstable ρ meson as a two-pion resonant state we assumed that the transition amplitudes were analytic functions of the square of the mass of the ρ meson with a branch point corresponding to the physical threshold for pion-pion scattering. We then required that the discontinuity across the cut $\omega \geq 4\mu^2$ satisfy a generalized unitarity condition. A similar condition was imposed on the discontinuity of the transition amplitude across the physical cut in the total energy square variable s. These two conditions, together with the requirements of analyticity of the amplitudes in s and ω, were shown to imply that the physical unitarity condition in the s channel is satisfied. The important thing to realize is that the requirement of physical unitarity is not sufficient to derive integral equations for the transition amplitudes in this problem which includes three-particle contributions. The reason is that the unitarity condition gives a relation for the simultaneous discontinuity of the amplitudes in both the s and the ω variable across their physical cuts. In order to evaluate these discontinuities separately generalized unitarity conditions have to be introduced. To simplify the resulting equations we have confined our discussion to the treatment of partial waves. Furthermore, we included only the pole contribution in the momentum transfer variable due to a single-pion exchange. This gives the longest range part of the interaction between pions, nucleons and ρ mesons, and is expected to be the dominant contribution for high partial waves. However, to illustrate the general procedure we have confined our remarks to S waves. At this stage we have written down dispersion relations in the s variable, keeping the ω variable fixed well below its physical threshold. In this case only normal thresholds occur, as is well known from the study of anomalous thresholds in perturbation theory. Using the analytic properties in ω we then proceed to increase the value of ω to its physical domain $4\mu^2<\omega<(s^{\frac{1}{2}}-M)^2$. The resulting equations, however, do not satisfy the discontinuity equations in ω. To preserve this condition it is necessary to include further contributions to our amplitudes in which the two pions in the $\pi+\pi+N$ state do not interact. A detailed discussion is given in the text and an example is discussed in perturbation theory in Appendix A. Finally, the resulting coupled integral equations are reduced by an extension of the ND^{-1} method to nonsingular Fredholm integral equations which can be solved by straightforward numerical methods.

This method, therefore, allows a generalization of the Chew-Mandelstam technique to include unstable particles. We hope this may prove useful in the treatment of the recent large number of such particles which have been discovered experimentally.

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APPENDIX A

In this Appendix we use a simple example from perturbation theory to illustrate the relation between the discontinuity equations in s (Eq. 2.10) and ω (Eq. 2.6) and the unitarity condition. In particular, we demonstrate that disconnected graphs produce discontinuities in ω, not in s, and that the complex singularities in the s plane for the production amplitude are closely connected with the phase of $\pi-\pi$ scattering.[25] The necessity of adding the nucleon pole diagram to satisfy the discontinuity equation in ω is also shown.

Consider the Feynman diagram in Fig. 7 for the T_{12} amplitude, where the three vertices correspond to the $\pi-N$ coupling constant, and constant scattering amplitudes T_{11} for $\pi-N$ scattering and f for $\pi-\pi$ scattering. From perturbation theory we find

$$T_{21}^{(1)}=iT_{11}fg\int \frac{d^4k}{(2\pi)^4}\{(k^2-\mu^2+i\epsilon)[(q_2-k)^2-\mu^2+i\epsilon][(p+q_1-k)^2-M^2+i\epsilon]\}^{-1}, \tag{A1}$$

where p, q_1, and q_2 are the four-momenta of the incident nucleon, pion, and outgoing $\pi-\pi$ system, respectively. For $q_2^2\equiv\omega$ sufficiently small, the discontinuity across the physical s cut is given by the imaginary part of $T_{21}(s_+,\omega)$ which can be obtained directly from Eq. (A1), by cutting the diagram in Fig. 7 along the line (a). The only contribution for $\omega<4\mu^2$ is

$$\mathrm{Im}T_{21}^{(1)}(s_+,\omega)=-\frac{fT_{11}g}{2}\int \frac{d^4k}{(2\pi)^4}2\pi\delta(k^2-\mu^2)\theta(k_0)2\pi\delta[(p+q_1-k)^2-M^2]\theta(p_0+q_{20}-k_0). \tag{A2}$$

Carrying out the integration over k we obtain

$$\mathrm{Im}T_{21}^{(1)}(s_+,\omega)=T_{11}\rho_1(s)fB(s,\omega),\quad s\geq(M+\mu)^2, \tag{A3}$$

where $B(s,\omega)$ is the Born term given in Eq. (4.1) and ρ_1 is the phase-space factor defined in Eq. (3.4). It follows from Eq. (A1) that $T_{21}^{(1)}(s,\omega)$ is (for $\omega<2\mu^2+\mu^3/M$) analytic in the s plane except for the cut $s\geq(M+\mu)^2$. Then using Eq. (A3) we obtain the integral representation

$$T_{21}^{(1)}(s_+,\omega)=\frac{fT_{11}}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\frac{\rho_1(s_+')B(s',\omega)}{(s'-s)}. \tag{A4}$$

[25] This connection has been pointed out by Blankenbecler and Tarski, reference 18.

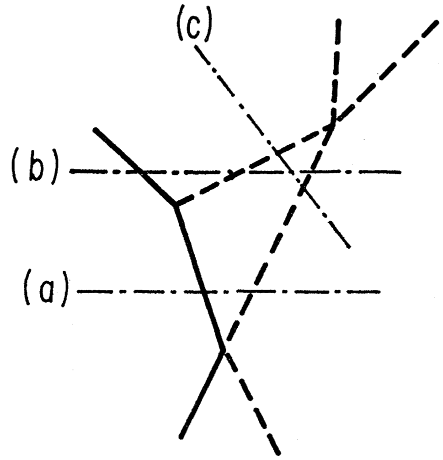

FIG. 7. Perturbation graph contributing to T_{21}.

If we now continue ω in the *upper* half-plane to physical values, the contour of integration in Eq. (A4) must be deformed downward to avoid the singularities in $B(s,\omega_+)$ as shown in Fig. 6. When this contour is collapsed around the cut in $B(s,\omega_+)$, $T_{21}^{(1)}$ may be written

$$T_{21}^{(1)}(s,\omega_+)=\frac{T_{11}f}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')B(s',\omega)}{(s'-s)}+2i\frac{T_{11}f}{\pi}\int_{s_+(\omega_+)}^{(M+\mu)^2}ds'\frac{\rho_1(s')\alpha(s',\omega)}{(s'-s)},\quad\text{(A5)}$$

where the only requirement on the contour in the second term of Eq. (A5) is that it does not cross the branch cuts of $\alpha(s,\omega)$ indicated in Fig. 6.

On the other hand, if ω is continued in the *lower* half-plane, the contour in Eq. (A4) must be deformed upward to avoid the cut in $B(s,\omega_-)$. The resulting expression for T_{21} is

$$T_{21}^{(1)}(s,\omega_-)=\frac{T_{11}f}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s_+')B(s',\omega)}{(s'-s)}-2i\frac{T_{11}f}{\pi}\int_{(M+\mu)^2}^{s_+(\omega_-)}ds'\frac{\rho_1(s')\alpha(s',\omega)}{(s'-s)}.\quad\text{(A6)}$$

Comparing Eqs. (A5) and (A6) it is clear that $T_{21}(s,\omega)$ has a branch point at $\omega=4\mu^2$, with a discontinuity in ω given by

$$T_{21}^{(1)}(s,\omega_+)-T_{21}^{(1)}(s,\omega_-)=2iB_1(s,\omega)f\rho(\omega)\quad\text{(A7)}$$

for $\omega>4\mu^2$, where

$$B_1(s,\omega)=\frac{T_{11}}{\pi\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)}ds'\frac{\rho_1(s')\alpha(s',\omega)}{(s'-s)}.\quad\text{(A8)}$$

This amplitude corresponds to the diagram in Fig. 8, in which the two pions are projected into an S wave in their own center-of-mass system. The discontinuity can, of course, be obtained directly from Eq. (A1) by cutting the diagram in Fig. 7 along line (c).

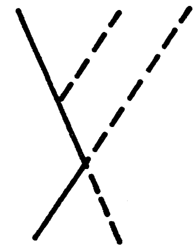

FIG. 8. Additional graph contributing to T_{21}.

Since $B_1(s,\omega)$ has no discontinuity in ω for $\omega\geq 4\mu^2$, then according to Eq. (A7) the first-order amplitude $T_{21}=B_1+T_{21}^{(1)}$ satisfies the discontinuity equation in ω, Eq. (2.6), to lowest order in f. This fact led us to the conjecture that a term analogous to B_1 must be added to the integral equations for the T_{ij} in order that these amplitudes satisfy the discontinuity equation in ω. This is shown in detail in Sec. V.

APPENDIX B. THE DEPENDENCE OF THE M_{ij} ON ω AND ω'

In Sec. VI we formulated a generalized N/D method for reducing the integral equations for the M_{ij} to tractable form. However, we considered there only the special case of $\omega=\omega'=m_\rho^2$. This is adequate for the calculation of M_{11}, which was our primary motive in undertaking this work. Nevertheless, once one has obtained the M_{ij}, n_{ij}, and d_{ij} for $\omega=\omega'=m_\rho^2$, it is only a matter of quadratures to calculate $M_{21}(\omega)$ for all values of ω. One can also calculate $M_{22}(s,\omega',\omega)$, but the procedure is more complicated. To show this we must derive the equations of Sec. VI in a more general form.

The equations for n_{11} and d_{11}, Eqs. (6.13a) and (6.14a), need not be changed. In order to calculate the rest of the n_{ij} and d_{ij}, we first derive the correct equations for ω, $\omega'<2\mu^2(1+\mu/2M)$. We shall then continue these equations to physically interesting values of ω and ω'. For $d_{12}(s,\omega)$ we make the ansatz

$$d_{12}(s,\omega)=-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s')n_{12}(s',\omega)}{s'-s}+\frac{1}{2\pi i}\int_{C(\omega)}ds'\frac{\mathrm{disc}[d_{12}(s',\omega);C(\omega)]}{s'-s}+\frac{1}{2\pi i}\int_{C(m_\rho^2)}ds'\frac{\mathrm{disc}[d_{12}(s',\omega);C(m_\rho^2)]}{s'-s}.\quad\text{(B1)}$$

We have been forced here to introduce a more elaborate notation for the discontinuity, one which specifies the contour. Then, observing from Eq. (5.5) that the discontinuity of $M_{21}(s,\omega)$ across $C(\omega)$ is

$$\mathrm{disc}[M_{21}(s,\omega);C(\omega)]=2i\alpha(s,\omega)\left[1-\frac{\rho_1(s)M_{11}(s)}{\rho(\omega)f(\omega)}\right],\quad\text{(B2)}$$

we calculate from Eq. (6.1b) that the discontinuity of $n_{12}(s,\omega)$ across $C(\omega)$ is

$$\mathrm{disc}[n_{12}(s,\omega);C(\omega)]=M_{11}(s)\,\mathrm{disc}[d_{12}(s,\omega);C(\omega)]+2i\alpha(s,\omega)\left[1-\frac{\rho_1(s)M_{11}(s)}{\rho(\omega)f(\omega)}\right]. \quad \text{(B3)}$$

In order to eliminate the dependence of n_{12} on M_{11} we choose

$$\mathrm{disc}[d_{12}(s,\omega);C(\omega)]=2i\alpha(s,\omega)\rho_1(s)/[\rho(\omega)f(\omega)]. \quad \text{(B4)}$$

Moreover, if we calculate the discontinuity of $n_{12}(s,\omega)$ across $C(m_\rho^2)$ we find

$$\mathrm{disc}[n_{12}(s,\omega);C(m_\rho^2)]=M_{11}(s)\,\mathrm{disc}[d_{12}(s,\omega);C(m_\rho^2)]+2i\alpha(s,m_\rho^2)[1+i\rho_1(s)M_{11}(s)]d_{22}(s,m_\rho^2,\omega), \quad \text{(B5)}$$

which implies that we should choose

$$\mathrm{disc}[d_{12}(s,\omega);C(m_\rho^2)]=2\alpha(s,m_\rho^2)\rho_1(s)d_{22}(s,m_\rho^2,\omega). \quad \text{(B6)}$$

This determines the form of the equations for n_{12} and d_{12}.

In order to calculate the discontinuities of n_{21} and n_{22} we shall need the following relations, derived from from Eq. (5.21)

$$\mathrm{disc}[M_{22}(s,\omega,m_\rho^2);C(m_\rho^2)]=-2\rho_1(s)M_{21}(s,\omega)\alpha(s,m_\rho^2), \quad \text{(B7)}$$

$$\mathrm{disc}[M_{22}(s,\omega,m_\rho^2);C(\omega)]=-2i\rho_1(s)M_{21}(s,m_\rho^2)\alpha(s,\omega)/[\rho(\omega)f(\omega)]. \quad \text{(B8)}$$

Using these relations, it is straightforward to show that

$$\mathrm{disc}[n_{21}(s,\omega);C(m_\rho^2)]=0, \quad \text{(B9)}$$

$$\mathrm{disc}[n_{21}(s,\omega);C(\omega)]=2i\alpha(s,\omega)[1+d_{11}(s)]-2i\alpha(s,\omega)\rho_1(s)n_{11}(s)/[\rho(\omega)f(\omega)], \quad \text{(B10)}$$

$$\mathrm{disc}[n_{22}(s,\omega',\omega);C(m_\rho^2)]=\mathrm{disc}[n_{22}(s,\omega',\omega);C(\omega)]=0, \quad \text{(B11)}$$

$$\mathrm{disc}[n_{22}(s,\omega',\omega);C(\omega')]=2i\alpha(s,\omega')d_{12}(s,\omega)-\frac{2i\alpha(s,\omega')\rho_1(s)n_{12}(s,\omega)}{\rho(\omega)f(\omega)}. \quad \text{(B12)}$$

Thus, we find that we are forced to add extra terms to the equations for n_{21} and n_{22}.

Let us now gather together the n and d equations which we have derived for ω, $\omega'<2\mu^2(1+\mu/2M)$:

$$d_{11}(s)=-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s')n_{11}(s')}{s'-s}-\frac{i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\alpha(s',m_\rho^2)\rho_1(s')d_{21}(s',m_\rho^2)}{s'-s}, \quad \text{(B13)}$$

$$d_{12}(s,\omega)=-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s')n_{12}(s',\omega)}{s'-s}+\frac{1}{\pi\rho(\omega)f(\omega)}\int_{C(\omega)}ds'\frac{\rho_1(s')\alpha(s',\omega)}{s'-s}-\frac{i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\rho_1(s')\alpha(s',m_\rho^2)d_{22}(s',m_\rho^2,\omega)}{s'-s}, \quad \text{(B14)}$$

$$d_{21}(s,\omega)=-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{\rho_2(s')n_{21}(s',\omega)}{s'-s}, \quad \text{(B15)}$$

$$d_{22}(s,\omega',\omega)=-\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty}ds'\frac{\rho_2(s')n_{22}(s',\omega',\omega)}{s'-s}, \quad \text{(B16)}$$

$$n_{11}(s)=\frac{1}{\pi}\int_{\Gamma(m_\rho^2)}ds'\frac{\alpha(s',m_\rho^2)d_{21}(s',m_\rho^2)}{s'-s}, \quad \text{(B17)}$$

$$n_{12}(s,\omega)=B(s,\omega)+\frac{1}{\pi}\int_{\Gamma(m_\rho^2)}ds'\frac{\alpha(s',m_\rho^2)d_{22}(s',m_\rho^2,\omega)}{s'-s}, \quad \text{(B18)}$$

$$n_{21}(s,\omega)=B(s,\omega)+\frac{1}{\pi}\int_{\Gamma(\omega)}ds'\frac{\alpha(s',\omega)d_{11}(s')}{s'-s}-\frac{1}{\pi\rho(\omega)f(\omega)}\int_{C(\omega)}ds'\frac{\alpha(s',\omega)\rho_1(s')n_{11}(s')}{s'-s}, \quad \text{(B19)}$$

$$n_{22}(s,\omega',\omega)=\frac{1}{\pi}\int_{\Gamma(\omega')}ds'\frac{\alpha(s',\omega')d_{12}(s',\omega)}{s'-s}-\frac{1}{\pi\rho(\omega')f(\omega')}\int_{C(\omega')}ds'\frac{\alpha(s',\omega)\rho_1(s')n_{12}(s',\omega)}{s'-s}. \quad \text{(B20)}$$

Now we proceed in a manner completely parallel to Sec. VI to derive integral equations for the n_{ij}. The results are, for ω, $\omega'<2\mu^2(1+\mu/2M)$,

$$n_{11}(s)=\int_{(M+2\mu)^2}^{\infty} ds'\rho_2(s')K(s,s';m_\rho^2)n_{21}(s',m_\rho^2), \tag{B21}$$

$$n_{12}(s,\omega)=B(s,\omega)+\int_{(M+2\mu)^2}^{\infty} ds'\rho_2(s')K(s,s';m_\rho^2)n_{22}(s',m_\rho^2,\omega), \tag{B22}$$

$$n_{21}(s,\omega)=B(s,\omega)+\int_{(M+2\mu)^2}^{\infty} ds'\rho_2(s')L(s,\omega;s',m_\rho^2)n_{21}(s',m_\rho^2), \tag{B23}$$

$$n_{22}(s,\omega',\omega)=\bar{B}(s,\omega',\omega)+\int_{(M+2\mu)^2}^{\infty} ds'\rho_2(s')L(s,\omega';s',m_\rho^2)n_{22}(s',m_\rho^2,\omega), \tag{B24}$$

where we have defined

$$K(s,s';\omega)\equiv\frac{B(s,\omega)-B(s',\omega)}{\pi(s-s')}, \tag{B25}$$

$$L(s,\omega;s',m_\rho^2)\equiv\int_{(M+\mu)^2}^{\infty} ds''K(s,s'';\omega)\rho_1(s'')K(s',s'';m_\rho^2)+\frac{i}{\pi}\int_{C(m_\rho^2)} ds''\frac{\rho_1(s'')K(s,s'';\omega)\alpha(s'',m_\rho^2)}{s''-s'}$$
$$-\frac{1}{\pi\rho(\omega)f(\omega)}\int_{C(\omega)} ds''\frac{\alpha(s'',\omega)\rho_1(s'')K(s'',s';m_\rho^2)}{s''-s}, \tag{B26}$$

$$\bar{B}(s,\omega',\omega)=\int_{(M+\mu)^2}^{\infty} ds''\rho_1(s'')K(s,s'';\omega')B(s'',\omega)-\frac{1}{\pi\rho(\omega')f(\omega')}\int_{C(\omega')} ds''\frac{\alpha(s'',\omega')\rho_1(s'')B(s'',\omega)}{s''-s}$$
$$-\frac{1}{\pi\rho(\omega)f(\omega)}\int_{C(\omega)} ds''\frac{\rho_1(s'')\alpha(s'',\omega)}{s''-s}[B(s'',\omega')-B(s,\omega')]. \tag{B27}$$

The remaining task is to continue these equations to ω, $\omega'>4\mu^2$. Since we have demonstrated the method repeatedly, we give only the results here. The integral equations for the n_{ij}, Eqs. (B21)–(B24), are unchanged, as well as the kernel K. The kernel L becomes

$$L(s,\omega;s',m_\rho^2)=\int_{(M+\mu)^2}^{\infty} ds''K(s,s'';\omega)\rho_1(s'')K(s',s'';m_\rho^2)+\frac{i}{\pi}\int_{C(m_\rho^2)} ds''\frac{\rho_1(s'')K(s,s'';\omega)\alpha(s'',m_\rho^2)}{s''-s}$$
$$-\frac{1}{\pi\rho(\omega)f(\omega)}\int_{s_-(\omega)}^{(M+\mu)^2} ds''\frac{\rho_1(s'')K(s'',s';m_\rho^2)\alpha(s'',\omega)}{s''-s}+\frac{1}{\pi\rho(\omega)f^*(\omega)}\int_{(M+\mu)^2}^{s_+(\omega)} ds''\frac{\rho_1(s'')K(s'',s';m_\rho^2)\alpha(s'',\omega)}{s''-s}. \tag{B28}$$

In order to facilitate the continuation, ω and m_ρ^2 have, as usual, been given small positive imaginary parts. Moreover, we choose $\mathrm{Im}\omega<\mathrm{Im}m_\rho^2$ to keep the cuts from crossing during the continuation. It can be shown that the result above agrees in the limit $\omega=m_\rho^2$ with Eq. (6.20) if one keeps track of where the kernel K is to be evaluated. Similarly, one finds

$$\bar{B}(s,\omega',\omega)=\int_{(M+\mu)^2}^{\infty} ds''\rho_1(s'')K(s,s'';\omega')B(s'',\omega)-\frac{1}{\pi\rho(\omega')f(\omega')}\int_{s_-(\omega')}^{(M+\mu)^2} ds''\frac{\rho_1(s'')\alpha(s'',\omega')B(s'',\omega)}{s''-s}$$
$$+\frac{1}{\pi\rho(\omega')f^*(\omega')}\int_{(M+\mu)^2}^{s_+(\omega')} ds''\frac{\rho_1(s'')\alpha(s'',\omega')B(s'',\omega)}{s''-s}-\frac{1}{\pi\rho(\omega)f(\omega)}\int_{s_-(\omega)}^{(M+\mu)^2} ds''\rho_1(s'')\alpha(s'',\omega)K(s,s'';\omega')$$
$$+\frac{1}{\pi\rho(\omega)f^*(\omega)}\int_{(M+\mu)^2}^{s_+(\omega)} ds''\rho_1(s'')\alpha(s'',\omega)K(s,s'';\omega'), \tag{B29}$$

where ω and ω' are assumed to have unequal positive imaginary parts, and the respective cuts are kept separated.

Now let us assume that we have solved the equations derived in Sec. V and, therefore, that n_{ij} and d_{ij} are known for $\omega=\omega'=m_\rho^2$. Then the equations we have derived up to this point permit us to calculate (by quadratures) $n_{21}(s,\omega)$ from Eq. (B23) and $n_{22}(s,\omega',m_\rho^2)$ from Eq. (B24). Once these quantities are known, we can calculate $M_{21}(s,\omega)$ and $M_{22}(s,\omega,m_\rho^2)=M_{22}(s,m_\rho^2,\omega)$ from the following equations, which follow from Eqs. (6.1):

$$M_{21}(s,\omega)=\frac{[1+d_{22}(s,m_\rho^2,m_\rho^2)]n_{21}(s,\omega)-n_{22}(s,\omega,m_\rho^2)d_{21}(s,m_\rho^2)}{D(s)}, \tag{B30}$$

$$M_{22}(s,\omega,m_\rho^2)=\frac{[1+d_{11}(s)]n_{22}(s,\omega,m_\rho^2)-n_{21}(s,\omega)d_{21}(s,m_\rho^2)}{D(s)}, \tag{B31}$$

where

$$D(s)=[1+d_{11}(s)][1+d_{22}(s,m_\rho^2,m_\rho^2)]-d_{12}(s,m_\rho^2)d_{21}(s,m_\rho^2). \tag{B32}$$

The quantity $M_{22}(s,\omega',\omega)$ seems to be more difficult to evaluate, since we must obtain $n_{22}(s,\omega',\omega)$ by solving Eq. (B24). We would also need $d_{12}(s,\omega)$, which we obtain from the continuation of Eq. (B14),

$$\begin{aligned} d_{12}(s,\omega)=&-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}ds'\frac{\rho_1(s')n_{12}(s',\omega)}{s'-s}-\frac{i}{\pi}\int_{C(m_\rho^2)}ds'\frac{\rho_1(s')\alpha(s',m_\rho^2)d_{22}(s',m_\rho^2,\omega)}{s'-s}\\ &+\frac{1}{\pi f(\omega)\rho(\omega)}\int_{s_-(\omega)}^{(M+\mu)^2}ds'\frac{\rho_1(s')\alpha(s',\omega)}{s'-s}+\frac{1}{\pi f^*(\omega)\rho(\omega)}\int_{s_+(\omega)}^{(M+\mu)^2}ds'\frac{\rho_1(s')\alpha(s',\omega)}{s'-s}. \end{aligned} \tag{B33}$$

Finally, one solves the algebraic Eqs. (6.1) for $M_{22}(s,\omega',\omega)$. Alternatively, one could evaluate this quantity via Eq. (5.20), using Eqs. (B30) and (B31).

It is interesting to note that it might be possible to use these amplitudes to improve upon the approximation made in reducing Eq. (3.2) to the form given in Eq. (3.8). In performing integrations over ω'' we used the sharply-peaked nature of $|f(\omega'')|$ to approximate $M_{21}(s,\omega'')$ by $M_{21}(s,m_\rho^2)$, and similarly for M_{22}. In this Appendix we have outlined a procedure for calculating the ω dependence, which could then be used to check or improve upon our approximation.