PHYSICAL REVIEW VOLUME 128, NUMBER 1 OCTOBER 1, 1962

High-Energy Potential Scattering*

S. ROSENDORFF†
Department of Physics, Washington University, St. Louis, Missouri

AND

SMIO TANI‡
Department of Physics, New York University, University Heights, New York
(Received August 8, 1961; revised manuscript received May 24, 1962)

The asymptotic expansion of the phase shift in inverse powers of the momentum p and the asymptotic expansion of the scattering amplitude in inverse powers of p and the momentum transfer q have been derived for a Dirac and Klein-Gordon particle. It is shown that at high energy (i) the higher-order phase shifts (proportional to α^k, $k\neq 1$, where α is the coupling constant) are very small and negligible in the calculation of the amplitude, (ii) the amplitude approaches the first Born approximation amplitude (linear in α) multiplied by a phase factor. Statement (ii) holds for spherically symmetric potentials $V(r)$ for which the first N derivatives (the phase shift is expanded asymptotically up to p^{-N}) exist for every real positive value of r, including $r=0$, and for which at least one derivative of odd order does not vanish at the origin. Statement (i) is probably correct also for potentials even at the origin [$V(r)=V(-r)$ for $r\approx 0$]. The upper limit on the coupling constant α is $\alpha\ll p\mu_1$ with the additional condition $p\mu_1\gg 1$. Here μ_1 is a characteristic length of the potential. The lower limit on the scattering angle θ is given by $\theta\gg(p\mu_2)^{-1}$, where μ_2 is another characteristic length of the potential and is usually of the same order of magnitude as μ_1. The problem of the model independence and other consequences of the theory are discussed.

1. INTRODUCTION

THE high-energy potential scattering of a Dirac or Klein-Gordon particle has been studied extensively.[1–7] In the present paper both the asymptotic expansion of the phase shifts in inverse powers of the energy and the asymptotic expansion of the scattering amplitude in inverse powers of the energy and the momentum transfer will be derived for a relativistic particle. It is known that the first Born approximation of the phase shift is very good at high energy.[2,7] The phase shift approaches the limit $\delta_\infty=-\alpha\int_0^\infty V(r)dr$ independent of the angular momentum l; $V(r)$ is the potential in coordinate space and α measures its strength (coupling constant). This is in contrast to the nonrelativistic case where the phase shift vanishes at high energy.

We will restrict ourselves to local, spherical symmetric potential functions $V(r)$. We shall assume that the first N derivatives (the phase shift is expanded asymptotically up to p^{-N}) of $V(r)$ exist for every positive real value of r, including $r=0$, and that at least one derivative of odd order does not vanish at the origin. Such potentials have been used extensively by Hofstadter[8] in his analysis of the high-energy electron nuclei scattering experiments. To see this, one has to realize that all smooth electric charge distributions (including distributions with a simple pole at the origin) which are not even functions of r at the origin give rise to potentials with the above mentioned properties; e.g., this is the case with the Yukawa or Fermi distribution. Potentials which correspond to charge distributions with a sharp edge (like the uniform distribution) will not be considered in the present analysis because they violate the required conditions. Potentials which are even at the origin (called here even potentials) and which correspond to even charge distributions, like the Gaussian distribution, will be discussed briefly at the end of the last section.

As to the scattering amplitude the following two statements will be shown to be valid at high energy: (i) The higher-order phase shifts are negligibly small in the calculation of the amplitude; (ii) the amplitude approaches the product of a phase factor with the linear term of the power series in α in which the amplitude is expanded. (The series is often called the Born series, and the linear term is usually referred to as the first Born approximation.) It will turn out that the coupling constant α is much less restricted in magnitude than in the Born approximation; its upper limit is given by $\alpha\ll p\mu_1$ with the additional condition $p\mu_1\gg 1$. Here p is the momentum of the scattered particle, and μ_1 is a characteristic length parameter of the potential. This means that the phase shifts are not small, in general, and therefore the expansion of the S matrix as a power series in the phase shift is not feasible because of its slow convergence. Statement (ii) is, therefore, a rather surprising

* Work supported in part by the U. S. Air Force Office of Scientific Research.
† Present address: Department of Physics and Astronomy, University of Rochester, Rochester, New York.
‡ Supported by the National Science Foundation. Part of this work was done while the author was staying at Washington University, St. Louis, Missouri.

[1] G. Molière, Z. Naturforsch. **2A**, 133 (1947).
[2] G. Parzen, Phys. Rev. **80**, 261 (1950).
[3] R. R. Lewis, Phys. Rev. **102**, 537 (1956).
[4] L. I. Schiff, Phys. Rev. **103**, 443 (1956). In this paper a more complete bibliography of the earlier work will be found.
[5] T. T. Wu, Phys. Rev. **108**, 466 (1957).
[6] R. J. Glauber, *Lectures in Theoretical Physics* (University of Colorado, Boulder, Colorado, 1958), Vol. 1.
[7] S. Rosendorff and S. Tani, Bull. Am. Phys. Soc. **5**, 426 (1960).
[8] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956); Ann. Rev. Nuclear Sci. **7**, 231 (1957).

result. It means that all the higher-order terms are equivalent to a phase factor. Of course, for the nonrelativistic wave equation the first-order Born approximation is always guaranteed, because the phase shift becomes small at high energies.

Although statement (i) has been proved analytically only for potentials which satisfy the conditions outlined above, it is probably true also for potentials which are even at $r=0$. In order to check this point, the scattering amplitude for the Gaussian potential is worked out numerically. The result given in Table I at the end of the last section, strongly supports our conjecture. [Statement (ii) is not valid for even potentials.]

The theory presented here is a large-angle scattering theory. [No attempt has been made to prove the equivalence of the present theory and Schiff's (reference 4) large-angle scattering theory.] The lower limit on the angle of scattering θ is given by $\theta \gg (p\mu_2)^{-1}$. Here, μ_2 is another parameter of the potential. For many potentials μ_1 and μ_2 are of the same order of magnitude. On the other hand, the Molière-Glauber[1,6] impact parameter representation is a high-energy, small-angle, scattering theory. However, both theories have a common property, namely the fact that the substitution of the first-order (linear in α) phase shift in the S matrix is well justified at high energy.

Recently, Nigam *et al.*[9] have criticized Molière's[1] scattering theory for being inconsistent. They contend that it yields only the first Born approximation consistently but not the higher Born approximation, because the higher-order phase shifts have been neglected in the calculation of the amplitude. This argument, though correct in principle, cannot be upheld for potentials subject to the conditions outlined above in the limit of high energies, as will become evident from the results of the present paper. Indeed, an indication to this effect appears in the paper by Lewis,[3] where it is shown that for charge distributions with a derivative at the origin the real part of the second-order amplitude (which comes from the second-order phase shift) is negligibly small compared to the first-order amplitude, at high energies.

In Sec. 2 the first-, second-, and third-order phase shifts are calculated at high energies for angular momenta $l < p\mu_2$. As the WKB method is very good in the high-energy limit, it has been used for the derivation of the higher-order phase shifts. (In Sec. 3 it will become evident that this method yields the amplitude exactly at high energy.) In Sec. 3 the asymptotic expansion of the scattering amplitude in inverse powers of the momentum and the momentum transfer is derived, by taking into account the results of Sec. 2. In Sec. 4 some consequences of the theory are discussed, and in particular the problem of the model independence is dealt with.

[9] B. P. Nigam, M. K. Sundaresan, and T. Y. Wu, Phys. Rev. **115**, 491 (1959).

In addition, the numerical example of the Gaussian potential is given. Because of the simple relationship[10] between the Dirac theory and the Klein-Gordon theory at high energy, for the most part only the latter theory is discussed.

2. ANGULAR MOMENTUM DEPENDENCE OF THE PHASE SHIFTS AT HIGH ENERGIES

Let us expand the phase shift of a Klein-Gordon particle in a power series of the coupling constant α:

$$\delta_l = \sum_{k=1}^{\infty} \alpha^k \delta_l^{(k)}.$$

We shall refer to $\delta_l^{(k)}$ as the kth-order phase shift.

In this section the behavior of the first three orders of the phase shifts will be derived as a function of the momentum p and the angular momentum l, at high energies.[11] Knowledge of the explicit dependence of the phase shift on p and l is necessary in order to derive the asymptotic expansion of the scattering amplitude at high energies, which is dealt with in Sec. 3.

It is well known that the WKB method for the computation of the phase shift is very good at high energies. In view of the results of references 2 and 7, we expect that the main contribution to the phase shift comes from the first order. We, therefore, will calculate the second- and third-order phase shifts in WKB approximation only. On the other hand, the first-order phase shift will be derived exactly, and its asymptotic expansion at high energies will be compared with the asymptotic expansion of the first-order WKB phase shift. This provides us with a method to extract from the WKB phase shift those angular-momentum-dependent terms which are essential for the calculation of the amplitude. Later it will turn out that only the terms which depend logarithmically on l contribute in the calculation of the amplitude.

The first-order Klein-Gordon phase shift is given by

$$\delta_l^{(1)} = -\frac{2p^2}{\beta} \int_0^{\infty} r^2 dr\, j_l^2(pr) V(r), \tag{1}$$

where β is the velocity of the particle. We will assume that the potential $\alpha V(r)$ satisfies the following four conditions: (i) It is spherical symmetric; (ii) the first N derivatives exist for every real, positive, and finite value of r, including $r=0$, if the phase shift is going to be expanded up to p^{-N}; (iii) it decreases faster than $1/r$ at infinity, and (iv) the first and/or third derivative does not vanish at the origin.

Now in order to find the behavior of $\delta_l^{(1)}$ for large momenta p, use will be made of the explicit depend-

[10] G. Parzen, Phys. Rev. **104**, 835 (1956).
[11] See also M. Verde, Nuovo cimento **6**, 340 (1957); **8**, 560 (1958).

ence[12] of $j_l(\rho)$ on ρ and l:

$$\rho^2 j_l^2(\rho)=\tfrac{1}{2}+\frac{l(l+1)}{(2\rho)^2}+3\frac{(l-1)l(l+1)(l+2)}{(2\rho)^4}+\cdots+\frac{(l+\frac{1}{2},l)^2}{2(2\rho)^{2l}}$$
$$+(-1)^l\cos 2\rho\left[-\tfrac{1}{2}+\frac{l(l+1)(l^2+l-1)}{(2\rho)^2}+\cdots-(-1)^l\frac{(l+\frac{1}{2},l)^2}{2(2\rho)^{2l}}\right]$$
$$+(-1)^l\frac{\sin 2\rho}{2\rho}\left[l(l+1)+\cdots-(-1)^l\frac{(l+\frac{1}{2},l-1)(l+\frac{1}{2},l)}{(2\rho)^{2l-2}}\right]. \quad (2)$$

Equation (2) represents the function $j_l(\rho)$ *exactly* for *every* value of ρ and l. The symbol $(l+\frac{1}{2},m)$ denotes some polynomial in l and is defined in reference 12.

The direct substitution of Eq. (2) in Eq. (1) will give rise to diverging integrals. In order to avoid this, we will introduce an auxiliary function $f(r)$, and rewrite (1) as

$$\beta\delta_l^{(1)}=-2p^2\int_0^\infty r^2dr\, j_l^2(pr)[V(r)-f(r)]$$
$$-2p^2\int_0^\infty r^2dr\, j_l^2(pr)f(r). \quad (1')$$

In order to expand $\delta_l^{(1)}$ in inverse powers of p up to the fourth order, the function $f(r)$ has to be chosen such that (i) $(V-f)$ has a zero of the fourth order at the origin[13] and (ii) the second integral in Eq. (1′) is explicitly known.

For $f(r)$, we take

$$f(r)=\frac{e^{-\lambda_1 r}-e^{-\lambda_2 r}}{\lambda_2-\lambda_1}[V(0)/r+a+br+cr^2], \quad (3)$$

where a, b, and c are uniquely determined by the above condition. They are explicitly given in Appendix A.

We also note that the sine and cosine terms do not contribute up to p^{-4} because of the well-known[14] asymptotic expansion

$$\int_0^\infty e^{2ipr}\varphi(r)dr=-\sum_{n=0}^{N-1} i^{n-1}\varphi^{(n)}(0)(2p)^{-n-1}+O((2p)^{-N}),$$

where it is assumed that $\varphi(r)$ and its first $(N-1)$ derivatives do exist for $0\leqslant r<\infty$.

In general, if one considers the asymptotic expansion of $\delta_l^{(1)}$ up to p^{-N}, then $f(r)$ has to be chosen such that $(V-f)$ has a zero of the Nth order at the origin. From the above expansion formula it follows then that the trigonometric terms contribute to terms of order p^{-N-2} and higher. In other words, the integrals involving $\cos(2\rho)$ and $\sin(2\rho)$ may be disregarded in the asymptotic expansion of $\delta_l^{(1)}$. We then obtain for $\delta_l^{(1)}$, using Eq. (2):

$$\beta\delta_l^{(1)}=-\int_0^\infty [V(r)-f(r)]dr-\frac{l(l+1)}{2p^2}\int_0^\infty \frac{dr}{r^2}[V(r)-f(r)]-\frac{3}{8}\frac{(l-1)l(l+1)(l+2)}{p^4}$$
$$\times\int_0^\infty \frac{dr}{r^4}[V(r)-f(r)]-2p^2\int_0^\infty r^2dr\, j_l^2(pr)f(r)+O(p^{-6}). \quad (4)$$

Using the expansion of the last integral in (4) which is given in Appendix A, one finds after some rearrangements, for the asymptotic expansion of the first-order phase shift

$$\beta\delta_l^{(1)}=-\int_0^\infty V(r)dr-\frac{V_0'}{(2p)^2}+\frac{V_0'''}{(2p)^4}+\frac{l(l+1)}{2p^2}\left[\int_0^\infty \ln r dr\, V''(r)+V_0'[\psi(l+1)-\ln 2p]-\tfrac{1}{2}V_0'-\frac{V_0'''}{(2p)^2}\right]$$
$$+\frac{(l-1)l(l+1)(l+2)}{(2p)^4}\left[\int_0^\infty \ln r dr\, V^{(4)}(r)+V_0'''[\psi(l+1)-\ln 2p]-(5/4)V_0'''\right]+O(p^{-6}). \quad (5)$$

The auxiliary function $f(r)$ cancels out as expected. $\psi(l)$ is the logarithmic derivative of the Γ function. The subscript 0 on V', etc., means these functions are to be evaluated at $r=0$. The above asymptotic expansion is obviously a better and better approximation to the exact value of $\delta_l^{(1)}$, the larger the value of the momentum p and the smaller the value of the angular momentum l (for a finite number of terms). The exact range of validity is difficult to define. However, one can get a rough idea about it and at the same time gain some

[12] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949), p. 117.

[13] M. Verde, Nuovo cimento 2, 1001 (1955).

[14] A. Erdélyi, *Asymptotic Expansions* (Dover Publication Inc. New York, 1956), p. 47.

physical insight by considering an example. Take, e.g., the superposed Yukawa potential $V=(e^{-\lambda_1 r}-e^{-\lambda_2 r})/r$ with $\lambda_1 \ll \lambda_2$. Then V_0, $(V_0')^{\frac{1}{2}}$, $(V_0''')^{\frac{1}{4}}$, etc. are all of the order of λ_2 which, magnitude-wise, is equal to the inverse of the radius of the charge distribution [see also Eq. (6)]. For this particular case, one reads from Eq. (5) that its range of validity is given by $p>\lambda_2$ as well as $p>l\lambda_2$. This means that both the wavelength of the scattered particle and the classical impact parameter $\rho=l/p$ should be small compared with the dimensions of the nuclear charge distribution. Putting $1/p=0$, one obtains Parzen's[2] limit of the phase shift at infinite energy:

$$\delta_\infty=-\int_0^\infty V(r)dr.$$

It is obvious that Parzen's limit gives the main contribution to the first-order phase shift, at high energies. Yet in Sec. 3 it will be shown that the only terms which contribute to the scattering amplitude are those proportional to $\psi(l+1)$ in Eq. (5). This fact enables one to extract from the phase shift those terms which are essential in the calculation of the amplitude.

The electromagnetic interactions are included in the present analysis, whereby one starts with a screened potential and takes a limit in which the screening vanishes, if necessary. In that case, one generally considers the electric charge distribution rather than the potential. It is well known that the nuclear charge distribution $\rho_c(r)$ is defined as the Fourier transform of the nuclear form factor $F(q)$. On the other hand, the Fourier transform of the potential is equal to $F(q)/q^2$. This gives rise to a simple connection between the potential and the charge distribution. It is easy to verify that the following relations hold:

$$V_0=\langle r^{-1}\rangle_N=4\pi\int_0^\infty rdr\,\rho_c(r), \tag{6}$$

$$2V_0'=-4\pi(r\rho_c)_0;\quad 3V_0''=-4\pi(r\rho_c)_0';\quad 4V_0'''=-4\pi(r\rho_c)_0'';\quad \cdots. \tag{6'}$$

The distribution $\rho_c(r)$ is normalized to unity when integrated over the whole space. Although $V(r)$ has to be regular at the origin, $\rho_c(r)$ may have a simple pole. It is also easy to show that to a potential even at $r=0$ corresponds an even charge density, and vice versa.

We turn now to the calculation of the phase shift in WKB approximation. It is well known that it is given by

$$\delta_l(\alpha)=\lim_{r_1\to\infty}\left[\int_{r_0(\alpha)}^{r_1}[F(r)+\alpha F_1(r)+\alpha^2F_2(r)]^{\frac{1}{2}}dr-\int_{r_0(0)}^{r_1}[F(r)]^{\frac{1}{2}}dr\right], \tag{7}$$

where[15]

$$F(r)=p^2-\frac{(l+\frac{1}{2})^2}{r^2};\quad F_1(r)=-2EV(r);\quad F_2=V^2(r);$$

and the lower limits $r_0(\alpha)$ and $r_0(0)$ are the zeros of the respective integrands.

Expanding Eq. (7) in a power series of α, one finds that the first-order phase shift is equal to

$$\delta_l^{(1)}=-\frac{1}{\beta}\int_0^\infty V[(r^2+\rho^2)^{\frac{1}{2}}]dr \tag{8}$$

Here, $\rho=(l+\frac{1}{2})/p$ is essentially the classical impact parameter. Equation (8) is just the phase shift used by Molière,[1] Schiff,[4] and others in their small angle, high-energy potential scattering theory.

Repeated integrations by parts of Eq. (8) and subsequent expansion of V', V'', and V''' gives the behavior of $\delta_l^{(1)}$ for small values of ρ:

$$\beta\delta_l^{(1)}=-\int_0^\infty V(r)dr+W_1(\rho^2)+\tfrac{1}{2}\rho^2\ln\rho[V_0'+\tfrac{1}{8}V_0'''\rho^2+\cdots], \tag{9}$$

where $W_1(\rho^2)$ is a power series of ρ^2, and $W_1(0)=0$. Equation (9) is derived in Appendix B. The above result shows that the main contribution to the first-order WKB phase shift, namely, the term $-(1/\beta)\int_0^\infty Vdr$ is identical with the corresponding term in the Born expansion, or, in other words, the WKB method is, as expected, very good for the computation of the phase shift, at high energies. But what is more important, it essentially yields the correct angular momentum dependence, and in particular, those terms which contribute in the calculation of the amplitude are almost exactly reproduced. These are the terms proportional to $\psi(l+1)$ and $\ln\rho$ in Eqs. (5) and (9), respectively. For not too small values of l, they are very close to each other because $(l+\frac{1}{2})^2\to l(l+1)$, $(l+\frac{1}{2})^4\to(l-1)l(l+1)(l+2)$ and $r=\psi(l+1)/\ln(l+\frac{1}{2})\to 1$. As a matter of fact,[16] for $l\geqslant 2$, r is almost equal to one.

Considering this fact, it suggests itself that the following simple substitutions should be made whenever the higher-order WKB phase shifts are used in the calculation of the amplitude:

$$\begin{aligned}\rho^2\ln(l+\tfrac{1}{2})&\to\frac{l(l+1)}{p^2}\psi(l+1),\\ \rho^4\ln(l+\tfrac{1}{2})&\to\frac{(l-1)l(l+1)(l+2)}{p^4}\psi(l+1).\end{aligned} \tag{10}$$

[15] Langer [R. E. Langer, Phys. Rev. **51**, 669 (1937)] and Molière (see reference 1) have given arguments why to substitute $(l+\frac{1}{2})$ for $l(l+1)$ in the centrifugal barrier term.

[16] For $l=2$, $r=1.007$, and for $l=3$, $r=1.003$.

This conjecture is completely borne out in the next section where it will be shown that the WKB phase shift in conjunction with (10) yields the first- and second-order amplitude exactly, at high energies.

The higher-order WKB phase shifts are obtained by taking the appropriate derivatives of $\delta_l(\alpha)$ [Eq. (7)] with respect to α. In order to avoid any divergence, care has to be taken not to differentiate under the integral sign in the vicinity of the lower limit $r_0(\alpha)$. This is accomplished by splitting the integral in two parts, from $r_0(\alpha)$ to r', and from r' to infinity. The parameter r' is arbitrary, but supposed to be close to $r_0(\alpha)$. It does not depend on α, and at the end it cancels out. The integrand of the first integral is then expanded in powers of $[r-r_0(\alpha)]$ and integrated term by term. Finally, the derivatives with respect to α are taken.

The second-order WKB phase shift calculated in this way is equal to

$$\delta_l^{(2)}=\frac{1}{2}\int_{r_0(0)}^{\infty}\left[F_2-\frac{1}{2}\frac{d}{dr}\left(\frac{F_1^2}{F'}\right)\right]\frac{dr}{F^{\frac{1}{2}}}+\left(\frac{F_1^2}{4F'F^{\frac{1}{2}}}\right)_\infty.$$

Substituting for the functions F, F_1, and F_2 from (7), $\delta_l^{(2)}$ becomes after some rearrangements:

$$\beta^2p\delta_l^{(2)}=\int_0^\infty\frac{dr}{(r^2+\rho^2)^{\frac{1}{2}}}\left(\frac{m^2}{p^2}r^2-\rho^2\right)\times V[(r^2+\rho^2)^{\frac{1}{2}}]V'[(r^2+\rho^2)^{\frac{1}{2}}],\quad(11)$$

where V' means the derivative with respect to the square root, m is the mass of the scattered particle, and β is again its velocity. Integration by parts of Eq. (11) and subsequent expansion of (VV') then yields $\delta_l^{(2)}$ for small values of ρ:

$$\beta^2p\delta_l^{(2)}=W_2(\rho^2)+\rho^2\ln\rho\times\left[\frac{3-\beta^2}{2}(VV')_0+\frac{5-\beta^2}{16}(VV')_0''\rho^2+\cdots\right],\quad(12)$$

where $W_2(\rho^2)$ is a power series of ρ^2 which vanishes at the origin.

The third-order WKB phase shift has been found in an analogous way. Its behavior for small values of ρ becomes

$$\beta^3p^2\delta_l^{(3)}=W_3(\rho^2)+\rho^2\ln\rho\left[3\frac{5-3\beta^2}{4}(V^2V')_0+5\frac{7-3\beta^2}{32}(V^2V')_0''\rho^2+\cdots\right],\quad(13)$$

where again W_3 is a power series of ρ^2 which vanishes at the origin. As has been pointed out before, the functions W_1, W_2, and W_3 do not contribute in the calculation of the amplitude. Their explicit dependence on ρ^2 is, therefore, omitted here.

In conclusion, the explicit dependence on p and l of the first three-order phase shifts have been found, at high energy. The result may be summarized as follows: (i) the higher-order phase shifts are much smaller than the first-order phase shift. (ii) Let $\delta_{\ln}^{(n)}$ denote the terms proportional to $\ln\rho$ in the nth-order phase shift, then $\delta_{\ln}^{(1)}=O(p^{-2k})$, in case $V_0^{(2k-1)}$ is the first nonvanishing odd derivative at the origin; $\delta_{\ln}^{(2)}=O(p^{-(2k+1)})$, in case $(VV')_0^{(2k-2)}$ is the first nonvanishing term; and $\delta_{\ln}^{(3)}=O(p^{-(2k+2)})$, in case $(V^2V)_0^{(2k-2)}$ is the first nonvanishing term. This follows immediately from Eqs. (5), (9), (12), and (13). We, therefore, conclude that $\delta_{\ln}^{(2)}/\delta_{\ln}^{(1)}=O(p^{-1})$, $\delta_{\ln}^{(3)}/\delta_{\ln}^{(1)}=O(p^{-2})$, etc. (provided $V_0\neq0$; if $V_0=0$, then these ratios are of higher order in $1/p$). This fact is decisive in the derivation of the amplitude as will be shown in the next section.

3. HIGH-ENERGY ASYMPTOTIC EXPANSION OF SCATTERING AMPLITUDE

We now wish to obtain the asymptotic expansion of the scattering amplitude in inverse powers of the momentum p and the momentum transfer $q=2p\sin(\theta/2)$. θ is the scattering angle. We shall expand the scattering amplitude in powers of the coupling constant α:

$$Q=\sum_{k=1}\alpha^kQ_k,$$

and call Q_k the kth-order (Born) amplitude. The differential cross section is proportional to

$$Q^\dagger Q=\alpha^2Q_1^2\left[1+2\alpha\frac{\mathrm{Re}Q_2}{Q_1}+\alpha^2\left(2\frac{\mathrm{Re}Q_3}{Q_1}+\frac{Q_2^\dagger Q_2}{Q_1^2}\right)+\cdots\right].$$

The first term is usually referred to as the first Born approximation cross section, the second term the second Born approximation cross section, etc.

Let us start the discussion with an example and find the first and second-order Klein-Gordon amplitude[17] for the superposed Yukawa potential:

$$\alpha V(r)=(\alpha/r)(e^{-\lambda_1r}-e^{-\lambda_2r}).\quad(14)$$

Equation (14) corresponds to a Yukawa-type charge distribution, $\rho_c=(\lambda_2^2/4\pi r)\exp(-\lambda_2r)$, for $\lambda_1=0$. The first-order amplitude for the above potential is equal to

$$Q_1=8\pi^2\left(\frac{1}{\lambda_1^2+q^2}-\frac{1}{\lambda_2^2+q^2}\right),\quad(15)$$

which for $q\gg\lambda_1$, λ_2 becomes

$$Q_1=8\pi^2\left(-\frac{\lambda_1^2-\lambda_2^2}{q^4}+\frac{\lambda_1^4-\lambda_2^4}{q^6}-\cdots\right).\quad(15')$$

The second-order amplitude for a Klein-Gordon particle

[17] At the end of this section it will be shown that the Dirac and Klein-Gordon amplitudes are simply related at high energy. (See also reference 10.)

is given by

$$Q_2=\frac{1}{(2\pi)^4}\int d^4k\,\langle p'|V|k\rangle\times\left[\frac{2E}{k^2-m^2+i\epsilon}-\frac{1}{2E}\right]\langle k|V|p\rangle, \quad (16)$$

where $\mathbf{p}'-\mathbf{p}=\mathbf{q}$, $p_0'=p_0=E$, the total energy, and

$$\langle p'|V|p''\rangle=\int e^{ip'\cdot x}V(x)e^{-ip''\cdot x}d^4x.$$

Q_2 has been evaluated by a method described in detail by Dalitz.[18] The explicit expression of Q_2 is a long and cumbersome function of p and q. Here, we are interested only in the leading term for $p,\ q\gg\lambda_1,\ \lambda_2$:

$$\mathrm{Re}Q_2=\frac{(4\pi)^2}{pq^4}\frac{3-\beta^2}{2\beta}(\lambda_1-\lambda_2)(\lambda_1^2-\lambda_2^2), \quad (17)$$

$$\mathrm{Im}Q_2=-\frac{(4\pi)^2}{\beta q^4}(\lambda_1^2-\lambda_2^2)\ln(\lambda_1/\lambda_2), \quad (17')$$

and, therefore, taking $\beta\to 1$,

$$\mathrm{Re}Q_2/Q_1=-2(\lambda_1-\lambda_2)/p, \quad (18)$$

$$\mathrm{Im}Q_2/Q_1=2\ln(\lambda_1/\lambda_2). \quad (18')$$

Suppose now we make a partial-wave expansion of the Klein-Gordon amplitude at high energies,

$$Q=\frac{4\pi^2 i}{p^2}\sum_l(l+\tfrac{1}{2})P_l(\cos\theta)[\exp 2i\sum_k\alpha^k\delta_l^{(k)}-1]$$

$$=-\frac{8\pi^2}{p^2}\alpha\sum_l(l+\tfrac{1}{2})P_l\delta_l^{(1)}-\frac{8\pi^2}{p^2}\alpha^2\sum_l(l+\tfrac{1}{2})P_l\delta_l^{(2)}-\frac{8\pi^2}{p^2}i\alpha^2\sum_l(l+\tfrac{1}{2})P_l[\delta_l^{(1)}]^2+\cdots$$

$$=\alpha Q_1+\alpha^2\,\mathrm{Re}Q_2+i\alpha^2\,\mathrm{Im}Q_2+\cdots,$$

then it follows immediately, that Q_1, $\mathrm{Re}Q_2$, and $\mathrm{Im}Q_2$ arise from $\delta_l^{(1)}$, $\delta_l^{(2)}$, and $(\delta_l^{(1)})^2$, respectively. Thus, Eq. (18) suggests that $\delta_l^{(2)}$ may be neglected in the calculation of the amplitude at high energies, at least in the case of the present example. In what follows it will become clear that not only the linear $\delta_l^{(2)}$ term, but also its higher powers (the same is true for $\delta_l^{(3)},\cdots$) may be neglected. Furthermore, we will see that the first Born approximation cross section becomes the dominant term at high energy. This seems to follow from Eq. (18), because $R=2(\mathrm{Re}Q_2)/Q_1\ll 1$, and this is just the ratio of the second to the first Born approximation cross section. But this conclusion is by no means self-evident since from Eq. (18′) it follows that the third and higher Born approximation cross sections are of the same order of magnitude as the first-order term.

Let us now turn to the general case. The first-order Klein-Gordon amplitude is given by the Fourier transform of the potential $V(r)$:

$$Q_1=2\pi\int e^{i\mathbf{q}\cdot\mathbf{r}}V(r)d^3r. \quad (19)$$

Its asymptotic expansion in inverse powers of q is readily obtained by repeated integrations by parts:

$$Q_1=(4\pi)^2\left[-\frac{V_0'}{q^4}+2\frac{V_0'''}{q^6}+\cdots\right]. \quad (20)$$

Obviously, Eq. (15′) agrees with Eq. (20). From the derivation of (20) from (19) follows: (i) The sufficient and necessary condition for the expansion of Q_1 up to q^{-N-1} is that $V(r)$ is N times continuously differentiable for $0\leqslant r<\infty$. (See reference 14 for a rigorous proof.) (ii) At least one derivative of odd order of $V(r)$ has to be different from zero at the origin. Since other cases can be treated in a similar way, we shall continue to assume that V_0' and/or V_0''' do not vanish. In case all odd derivatives of $V(r)$ do vanish at the origin, i.e., potentials which are an even function of r at the origin, an asymptotic expansion of Q_1 in inverse powers of q does not exist. In that case Q_1 falls off faster than any power of q, i.e., at least exponentially. A potential of this class will be discussed at the end of Sec. 4.

We now proceed to calculate the asymptotic expansion of the scattering amplitude Q in inverse powers of the momentum p and the momentum transfer q. Use will be made of the following six infinite series, valid for $\theta\neq 0$:

$$F_n=\lim_{x\to 1,\,x<1}\sum_l(l+\tfrac{1}{2})f_n(l)x^lP_l(\cos\theta)=0, \quad (21)$$

$$G_n=\lim_{x\to 1,\,x<1}\sum_l(l+\tfrac{1}{2})f_n(l)x^l\psi(l+1)P_l(\cos\theta)=(-1)^{\frac{1}{2}n+1}\tfrac{1}{4}[(\tfrac{1}{2}n)!]^2\sin^{-n-2}(\theta/2), \quad (21')$$

[18] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).

where

$$f_0(l)=1, \quad f_2(l)=l(l+1), \quad f_4=(l-1)l(l+1)(l+2).$$

The proof of Eqs. (21) and (21′) is given in Appendix C.[19]

We start with the partial wave expansion:

$$Q=\frac{4\pi^2 i\beta}{p^2}\sum_l (l+\tfrac{1}{2})P_l(\cos\theta)[\exp(2i\sum_k \alpha^k\delta_l^{(k)})-1].$$

Inside the square brackets only the exponential function may be retained because of Eq. (21). It is, therefore, possible to factor out the phase factor which comes from the l-independent term of δ. Expansion of the exponential function then gives

$$Q=\frac{8\pi^2 i\beta}{p^2}\eta\sum_l (l+\tfrac{1}{2})P_l\left[i\alpha\left(\frac{1}{\beta}\int_0^\infty Vdr+\delta_l^{(1)}\right)+i\alpha^2\delta_l^{(2)}+i\alpha^3\delta_l^{(3)}-\alpha^2\left(\frac{1}{\beta}\int_0^\infty Vdr+\delta_l^{(1)}\right)^2+\cdots\right],$$

where η is the phase factor $\eta=\exp[-2i\alpha\int_0^\infty Vdr+O(p^{-2})]$. If now the asymptotic expansions of the phase shifts, (obtained in Sec. 2) are introduced[20] and use is made of Eqs. (21) and (21′), one obtains

$$Q=\frac{4\pi^2 i\beta}{p^2}\eta\left[i\alpha\frac{V_0'}{4\beta p^2}\sin^{-4}(\theta/2)+i\alpha^2\frac{(VV')_0}{2p^3}\frac{3-\beta^2}{2\beta^2}\sin^{-4}(\theta/2)+\text{terms of order } p^{-4} \text{ and higher}\right],$$

which becomes

$$Q=-\frac{(4\pi)^2\alpha V_0'}{q^4}\eta\left[1+\alpha\frac{3-\beta^2}{\beta}\frac{V_0}{p}+\tfrac{3}{2}\alpha^2\frac{5-3\beta^2}{\beta^2}\frac{V_0^2}{p^2}\right.$$
$$\left.-2\frac{V_0'''/V_0'}{q^2}-2\alpha\frac{5-\beta^2}{\beta}\frac{(VV')_0''/V_0'}{pq^2}+\frac{2i\alpha}{\beta}U\left(\frac{8}{q^2}-\frac{1}{p^2}\right)+i\alpha\frac{V_0'}{\beta p^2}[f(\theta)-1]+O(p^{-3})\right]. \quad (22)$$

The function $f(\theta)$ has not been calculated explicitly. It arises from $[\psi(l+1)]^2$ of $(\delta_l^{(1)})^2$. The term proportional to U also comes from $(\delta_l^{(1)})^2$. U is equal to [cf. Appendix B and Eq. (5)]

$$U=V_0'(\tfrac{1}{2}+\ln 2p)-\int_0^\infty V''(r)\ln r dr.$$

The first and fourth terms in Eq. (22) are derived from $\delta^{(1)}$. The second and fifth terms come from $\delta^{(2)}$ and from all cross terms between $\delta^{(1)}$ and $\delta^{(2)}$, which are linear in $\delta^{(2)}$. Finally, the third term comes from $\delta^{(3)}$ and again from cross terms between $\delta^{(1)}$ and $\delta^{(3)}$, linear in $\delta^{(3)}$. We therefore conclude that $\delta^{(1)}$ and only $\delta^{(1)}$ contributes to the leading term in the asymptotic expansion of the amplitude. Its magnitude is given by the first-order amplitude (first Born approximation). The reason for this is simple. The only contributing terms of the phase shift in the calculation of the amplitude are the terms $\delta_{\ln}$ defined at the end of the last section. Now $\delta_{\ln}^{(1)}=O(p^{-2})$, $\delta_{\ln}^{(2)}=O(p^{-3})$, etc. This, however, is the sufficient condition for the validity of the first Born approximation, and therefore explains why at high energies the magnitude of the dominant term of the scattering amplitude is given by the first Born approximation.

From Eq. (22), it follows that

(i) The term linear in the coupling constant α is identical with Eq. (20).

(ii) The term proportional to $i\alpha^2$ is given by

$$\text{Im}Q_2=\frac{2(4\pi)^2}{\beta q^4}V_0'\int_0^\infty Vdr+O(p^{-6}). \quad (23)$$

This expression agrees with the corresponding expression, Eq. (17′) derived for the potential (14), since $V_0'=(\lambda_1^2-\lambda_2^2)/2$ and $\int_0^\infty Vdr=-\ln(\lambda_1/\lambda_2)$.

[19] The notation $\lim_{x\to 1, x<1}$ means that we include the converging factor x^l to begin and take the limit $x\to 1$ at the end. This defines the limit of the series on its radius of convergence (Abel's sum). See, e.g., G. H. Hardy, *Divergent Series* (Oxford University Press, New York, 1949), p. 7. The definition of the amplitude Q just below should be understood in this sense although we omit the limit symbol.

[20] The convergence of the expression $\sum_l(l+\frac{1}{2})(\delta_l)^n P_l(\cos\theta)$ is realized by the rapid oscillation of P_l for large l. The asymptotic expansion of this expression in inverse powers of q is completely analogous to the asymptotic expansion of the Fourier integral $\int_0^\infty f(\rho)\cos\rho q d\rho$ in inverse powers of q. It has been rigorously shown that the latter is obtained by expanding the slowly varying function $f(\rho)$ as an ascending power series of ρ [provided $f(\rho)$ and all its derivatives vanish at infinity], and integrating term by term. See, e.g., reference 14. This result is true even if the expansion of $f(\rho)$ is not valid for large values of ρ. Here, $f(\rho)$ and $\cos\rho q$ correspond to $(l+\frac{1}{2})(\delta_l)^n$ and $P_l(\cos\theta)$, respectively. Note that the asymptotic expansion of δ_l is essentially an ascending power series of l. The summability of the series is guaranteed in the sense that Eqs. (21) and (21′) can be used.

(iii) The term proportional to α^2 is given by

$$\mathrm{Re}Q_2=-\frac{2(4\pi)^2}{pq^4}\left[\frac{3-\beta^2}{2\beta}(VV')_0-\frac{5-\beta^2}{\beta}\frac{(VV')_0''}{q^2}\right]. \quad (24)$$

Again the leading term agrees with the corresponding term Eq. (17), because $V_0=-(\lambda_1-\lambda_2)$. In case, $V_0'=0$, Eq. (24) becomes

$$\mathrm{Re}Q_2=2(4\pi)^2\frac{5-\beta^2}{\beta}\frac{(VV''')_0}{pq^6}. \quad (24')$$

The corresponding first-order term is according to Eq. (20) given by

$$Q_1=2(4\pi)^2V_0'''/q^6,$$

and therefore the ratio, assuming the velocity $\beta\to 1$

$$R=2\frac{\mathrm{Re}Q_2}{Q_1}=8\frac{V_0}{p}=8\frac{\langle r^{-1}\rangle_N}{p}. \quad (25)$$

In the last step Eq. (6) has been used. This result coincides with the expression calculated exactly by Lewis[3] for charge distributions with a finite derivative at the origin; the latter means $V_0'=0$, according to (6′). Therefore, as mentioned already in the last section, the WKB phase shifts in conjunction with conjecture (10) indeed yield the correct amplitude, at high energy.

Finally, a word about the Dirac amplitude. Parzen[10] has shown that at high energy the Dirac phase shifts η_l are related to the Klein-Gordon phase shifts δ_l by $2\eta_l=\delta_l+\delta_{l+1}$, provided $\delta_{l+1}-\delta_l\ll 1$. This condition is obviously satisfied by the phase shifts discussed in Sec. 2. The Dirac amplitude at high energies is given[10] by

$$Q_D=\frac{4\pi^2 i}{p^2}\sec(\theta/2)\times\sum_l\left[(l+1)(e^{2i\eta_l}-1)+l(e^{2i\eta_{l-1}}-1)\right]P_l(\cos\theta).$$

This simplified expression for the Dirac amplitude follows from the fact that at high energy the helicity is a constant of the motion. If now use is made of the asymptotic expansion of δ_l derived in Sec. 2, it is easy to verify that

$$Q_D=Q_K\cos(\theta/2), \quad (26)$$

where Q_K, the Klein-Gordon amplitude, is given by Eq. (22). From (26) then the relationship between the respective cross sections,

$$\sigma_D=\sigma_K\cos^2(\theta/2),$$

follows. This has also been found by Schiff.[4]

4. DISCUSSION

It follows from the foregoing analysis that at high energy: (i) The higher-order phase shifts do not contribute to the dominant term of the scattering amplitude, or, in other words, the substitution of $\delta^{(1)}$ for δ in the S matrix is a good approximation. (ii) The dominant term of the amplitude is given by the first Born approximation term, multiplied by a phase factor. However, this result is independent of the magnitude of the phase shift.

If the cross section is measured at all angles, the phase factor η may be determined by the unitarity condition for the scattering amplitude.[21,22]

Statement (ii) is correct for all potentials subject to the conditions outlined at the beginning of Sec. 2. Statement (i) is probably correct also for potentials which are even at the origin. Also, from Eq. (22) it follows that $V(r)$ may have a $1/r$ dependence at infinity, because only the phase η will be affected. It becomes logarithmically divergent.

Conclusion (i) seems trivial on first sight, because it has been proved in Sec. 2 that δ is well approximated by $\delta^{(1)}$. However, Ravenhall and Yennie[23] have shown, with the aid of some numerical examples, that tiny differences in the phase shifts may give rise to considerable differences in the amplitude, and vice versa. Conclusion (i), therefore, means that small differences in δ due to the higher-order phase shifts have only a very small effect on the amplitude. The sensitive dependence of Q on δ observed in reference 23 is explained as follows: According to Eq. (5) the value of δ almost entirely comes from $-\alpha\int_0^\infty Vdr$. On the other hand, only the small "correction" terms (δ_{ln}) proportional to $\psi(l+1)$ do contribute to Q. Therefore, charge distributions which give rise to the same $\alpha\int_0^\infty Vdr$, but to different $\psi(l+1)$ terms may have quite different amplitudes, although their corresponding phase shifts are almost identical. Conclusion (i) also implies that the cross section is independent of the sign of the coupling constant α at very high energy.

Another interesting result which follows from Eq. (22) is what one may call the conditional model independence. By this we mean that there is a limitation on the amount of detail of the nuclear-charge distribution which can be obtained in the very high energy region. For instance, the scattering amplitude will have the same $1/q^4$ dependence for all charge distributions for which $V_0'\neq 0$ is the same, or, it will have a $1/q^6$ dependence for all distributions for which $V_0'=0$, but $V_0'''\neq 0$ is the same, etc. Of course, the above statements are true only for a limited group of potentials, namely, those subject to the conditions outlined in Sec. 2. Potentials which do not satisfy these conditions, like the potentials which are even at $r=0$, or potentials which correspond to charge distributions with a sharp edge (e.g., uniform or shell) most probably are not subject to

[21] L. Puzikov, R. Ryndin, and I. Smorodinskii, J. Exptl. Theoret. Phys. U.S.S.R. **5**, 489 (1957).

[22] We are indebted to Dr. E. C. G. Sudarshan for this remark.

[23] D. G. Ravenhall and D. R. Yennie, Proc. Phys. Soc. (London) **A70**, 857 (1957).

TABLE I. The different amplitudes as explained in the text as a function of the coupling constant α. The energy and scattering angle are chosen such that $q^2/4a^2=10$ and $p/q=10$.

α	$Q_1 \propto 4a^2\alpha\langle\delta^{(1)}\rangle$	$\text{Re}Q_2 \propto 4a^2\alpha^2\langle\delta^{(2)}\rangle$	$2a^2\lvert[e^{2i\alpha\delta^{(1)}}-1]\rvert$	$2a^2\lvert[e^{2i(\alpha\delta^{(1)}+\alpha^2\delta^{(2)})}-1]\rvert$
0.01	0.091×10^{-5}	0.0034×10^{-5}	0.112×10^{-5}	0.115×10^{-5}
0.10	0.091×10^{-4}	0.0340×10^{-4}	0.663×10^{-4}	0.670×10^{-4}
1.00	0.091×10^{-3}	0.3400×10^{-3}	$10.33\ \times10^{-3}$	10.390×10^{-3}

any model independence. This has been demonstrated in reference 23. From what has been shown in the present paper, it is obvious that the arguments given by Reignier[24] and others in favor of model independence, essentially the equality of the values of $\alpha\int_0^\infty Vdr$, do not hold. The equality of $\alpha\int_0^\infty Vdr$ is no guarantee whatsoever of the equality of the amplitudes.

As to the range of validity of our result, no attempt has been made to find the exact conditions. Obviously the scattering amplitude Eq. (22) is an asymptotic expansion in both the momentum p and the momentum transfer q. The reason why its validity is limited to large values of q is that the expansion of the phase shift is valid only for a certain range of l from 0 to some value $l_{\max}$. It is therefore reasonable to assume that both p and q are much larger than some characteristic inverse length of the potential. Call it $1/\mu_2$. It then follows that: (i) $\theta_0=(p\mu_2)^{-1}\ll1$ and $\theta\gg\theta_0$. (ii) The higher-order phase shifts in Eq. (22) are negligible for $\alpha/\beta p\mu_1\ll1$, where μ_1 is another characteristic length of the potential. This means that our result is valid for a large range of the coupling constant α. For many potentials the two parameters μ_1 and μ_2 are of the same order of magnitude. When the potential is such that V_0, $V_0'\neq0$, the corresponding charge distribution has a simple pole at the origin. In this case it follows from Eqs. (22) and (6′) that

$$\mu_2^2=V_0'/V_0'''=2(r\rho_c)_0/(r\rho_c)_0'',$$

and from Eqs. (22) and (6) that

$$1/\mu_1=V_0=\langle r^{-1}\rangle_N.$$

Similar conditions hold in the case that $V_0'=0$ $V_0'''\neq0$, for which the charge distribution has a finite first-order derivative at the origin. Extension to other cases is obvious. The condition (ii) means that the energy should be much larger than the potential depth. Our result is similar to the nonrelativistic Coulomb scattering amplitude which also, for every value of α, is given by the first-order term times a phase factor. Finally, it should be mentioned, that the often quoted condition for the validity of the Born approximation, namely, $|\alpha\int_0^\infty Vdr|\ll1$, is sufficient but is by no means necessary as follows from the results of the present paper.

So far, we have dealt with potentials which have at least one nonvanishing odd derivative at the origin. For potentials which are even at the origin the first Born approximation for the amplitude most probably is not valid in the high-energy limit (except, of course, for very small coupling constants). This follows from the calculations performed for the Gaussian charge distribution by Lewis,[3] and for the Gaussian potential by Wu,[25] who showed that, in contrast to the noneven potentials, the ratio $(\text{Re}Q_2)/Q_1$ increases exponentially with the energy, at high energies. This seems to be an indication that the second-order phase shift contributes more to the amplitude than does the first-order phase shift. On the other hand, calculations analogous to those of Sec. 2 show, that also for potentials which are even at the origin the phase shift at high energies is well approximated by the first term of its expansion in a power series of the coupling constant α. (The main difference between the noneven and even potentials is that in the latter case no $\psi(l+1)$-dependent terms appear in the asymptotic expansion of the phase shifts.) Probably the answer to the problem is that the first Born approximation amplitude provides a much too small value to the exact amplitude. However there are strong indications that the higher-order phase shifts are negligible in the calculation of the amplitude, just as in the case of noneven potentials, and that the substitution of $\delta^{(1)}$ for δ is a very good approximation.

We would like to demonstrate these points by a numerical example. Let us take the Gaussian potential:

$$\alpha V(r)=-(2a/\sqrt{\pi})\alpha e^{-a^2r^2}. \tag{27}$$

It will suffice to use the WKB method for the computation of the phase shifts. The first- and second-order phase shifts are then given by

$$\delta_l^{(1)}=e^{-a^2l^2/p^2}, \tag{28}$$

and

$$\delta_l^{(2)}=-(8/\pi)^{\frac{1}{2}}(a/p)^3l^2e^{-2a^2l^2/p^2}, \tag{28′}$$

using Eqs. (8) and (11), respectively. Thus, $\delta^{(2)}/\delta^{(1)}\to0$ when $p\to\infty$ for every l. In the evaluation of the amplitude the summation over l has been approximated by an integration, which should provide a very good approximation for small scattering angles. We have calculated (i) the amplitude in which δ is approximated by $\alpha\delta^{(1)}$, (ii) the amplitude in which δ is approximated by $(\alpha\delta^{(1)}+\alpha^2\delta^{(2)})$, as well as (iii) the first-order amplitude and the real part of the second-order amplitude, for comparison. This has been done for three different values of α. The results are given in Table I. There is a difference of less than 1% between the amplitude de-

[24] J. Reignier, Nuclear Phys. 3, 340 (1957).

[25] T. Y. Wu, Phys. Rev. 73, 934 (1948).

rived from $[\exp(2i\alpha\delta^{(1)})-1]$ and the amplitude derived from $[\exp 2i(\alpha\delta^{(1)}+\alpha^2\delta^{(2)})-1]$, thus indicating strongly that the higher-order phase shifts are negligible in the calculation of the amplitude, at high energies. On the other hand, there is a marked discrepancy between the exact amplitude and its first-order term.

Even for $\alpha=0.01$ there is some discrepancy. This might be an indication that for potentials which are even at the origin the first Born approximation is rather bad even for relatively weak couplings. But more work is needed to clarify this point.

APPENDIX A

The parameters a, b, and c of Eq. (3) are given by

$$2a=(\lambda_1+\lambda_2)V+2V', \qquad (A1)$$

$$12b=[(\lambda_1+\lambda_2)^2+2\lambda_1\lambda_2]V+6(\lambda_1+\lambda_2)V'+6V'', \qquad (A1')$$

$$12c=\lambda_1\lambda_2(\lambda_1+\lambda_2)V+[(\lambda_1+\lambda_2)^2+2\lambda_1\lambda_2]V' +3(\lambda_1+\lambda_2)V''+2V'''. \qquad (A1'')$$

V, V', V'', and V''' are all taken at $r=0$.

The last integral in Eq. (4) is given by

$$2p^2(\lambda_2-\lambda_1)\int_0^\infty r^2dr\, j_l^2(pr)f(r)=V(Q_{l1}-Q_{l2})+a\left[-\frac{dQ_{l1}}{d\lambda_1}+\frac{dQ_{l2}}{d\lambda_2}\right]+b\left[\frac{d^2Q_{l1}}{d\lambda_1^2}-\frac{d^2Q_{l2}}{d\lambda_2^2}\right]+c\left[-\frac{d^3Q_{l1}}{d\lambda_1^3}+\frac{d^3Q_{l2}}{d\lambda_2^3}\right], \qquad (A2)$$

where Q_l is the Legendre function of the second kind, and $Q_{li}=Q_l(1+\lambda_i^2/2p^2)$. The asymptotic expansion of Q_l is

$$Q_l(1+z)=\left(\tfrac{1}{2}\ln\frac{2}{z}-\sum_1^l\frac{1}{n}\right)+\frac{z}{4}\left[1+l(l+1)\left(2+\ln\frac{2}{z}-2\sum_1^l\frac{1}{n}\right)\right]$$
$$+\left(\frac{z}{4}\right)^2\left[-1+2l(l+1)+(l-1)l(l+1)(l+2)\left(\tfrac{3}{2}+\tfrac{1}{2}\ln\frac{2}{z}-\sum_1^l\frac{1}{n}\right)\right]+O(z^3). \qquad (A3)$$

APPENDIX B

We wish to obtain the behavior of the first-order WKB phase shift $\delta_l^{(1)}$ for small values of ρ. Calling $s=(r^2+\rho^2)^{\frac{1}{2}}$ and assuming that $V(r)$ decreases faster than $1/r$ at infinity, one finds on integrating Eq. (8) by parts twice, that

$$\beta\delta_l^{(1)}=-\int_0^\infty V(s)dr=\int_0^\infty\frac{r^2dr}{s}V'(s)$$
$$=\tfrac{1}{2}\rho^2V'(\rho)\ln\rho$$
$$-\frac{1}{2}\int_0^\infty[rs-\rho^2\ln(r+s)]\frac{rV''(s)}{s}dr. \qquad (B1)$$

Additional integration by parts of the last integral gives

$$\beta\delta_l^{(1)}=\tfrac{1}{2}\rho^2V'(\rho)\ln\rho-\tfrac{1}{2}\rho^3V''(\rho)\ln\rho$$
$$+\frac{1}{2}\int_0^\infty[\tfrac{1}{3}r^3+(r-s\ln(r+s))\rho^2]\frac{rV'''(s)}{s}dr. \qquad (B2)$$

The last integral becomes, at $\rho^2=0$,

$$\frac{1}{6}\int_0^\infty r^3V'''(r)dr=-\int_0^\infty V(r)dr. \qquad (B3)$$

Therefore, the consistent expansion of $\delta_l^{(1)}$ up to ρ^2 is given by

$$\beta\delta_l^{(1)}=\tfrac{1}{2}\rho^2V_0'\ln\rho+\bar{W}_1(\rho^2), \qquad (B4)$$

where

$$\bar{W}_1(\rho^2)=\frac{1}{2}\int_0^\infty\{\tfrac{1}{3}r^3+[r-s\ln(r+s)]\rho^2\}\frac{rV'''(s)}{s}dr$$
$$=-\int_0^\infty V(r)dr$$
$$+\frac{1}{2}\left(-\tfrac{1}{2}V_0'+\int_0^\infty V''(r)\ln 2r\,dr\right)\rho^2+\cdots. \qquad (B5)$$

By continuing this process one finds that the next term of $\bar{W}_1$ is of the form $(a_4+b_4\ln\rho)\rho^4$ with $b_4=V_0'''/16$, the next term of the form $(a_6+b_6\ln\rho)\rho^6$, etc. In this way one finds the result quoted in Eq. (9). The expansions of $\delta^{(2)}$ and $\delta^{(3)}$ given by Eqs. (12) and (13) are obtained by the same method.

APPENDIX C

In order to prove Eqs. (21) and (21′), we start with the generating function of the Legendre polynomials, $P_l(\cos\theta)$:

$$f(x)=\sum_l x^lP_l(\cos\theta)=(1+x^2-2x\cos\theta)^{-\frac{1}{2}}, \qquad (C1)$$

valid for $x<1$. Hereafter we will always assume that $\theta\neq0$. From (C1) it follows immediately that

$$F_0(x)=\sum_l(l+\tfrac{1}{2})x^lP_l(\cos\theta) \qquad (C2)$$

is given by

$$xf'(x)+\tfrac{1}{2}f(x)=\frac{1-x^2}{2(1+x^2-2x\cos\theta)^{\frac{3}{2}}}. \qquad (C2')$$

This takes care of the first equation of (21). The second equation of (21) is

$$F_2(x)=\sum_l(l+\tfrac{1}{2})l(l+1)x^lP_l(\cos\theta)$$
$$=x^2F_0''(x)+2xF_0'(x), \quad \text{(C3)}$$

which gives $F_2(1)=0$. In a similar way it is found that $F_4(1)=0$.

Now, in order to derive Eq. (21′) use will be made of the integral representation of $\psi(l+1)$:

$$\psi(l+1)=\int_0^\infty[e^{-t}-(1+t)^{-l-1}]\frac{dt}{t}. \quad \text{(C4)}$$

Therefore,

$$G_0(x)=\sum_l(l+\tfrac{1}{2})\psi(l+1)x^lP_l(\cos\theta)$$
$$=\lim_{\epsilon=0}\left[\int_\epsilon^\infty\frac{e^{-t}}{t}dt\,F_0(x)-\int_\epsilon^\infty\frac{F_0[x/(1+t)]}{t(1+t)}dt\right]$$
$$=-\lim_{\epsilon=0}\left[F_0(x)\,\mathrm{Ei}(-\epsilon)+\int_0^{x(1-\epsilon)}\frac{F_0(t)dt}{x-t}\right]. \quad \text{(C5)}$$

The calculation of the integral is quite long but straightforward. After inserting (C2′) for $F_0(t)$ it becomes

$$\frac{1}{2}\int_0^{x(1-\epsilon)}\frac{(1-t^2)dt}{(x-t)\Lambda^3(t)}$$
$$=\Lambda^{-2}(x)-F_0(x)\left[2+\ln\frac{\epsilon(\Lambda+1-x\cos\theta)}{\Lambda^2}\right], \quad \text{(C6)}$$

where $\Lambda^2(x)=1+x^2-2x\cos\theta$. Finally, the desired function $G_0(x)$ becomes

$$G_0(x)=F_0(x)\left[2+\ln\frac{\Lambda+1-x\cos\theta}{2\gamma\Lambda^2}\right]-\Lambda^{-2}(x), \quad \text{(C7)}$$

where $\ln\gamma=0.577$ is "Euler's constant." The first equation of (21′) then follows directly: $G_0(1)=-[\sin^{-2}(\theta/2)]/4$, because $F_0(1)=0$. G_2 and G_4 are derived from G_0 by the same method as F_2 and F_4 were derived from F_0. It should be mentioned that the integral representation (A4) and (A4″) in reference 1 coincide with the present formulas for small values of θ and $x=1$.