Pionic Contributions to the Magnetic Moment of the Muon*

LOYAL DURAND, III
Department of Physics, Yale University, New Haven, Connecticut
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The anomalous magnetic moment κ of the muon is affected in order α^2 by vacuum polarization corrections to the photon propagator which involve strongly interacting particles. The contributions of the two- and three-pion systems to κ are estimated using the present experimental information on the $J=1^-$ two-pion and three-pion resonances (the ρ and ω mesons). The analysis is based on the use of dispersion relations for the muon vertex functions, and of the Lehmann-Källén representation for the photon propagator. If it is assumed that the ρ-meson resonance gives the dominant contribution to the electromagnetic form factor of the pion, the modifications of the photon propagator associated with the two-pion intermediate state lead to a change in κ: $(\Delta\kappa)_{2\pi} \sim(\alpha^2/36\pi)(m_\rho^2-4m_\pi^2)^{3/2}m_\mu^2m_\rho^{-4}\Gamma_\rho^{-1}$, where Γ_ρ is the width of the ρ resonance in pion-pion scattering. In the limit of Gell-Mann's unitary symmetry for the meson-baryon interactions, the ω-meson contributions to κ are one-third of the ρ-meson contributions. With $m_\rho=750$ MeV and the recent Yale value of 60 MeV for Γ_ρ, one obtains a combined result $\Delta\kappa\sim 1.1\times 10^{-7}$ which is an order of magnitude larger than (1) would be estimated using perturbation theory and (2) the nominal size of the sixth-order electrodynamic corrections; but which is still negligible in comparison with the quoted limits of error of $\pm 5\times 10^{-6}$ in the best measurement of κ, that of Charpak *et al.* The possibility of obtaining additional large contributions from $\rho-\pi$ and $\omega-\pi$ configurations in the three- and four-pion states is discussed briefly.

I. INTRODUCTION

THE anomalous magnetic moment of the muon, when calculated to fourth order in e including all electrodynamic processes which involve only muons, electrons, and photons, is found to be[1]

$$\kappa=(\alpha/2\pi)+0.75(\alpha/\pi)^2=11655\times 10^{-7}.$$

Recent measurements of κ clearly favor the correctness of the $(\alpha/2\pi)$ term, and are nearly accurate enough to test the α^2 term as well.[2,3] Thus, Charpak *et al.*,[2] by a direct measurement of $g-2$ for the muon, obtain $\kappa=(11620\pm 50)\times 10^{-7}$, while the α^2 contribution to κ is 41×10^{-7}. In this situation, it is of interest to consider the contribution to κ of those terms of order α^2 which depend on the properties of strongly interacting particles, and which have so far been neglected. These arise from vacuum polarization corrections to the propagator of the internal photon in the familiar triangle diagram of perturbation theory; the intermediate state may involve, for example, pion pairs, triples, and so forth. It is well known that the effects of these vacuum polarization diagrams are suppressed for heavy intermediate states by a factor $(m_\mu/M)^2$, where M is a mass characteristic of the system considered. Furthermore, the contribution to κ of the least massive system of strongly interacting particles, that of two pions, is forbiddingly small when calculated in Born approximation,

$$(\Delta\kappa)_B=(\alpha^2/360\pi^2)(m_\mu/m_\pi)^2[1-0.18+O(m_\mu^4/m_\pi^4)] \cong 0.7\times 10^{-8}.$$

It has, therefore, been customary to ignore these effects.

The situation is changed considerably when account is taken of the recently discovered ρ-meson[4] [$J=1^-$, $T=1$] and ω-meson[5] [$J=1^-$, $T=0$] resonances in the two- and three-pion systems, respectively. These resonances result in a considerable increase in the pionic contribution to the photon propagator, and lead, despite the large masses involved [$m_\rho\sim 750$ MeV, $m_\omega\sim 785$ MeV], to a change in κ an order of magnitude larger than that obtained for noninteracting pions. Thus, if it is assumed that the ρ-meson resonance gives the dominant contribution to the electromagnetic form factor of the pion, the contribution to κ associated with the two-pion intermediate state is given by

$$(\Delta\kappa)_{2\pi}\sim(\alpha^2/36\pi)(m_\rho^2-4m_\pi^2)^{3/2}m_\mu^2m_\rho^{-4}\Gamma_\rho^{-1},$$

where Γ_ρ is the width of the ρ-meson resonance in pion-pion scattering. In the limit of Gell-Mann's unitary symmetry model for the meson-baryon inter-

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[1] J. Schwinger, Phys. Rev. **73**, 416 (1948); R. Karplus and N. M. Kroll, *ibid.* **77**, 536 (1950); H. Suura and E. H. Wichmann, *ibid.* **105**, 1930 (1957); A. Petermann, *ibid.* **105**, 1931 (1957); C. M. Sommerfield, *ibid.* **107**, 328 (1957); A. Petermann, Helv. Phys. Acta **30**, 407 (1957). The α^2 term in the second reference is incorrect, and this error was propagated in the following two papers. The correct result was first obtained by Sommerfield.

[2] G. Charpak, F. J. M. Farley, R. L. Garwin, T. Muller, J. C. Sens, and A. Zichichi, Phys. Letters **1**, 16 (1962); G. Charpak, F. J. M. Farley, R. L. Garwin, T. Muller, J. C. Sens, V. L. Telegdi, and A. Zichichi, Phys. Rev. Letters **6**, 128 (1961).

[3] D. P. Hutchinson, Columbia University dissertation and Nevis Cyclotron Laboratory Report No. 103 (unpublished). It is, unfortunately, not possible to obtain a precise value of κ from this very accurate (± 13 ppm) measurement of the total magnetic moment of the muon because of the large uncertainties (± 100 ppm) in the muon mass.

[4] A. R. Erwin, R. March, W. D. Walker, and E. West, Phys. Rev. Letters **6**, 628 (1961); D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler, H. Kraybill, J. Sandweiss, J. Sanford, and H. Taft, *ibid.* **6**, 624 (1961). This result clearly establishes the isotopic spin of the resonance, $T=1$. E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. Letters **7**, 192 (1961); D. McLeod, S. Richert, and A. Silverman, *ibid.* **7**, 383 (1961). D. D. Carmony and R. T. Van de Walle, *ibid.* **8**, 73 (1962). This paper presents evidence for the spin assignment $J=1^-$. J. Button, G. R. Kalbfleisch, G. R. Lynch, B. C. Maglić, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. **126**, 1858 (1962).

[5] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961); N. H. Xuong and G. R. Lynch, *ibid.* **7**, 327 (1961); A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, *ibid.* **7**, 421 (1961). M. L. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, Phys. Rev. **125**, 687 (1962). This paper shows the spin and parity of the ω to be $J=1^-$.

actions,[6] the ω-meson contributions to κ are one-third of the ρ-meson contributions. Using the recent Yale value of 60 MeV for Γ_ρ[7] one obtains a combined result $\Delta\kappa\sim 1.1\times 10^{-7}$. This is larger by a factor of 13 than the estimate based on perturbation theory, but is still negligible in comparison with the quoted limits of error $[\pm 50\times 10^{-7}]$ in the best measurement of κ to date.[2] However, this result is also an order of magnitude larger than the expected sixth-order electrodynamic corrections to κ, and consequently limits the validity of "pure" electrodynamics for the muon-electron-photon system to low-order processes. Detection of the mesonic contributions to κ would be of great interest.

The calculation of the pionic corrections to κ are conveniently carried out using dispersion relations for the muon vertex functions and the Lehmann-Källén[8] representation for the photon propagator. The relations necessary for the specification of $\Delta\kappa$ in terms of the pionic contributions to the spectral function for the photon propagator are derived in Sec. II. The structure of the spectral function is discussed in Sec. III, with the main emphasis on the contributions of the ρ and ω resonances in the two- and three-pion systems. The possibility that $\rho-\pi$ and $\omega-\pi$ configurations in the three- and four-pion systems contribute strongly to the spectral function is discussed in a rather speculative manner, and it is concluded that such contributions are probably not important. The results for $\Delta\kappa$ are discussed in Sec. IV.

II. THE ELECTROMAGNETIC VERTEX FUNCTION FOR THE MUON

The most general form for the momentum-space matrix element of the electromagnetic current operator j_λ between single-particle states of a spin $\frac{1}{2}$ particle is easily shown to be[9]

$$\langle p'|j_\lambda|p\rangle = -ie\bar{u}(p')[f_1(q^2)\gamma_\lambda + (2m)^{-1}f_2(q^2)\sigma_{\lambda\nu}(p'-p)_\nu]u(p), \quad (1)$$

where $q=p'-p$ is the four-momentum transfer at the vertex, and m is the mass of the particle. The Dirac and Pauli form factors $f_1(q^2)$ and $f_2(q^2)$ are functions only of the invariant square of the four-momentum transfer, and are normalized so that $f_1(0)$ is equal to unity, while $f_2(0)$ is equal to κ, the anomalous magnetic moment of the particle in units of the particle magneton. We will use the covariant normalization for the spinors,

$$\bar{u}(p)\gamma_\lambda u(p) = ip_\lambda, \quad (2)$$

corresponding to the invariant inner product in the Hilbert space

$$\langle \mathbf{p}',\lambda'|\mathbf{p},\lambda\rangle = (2\pi)^3 p_0 \delta^3(\mathbf{p}-\mathbf{p}')\delta_{\lambda',\lambda}, \quad (3)$$

and a density of states in momentum space $(2\pi)^{-3}p_0^{-1}d^3p$.

[6] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[7] D. Stonehill, Yale University dissertation, 1962 (unpublished.)
[8] G. Källén, Helv. Phys. Acta. **25**, 417 (1952). H. Lehmann, Nuovo cimento **11**, 342 (1954).
[9] G. Salzman, Phys. Rev. **99**, 973 (1955).

The form factors f_1 and f_2 are assumed to satisfy the simple dispersion relations

$$f_1(t) = 1 + \frac{t}{\pi}\int_0^\infty \frac{A_1(t')}{t'(t'-t)}dt', \quad (4)$$

$$f_2(t) = \frac{1}{\pi}\int_0^\infty \frac{A_2(t')}{t'-t}dt', \quad (5)$$

where $t=-q^2+i\epsilon$, $\epsilon\to 0+$, and $A_i(t')=\mathrm{Im}f_i(t')$. The subtraction in the dispersion relation for $f_1(t)$ insures that this function satisfies the proper boundary condition for $t=0$. The absorptive parts in Eqs. (4) and (5) may be determined by standard methods. We begin, not with the matrix element in Eq. (1), but with the related matrix element $\langle p\bar{p}^{\text{out}}|j_\lambda|0\rangle$, which we write, following Lehmann, Symanzik, and Zimmermann,[10] in the form

$$\langle p\bar{p}^{\text{out}}|j_\lambda|0\rangle = i\int dx\, \bar{u}(p)e^{-ip\cdot x} \times\langle\bar{p}|\theta(x_0)[f(x),j_\lambda(0)]|0\rangle + \cdots, \quad (6)$$

where

$$f(x) = (\gamma\cdot\partial + m)\psi(x), \quad (7)$$

and $\psi(x)$ is the field operator for the particle in question. We have omitted an equal-time commutator which affects only the constant term in Eq. (4). The imaginary (or absorptive) parts of the function $f_1(t)$ and $f_2(t)$, $t=-(p+\bar{p})^2$, can now be extracted using the observation that, under the operation of Wigner time inversion, $f_1(t)\to f_1^*(t)$, $f_2(t)\to f_2^*(t)$, t real. Upon performing the necessary manipulations, the absorptive part is obtained as the familiar sum over intermediate states,

$$-ie\bar{u}(p)[A_1(t)\gamma_\lambda + (2m)^{-1}A_2(t)\sigma_{\lambda\nu}(p+\bar{p})_\nu]v(\bar{p}) = \tfrac{1}{2}(2\pi)^4\sum_\alpha \delta^4(p+\bar{p}-\alpha)\bar{u}(p) \times\langle\bar{p}|f(0)|\alpha^{\text{out}}\rangle\langle\alpha^{\text{out}}|j_\lambda(0)|0\rangle. \quad (8)$$

For convenience, we have the *out* states for our complete set.

It should be remarked that the form factors as defined in Eq. (1) are associated with the full vertex function $\langle p'|j_\lambda|p\rangle$ rather than with the truncated vertex function[11]

$$\langle p'|\Gamma_\lambda|p\rangle = D(q^2)D'^{-1}(q^2)\langle p'|j_\lambda|p\rangle, \quad (9)$$

and consequently include contributions from vacuum polarization corrections to the incoming photon line. However, these do not affect the static value of the anomalous magnetic moment since the factor $D(q^2)D'^{-1}(q^2)$ in Eq. (9) is equal to unity for $q^2=0$. The vacuum polarization corrections are easily identi-

[10] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955); **6**, 319 (1957).
[11] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **2**, 425 (1955).

fied in a diagrammatic analysis of the sum in Eq. (8), and will be omitted in the ensuing discussion.

The small size of the fine structure constant α allows us to reduce the infinite sum in Eq. (8) to a few manageable terms.[12] The contribution to the sum of lowest order in e $[O(e^3)]$ arises from the intermediate muon pair state, with the matrix elements $\langle\bar{p}|f(0)|p'\bar{p}'^{\text{out}}\rangle$ and $\langle p'\bar{p}'^{\text{out}}|j_\lambda(0)|0\rangle$ evaluated in first Born approximation. This term yields the corrections to the form factors of order α, and the value $(\alpha/2\pi)$ for the anomalous magnetic moment of the muon. The corrections of order α^2 are obtained by calculating these matrix elements to the next higher order and including also the lowest order contribution to the sum of the intermediate state which consists of a muon pair plus a single photon. Corrections to the form factors which involve strongly interacting particles first appear in order α^2, and result from the replacement of the free photon propagator in the first Born approximation for the matrix element $\langle\bar{p}|f(0)|p'\bar{p}'^{\text{out}}\rangle$ by the complete propagator. We will, therefore, confine our attention to the intermediate muon pair state in Eq. (8), and replace the vertex function $\langle p'\bar{p}'^{\text{out}}|j_\lambda(0)|0\rangle$ by its leading term,

$$\langle p'\bar{p}'^{\text{out}}|j_\lambda(0)|0\rangle \to -ie\bar{u}(p')\gamma_\lambda v(\bar{p}'). \tag{10}$$

We will, furthermore, retain only that contribution to the matrix element $\langle\bar{p}|f(0)|p'\bar{p}'^{\text{out}}\rangle$ which contains the pionic corrections of order α,

$$\bar{u}(p)\langle\bar{p}|f(0)|p'\bar{p}'^{\text{out}}\rangle \to e^2\bar{u}(p)\gamma_\beta u(p')\bar{v}(\bar{p}')\gamma_\beta v(\bar{p})d_\pi((p-p')^2). \tag{11}$$

The function $d_\pi(k^2)$, which represents the pionic contribution to the photon propagator, may be identified using the Lehman-Källén representation for the complete propagator $D_{\mu\nu}'$,

$$D_{\mu\nu}'(k^2)=\delta_{\mu\nu}/k^2+[\delta_{\mu\nu}-k_\mu k_\nu/k^2]\int_0^\infty \frac{s(x)}{x+k^2}dx, \tag{12}$$

$$k^4 s(-k^2)=\tfrac{1}{3}(2\pi)^3\textstyle\sum_\alpha\langle 0|j_\mu(0)|\alpha\rangle\langle\alpha|j_\mu(0)|0\rangle\delta^4(\alpha-k).$$

Restricting the sum to those intermediate states which involve pions, we obtain to the appropriate order in α

$$d_\pi(k^2)=\int_{4m_\pi^2}^\infty \frac{s_\pi(x)}{x+k^2}dx, \tag{13}$$

where

$$k^4 s_\pi(-k^2)=\tfrac{1}{3}(2\pi)^3\sum_{n\pi}\langle 0|j_\mu(0)|n\pi\rangle\langle n\pi|j_\mu(0)|0\rangle \times\delta^4(p_1+\cdots+p_n-k). \tag{14}$$

Since j_μ is a vector operator in x space, and a sum of scalar and vector operators in isotopic spin space, the summation in Eq. (14) is actually restricted to the n-pion states with $J=1^-$ and $T=0$ or 1. Furthermore, the n-pion state contributes to s_π only for $-k^2$ larger than the threshold value, $-k^2=(nm_\pi)^2$. We shall discuss this function in detail in Sec. III.

Using Eqs. (8), (10), and (11), one finds that the pionic contributions $A_{i,\pi}(t)$ to the absorptive parts of the form factors f_1 and f_2 are given to lowest order in α by

$$\begin{aligned}\bar{u}(p)[A_{1,\pi}(t)\gamma_\lambda&+(2m)^{-1}A_{2,\pi}(t)\sigma_{\lambda\nu}(p+\bar{p})_\nu]v(\bar{p})\\ &=\tfrac{1}{2}e^2(2\pi)^4\textstyle\sum\delta^4(p+\bar{p}-p'-\bar{p}')d_\pi((p-p')^2)\\ &\quad\times\theta(t-4m_\mu^2)\bar{u}(p)\gamma_\beta u(p')\bar{u}(p')\gamma_\lambda v(\bar{p}')\bar{v}(\bar{p}')\gamma_\beta v(\bar{p})\\ &=(\alpha/16\pi)(t-4m_\mu^2)^{1/2}t^{-1/2}\theta(t-4m_\mu^2)\\ &\quad\times\int d\Omega d_\pi((p-p')^2)\bar{u}(p)\gamma_\beta(-i\gamma\cdot p'+m)\gamma_\lambda\\ &\qquad\times[-i\gamma\cdot(p+\bar{p}-p')-m]\gamma_\beta v(\bar{p}),\end{aligned} \tag{15}$$

where $t=-(p+\bar{p})^2$, and the remaining integration is over the directions of the muon momentum in the center-of-mass system of the pair. After the azimuthal integration is performed, the result can be reduced to standard form, and the functions $A_{1,\pi}(t)$ and $A_{2,\pi}(t)$ extracted, through the use of the Dirac equations for the spinors u and v. Substituting for $d_\pi((p-p')^2)$ the integral representation given in Eq. (13), and replacing the integration over θ by an integration over u, $\cos\theta=1-2u$, one obtains

$$\begin{aligned}A_{1,\pi}(t)&=\tfrac{1}{4}\alpha\theta(t-4m_\mu^2)(t-4m_\mu^2)^{\frac{1}{2}}t^{-\frac{1}{2}}\int_{4m_\pi^2}^\infty dx\,s_\pi(x)\\ &\quad\times\int_0^1 du\,[x+(t-4m_\mu^2)u]^{-1}[2(t-4m_\mu^2)(1-u)^2\\ &\qquad+4m_\mu^2(1-4u+6u^2)],\end{aligned} \tag{16}$$

$$\begin{aligned}A_{2,\pi}(t)&=2\alpha m_\mu^2\theta(t-4m_\mu^2)(t-4m_\mu^2)^{\frac{1}{2}}t^{-\frac{1}{2}}\int_{4m_\pi^2}^\infty dx\,s_\pi(x)\\ &\quad\times\int_0^1 du\,u(2-3u)[x+(t-4m_\mu^2)u]^{-1}.\end{aligned} \tag{17}$$

The pionic correction $\Delta\kappa$ to the anomalous magnetic moment of the muon may be expressed in terms of $s_\pi(x)$ using Eqs. (5) and (17) in conjunction with the definition $\kappa=f_2(0)$,

$$\Delta\kappa=\frac{1}{\pi}\int_{4m_\pi^2}^\infty\frac{A_{2\pi}(t)}{t+q^2}dt. \tag{18}$$

After a lengthy, but straightforward calculation, one obtains the desired result,

$$\Delta\kappa=(\alpha/2\pi)\int_{4m_\pi^2}^\infty g(4m_\mu^2/x)s_\pi(x)dx, \tag{19}$$

[12] It is amusing to note the contrast between the present approximation and that often used in problems involving only strongly interacting particles. In the latter, one retains only the states of the lowest total mass, and hopes that the denominator in the dispersion relations will suppress the effects of the high mass states. The state of the lowest mass in the present problem is that of three photons, which first enters in order α^3.

with

$$g(z)=\frac{2}{z}\left\{4\left(\frac{2}{z}-1\right)\ln\frac{4}{z}-\left[\frac{8}{z}-8+z\right]\frac{1}{(1-z)^{1/2}}\times\ln\frac{1+(1-z)^{1/2}}{1-(1-z)^{1/2}}+\left(\frac{3}{z}-2\right)\ln(1-z)-1\right\}. \quad (20)$$

As will be shown, the main contributions to the remaining integral probably arise from values of x for which $z=4m_\mu^2/x\ll1$. In this region, $g(z)$ may be expanded as follows:

$$g(z)\to z\left[\frac{1}{6}+z\left(\frac{25}{96}-\frac{1}{8}\ln\frac{4}{z}\right)+z^2\left(\frac{97}{320}-\frac{3}{16}\ln\frac{4}{z}\right)+z^3\left(\frac{13}{40}-\frac{7}{32}\ln\frac{4}{z}\right)+\cdots\right]. \quad (21)$$

This series is perhaps most easily derived by performing the integration over t in Eq. (18) (for $q^2=0$), and expanding the result in powers of zu before performing the integration over u. Using this expansion in Eq. (19), we obtain as a first approximation for $\Delta\kappa$

$$\Delta\kappa\sim(\alpha/3\pi)\int_{4m_\pi^2}^{\infty}(m_\mu^2/x)s_\pi(x)dx. \quad (22)$$

This result exhibits clearly the dependence of $\Delta\kappa$ on the effective mass of the intermediate state, $\Delta\kappa\propto m_\mu^2/M^2$ for $m_\mu^2/M^2\ll1$.

III. PIONIC CORRECTIONS TO THE PHOTON PROPAGATOR

The calculation of the pionic corrections to the anomalous magnetic moment of the muon is reduced at this point to the calculation of the spectral function $s_\pi(t)$. This was defined in Eq. (14) as a sum over the n-pion intermediate states with $J=1^-$ and $T=0$ or 1,[13]

$$x^2s_\pi(x)=\tfrac{1}{3}(2\pi)^3\sum\langle0|j_\mu(0)|n\pi\rangle\langle n\pi|j_\mu(0)|0\rangle\times\delta^4(p_1+\cdots+p_n-k),\quad x=-k^2. \quad (14')$$

There is, at present, very little information about the properties of multipion systems, considerably less theoretically than experimentally. It is known that there are $J=1^-$ resonances in the two-pion $(T=1)$[4] and three-pion $(T=0)$[5] systems at masses of 750 MeV and 785 MeV, respectively, and perhaps at lower masses as well.[14,15] The possibility exists that these resonances dominate the two- and three-pion contributions to $s_\pi(x)$; we shall assume that this is the case. Unfortunately the resonant energies lie above the five-pion threshold and it is by no means clear that the contributions of the states containing four, five, or more pions can be neglected. We shall make some rough estimates with respect to the four-pion state, but shall not attempt to explore this question in detail.

The contribution of the two-pion state to the sum in Eq. (14′) can be expressed in terms of the pion form factor $F_\pi(x)$,

$$\langle p,i;q,j^{\rm out}|j_\mu(0)|0\rangle=ie(p-q)_\mu(4p_0q_0)^{-1/2}F_\pi(x)\epsilon_{ij3}, \quad (23)$$

where $x=-(p+q)^2$, i and j are the isotopic spin indices of the pions, and ϵ_{ijk} is the completely antisymmetric tensor in three dimensions. Substituting this definition in Eq. (14′), summing over the isotopic spin indices, and integrating over the allowed phase space, one obtains for the two-pion contribution to $s_\pi(x)$,

$$s_\pi^{(2)}(x)=(\alpha/12\pi)(x-4m_\pi^2)^{3/2}x^{-5/2}\times|F_\pi(x)|^2\theta(x-4m_\pi^2). \quad (24)$$

While there is as yet no experimental information on the pion form factor,[13] it is expected that the structure of $F_\pi(x)$ will be determined to a considerable extent by the observed $T=J=1$ resonance in pion-pion scattering (the "ρ meson").[4,16] The pole of the S matrix which leads to the resonance appears also in the contribution of the two-pion intermediate state to the pion form factor.[17] If, as expected, this state is the most important, $F_\pi(x)$ will be strongly peaked for values of x near the scattering resonance, $x\sim m_\rho^2\sim(750\text{ MeV})$,[2] and may be approximated in this region by the expression

$$F_\pi(x)\sim m_\rho^2R(m_\rho^2-x-im_\rho\Gamma_\rho)^{-1}. \quad (25)$$

Here, Γ_ρ is the full width at half maximum of the ρ-meson resonance in pion-pion scattering, 60 MeV $\lesssim\Gamma_\rho\lesssim$150 MeV,[4,7] and m_ρ^2R is the residue of $F_\pi(x)$ at the ρ pole. It is interesting to examine this result from the point of view of Gell-Mann and Zachariasen.[18] Those authors consider the ρ meson as an unstable elementary particle coupled to the vector isotopic spin currents of the mesons and baryons. Since the electromagnetic field is also coupled to the conserved isotopic spin current, there is an effective coupling between the

[13] The spectral function $s_\pi(x)$ enters as a factor in the cross section for the annihilation process $e+\bar{e}\to$ pions, and, in principle, can be determined by the measurement of this cross section in an electron-positron colliding-beams experiment. The individual terms $s_\pi^{(n)}(x)$ in the sum in Eq. (14′) enter as factors in the partial annihilation cross sections for $e+\bar{e}\to n\pi$. See, for example, N. Cabibbo and R. Gatto, Phys. Rev. Letters 4, 313 (1960).

[14] A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961). P. L. Bastien, J. P. Berge, O. I. Dahl, M. Ferro-Luzzi, D. H. Miller, J. J. Murray, A. H. Rosenfeld, and M. B. Watson, *ibid.* **8**, 114 (1962); E. Pickup, D. K. Robinson, and E. O. Salant, *ibid.* **8**, 329 (1962).

[15] R. Barloutaud, J. Heughebaert, A. Leveque, J. Meyer, and R. Omnes, Phys. Rev. Letters **8**, 32 (1962). A. R. Erwin, R. March, W. D. Walker, and E. West, *ibid.* **6**, 628 (1961). C. C. Peck, L. W. Jones, and M. L. Perl Phys. Rev. **126**, 1836 (1962). B. Sechi Zorn, Phys. Rev. Letters **8**, 282 (1962).

[16] W. Frazer and J. Fulco, Phys. Rev. Letters **2**, 365 (1959); Phys. Rev. **117**, 1609 (1960). An extensive bibliography of theoretical papers on the pion-pion interaction is given by Button *et al.*, reference 4.

[17] R. Oehme, Phys. Rev. **121**, 1840 (1961).

[18] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

neutral ρ meson and the photon, $\rho^0 \leftrightarrow \gamma$, with a coupling constant defined as $(em_\rho^2/2\gamma_\rho)$. Denoting the coupling constant for the coupling of the ρ meson to the isotopic vector current of the two-pion system by $2\gamma_{\rho\pi\pi}$, and retaining only those contributions to the pion form factor which involve an intermediate ρ meson, one obtains for $F_\pi(x)$ the result given in Eq. (25), but with R replaced by the ratio[6,18]

$$R=\gamma_{\rho\pi\pi}/\gamma_\rho. \tag{26}$$

With this identification of R, Eq. (25) assumes a form familiar in the theory of multichannel nuclear reactions in which the residue at a Breit-Wigner pole is a product of two factors, one characteristic of the incident channel, the other, of the exit channel.[19] The denominator depends, furthermore, on the full width Γ_ρ of the ρ meson resonance,

$$\Gamma_\rho=\Gamma_{\rho,2\pi}+\Gamma_{\rho,4\pi}+\cdots, \tag{27}$$

and incorporates thereby the effects on $F_\pi(x)$ of all intermediate states which couple to two pions through the ρ resonance. Assuming for the moment that the ρ meson contributions to $F_\pi(x)$ are dominant, and substituting Eqs. (25) and (26) in Eq. (24), we obtain for $s_\pi^{(2)}(x)$ the approximation

$$s_\pi^{(2)}(x)\sim(\alpha/12\pi)(\gamma_{\rho\pi\pi}/\gamma_\rho)^2 m_\rho^4 x^{-5/2}(x-4m_\pi^2)^{3/2} \times[(x-m_\rho^2)^2+m_\rho^2\Gamma_\rho^2]^{-1}\theta(x-4m_\pi^2). \tag{28}$$

Since $\Gamma_\rho \ll m_\rho$, the last factor in this expression can be replaced by a delta function of weight $(\pi/m_\rho\Gamma_\rho)$, an approximation accurate to better than 1% in Eq. (22), and we write finally

$$\begin{aligned} s_\pi^{(2)}(x)&\sim(\alpha/12)(\gamma_{\rho\pi\pi}/\gamma_\rho)^2\Gamma_\rho^{-1}m_\rho^{-2} \times(m_\rho^2-4m_\pi^2)^{3/2}\delta(x-m_\rho^2)\\ &=(\alpha/4)(4\pi/\gamma_\rho^2)(\Gamma_{\rho,2\pi}/\Gamma_\rho)\delta(x-m_\rho^2). \end{aligned} \tag{29}$$

In the last step we have observed that the partial width for the strong $\rho\rightarrow 2\pi$ decay mode is given in terms of $\gamma_{\rho\pi\pi}$ by

$$\Gamma_{\rho,2\pi}=\tfrac{1}{3}(\gamma_{\rho\pi\pi}^2/4\pi)(m_\rho^2-4m_\pi^2)^{3/2}m_\rho^{-2}. \tag{30}$$

It is clear from Eqs. (22) and (24) that $\Delta\kappa$ is most sensitive to the behavior of $F_\pi(x)$ for values of x near the two-pion threshold at $x=4m_\pi^2$. It has been suggested that there is a second $T=J=1$ resonance (the ζ meson) in the two-pion system at an energy of about 570 MeV,[15] but it is, as yet, not clear that the resonance is real;[7] the spin and parity assignments are purely speculative.[20] Should it in fact exist, it would contribute a second term of the form given in Eq. (25) to $F_\pi(x)$. There is otherwise no evidence at present for any strong two-pion interaction in the $T=J=1$ state for $x<m_\rho^2$, and we will therefore assume that $F_\pi(x)$ is relatively small (~ 1) in this region. On the other hand, it has been suggested by Blankenbecler[21] that there may be important contributions to the pion-pion scattering amplitude, hence also to the pion form factor, associated with four-pion intermediate states in which three of the pions are in resonance in the $T=0$, $J=1^-$ state (the ω meson resonance[5]). This effect can be calculated in a rather speculative manner using the ideas of Gell-Mann and Zachariasen[18] and Gell-Mann *et al.*[22] One introduces an effective $\rho-\omega-\pi$ coupling. The $\omega-\pi$ system is then coupled to the electromagnetic field and to the two-pion system through intermediate ρ meson states. A rough calculation of a type to be sketched later then shows that, for the value of the $\rho-\omega-\pi$ coupling constant consistent with the $\pi^0\rightarrow 2\gamma$ decay rate,[22] the $\omega-\pi$ contributions to $F_\pi(0)$ are at most of the order of a few percent of its known value, unity, hence, that some other state, presumably the ρ meson, must give the dominant contribution to this quantity. It is easily shown that the $\omega-\pi$ contributions to $s_\pi^{(2)}$ are also small. We shall therefore assume that $F_\pi(x)$ is well approximated by the expression in Eq. (25), and $s_\pi^{(2)}(x)$, by that in Eq. (29). The boundary condition $F_\pi(0)=1$ then requires that $R=\gamma_{\rho\pi\pi}/\gamma_\rho\sim 1$.

The vector meson approach provides what is at present the only tractable method for estimating the contributions of the three-pion intermediate state to $s_\pi(x)$. We will therefore assume that the $T=0$, $J=1^-\omega$ resonance gives the main contribution to $s_\pi^{(3)}$, that is, that the matrix element $\langle 3\pi|j_\mu(0)|0\rangle$ in Eq. (14′) is large only for values of $\sqrt{x}$ near the mass of the ω meson, $m_\omega\sim 785$ MeV.[5] This assumption can, in principle, be checked by measuring the cross section for the annihilation process $e+\bar{e}\rightarrow\pi^++\pi^-+\pi^0$ in a colliding beams experiment: The cross section contains $s_\pi^{(3)}(x)$ as a factor.[13] If the ω meson is coupled to the scalar isotopic spin current, it will also be coupled to the electromagnetic field; following Gell-Mann and Zachariasen,[18] the effective $\omega-\gamma$ coupling constant will be denoted by $(em_\omega^2/2\sqrt{3}\gamma_\omega)$. The factor $\sqrt{3}$ is introduced so that $\gamma_\omega\sim\gamma_\rho$ in the limit of unitary symmetry for the meson-baryon interactions.[6] With this convention, the ω-meson contribution to $s_\pi^{(3)}(x)$ is readily found to be

$$s_\pi^{(3)}(x)\sim(\alpha/12)(4\pi/\gamma_\omega^2)(\Gamma_{\omega,3\pi}/\Gamma_\omega)\delta(x-m_\omega^2), \tag{31}$$

where $\Gamma_{\omega,3\pi}$ is the partial width for the $\omega\rightarrow\pi^++\pi^-+\pi^0$ decay mode, and Γ_ω is the full width of the ω-meson resonance. It is possible that there are significant contributions to $s_\pi^{(3)}$ which arise from other configurations of the three-pion system. One such configuration may be associated with the η meson with a mass of 550 MeV.[14] However, the spin and parity assignment for this particle are still in doubt, with $J=0^-$ and positive G parity apparently favored over 1^- and

[19] See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. 10.

[20] For possible consequences of different spin assignments, see for example G. Feinberg and A. Pais, Phys. Rev. Letters **8**, 341 (1962).

[21] R. Blankenbecler, Phys. Rev. **125**, 755 (1962).

[22] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

negative G parity.[23] If the first assignment is correct, the η cannot contribute to $s_\pi^{(3)}$. Of the more complex configurations of three pions, perhaps the most likely to be important consists of a ρ meson (resonant two-pion system) plus a single pion. It is amusing to attempt to estimate the effects of this state, treating the meson as a stable particle. The $\gamma-\rho-\pi$ matrix element $\langle\rho\pi|j_\mu|0\rangle$ will be taken in the form

$$F_{\rho\pi\gamma}(x)m_\omega^{-1}(4p_0k_0)^{-1/2}\epsilon_{\mu\nu\sigma\lambda}p_\nu\xi_\sigma k_\lambda\delta_{ij}, \qquad (32)$$

where $F_{\rho\pi\gamma}(x)$ is the form factor for the $\rho-\pi-\gamma$ vertex, p and k are the four-momenta of the ρ and π mesons, i and j denote their isotopic spin indices, and ξ is the spin vector of the ρ. If we follow the speculations of Gell-Mann *et al.*,[22] the $\rho-\pi-\gamma$ coupling proceeds through a $\omega-\gamma$ electromagnetic coupling, followed by a strong $\omega-\rho-\pi$ coupling. Accordingly,[24]

$$F_{\rho\pi\gamma}(x)\sim e(2\sqrt{3}\gamma_\omega)^{-1}f_{\rho\omega\pi}m_\omega^2(m_\omega^2-x)^{-1}. \qquad (33)$$

The ratio $(f_{\rho\omega\pi}^2/\gamma_\omega^2)$ can be estimated in the same speculative manner by using a $\pi^0\to\rho^0+\omega$, $\rho^0\to\gamma$, $\omega\to\gamma$ chain to calculate the rate for the $\pi^0\to 2\gamma$ decay.[22] Using $(\gamma_\rho^2/4\pi)\sim 0.3$, corresponding to a ρ meson width $\Gamma_\rho\sim 60$ MeV,[7] one finds that $(f_{\rho\omega\pi}^2/\gamma_\omega^2)\sim 0.7$, a result which yields for the effective $\rho-\pi-\gamma$ coupling constant the plausible value $F_{\rho\pi\gamma}(0)\sim e/4$. A brief calculation in which m_π, the width of the ρ resonance, and the $\rho-\omega$ difference are neglected relative to m_ω leads from Eqs. (32) and (33) to the result

$$\Delta s_\pi^{(3)}(x)\sim(\alpha/96\pi)(f_{\rho\omega\pi}/\gamma_\omega)^2 \times m_\omega^2x^{-3}(x-m_\omega^2)\theta(x-m_\omega^2). \qquad (34)$$

The corresponding contribution to $\Delta\kappa$ is about 1% of that of the ω meson term, Eq. (31). While this estimate could be grossly in error, it may, perhaps, make more plausible the assumption that ω meson resonance indeed gives the most important contribution to $s_\pi^{(3)}$.

We turn finally to the four-pion intermediate state, about which our knowledge is essentially nil. It has been suggested by Blankenbecler[21] that this state, or in particular, the configuration consisting of an ω meson plus a pion, may significantly influence the pion-pion scattering amplitude. It would, therefore, be expected to contribute significantly to $s_\pi(x)$ through a $\gamma\to\rho^0$, $\rho^0\to\omega+\pi^0$ chain. However, a calculation of the type performed above for the $\rho-\pi$ contribution to $s_\pi^{(3)}(x)$, using the coupling constant $f_{\rho\omega\pi}$ derived from the $\pi^0\to 2\gamma$ decay, leads again to negligible changes in $\Delta\kappa$. A much more important effect, and one which can be estimated fairly reliable, is associated with the ρ-meson resonance, which lies well above the four-pion threshold. It is consequently possible to have a resonance in the four-pion system corresponding to this "compound state." One readily verifies that the resulting contribution to $s_\pi^{(4)}(x)$ is equal to $(\Gamma_{\rho,4\pi}/\Gamma_{\rho,2\pi})$ times the ρ-meson contribution to $s_\pi^{(2)}(x)$, where $\Gamma_{\rho,2\pi}$ and $\Gamma_{\rho,4\pi}$ are the partial widths for the two- and four-pion decay modes of the ρ. The four-pion decay mode has not yet been detected, but could possibly represent a significant fraction of the total decays.[25] If, as seems likely, the strong two-pion and four-pion decay modes are the only ones which are significant for the ρ meson, then

$$\Gamma_\rho\sim\Gamma_{\rho,2\pi}+\Gamma_{\rho,4\pi},$$

and, combining the two-pion and four-pion contributions to $s_\pi(x)$, we obtain the total contribution associated with the ρ meson,

$$s_\pi^\rho(x)\sim(\alpha/4)(4\pi/\gamma_\rho^2)\delta(x-m_\rho^2). \qquad (35)$$

If $\Gamma_{\rho,2\pi}\gg\Gamma_{\rho,4\pi}$, and if the ρ-meson state gives the largest contributions to $F_\pi(x)$, then $\gamma_\rho\sim\gamma_{\rho\pi\pi}$, and the latter may be related to Γ_ρ. It is amusing to note that Eq. (35) can be obtained immediately in the vector meson theory[6,18,22] by considering the chain $\gamma\to\rho^0\to\gamma$, and foregoing an analysis in terms of the physical intermediate states. The more detailed analysis may, however, cast some light on the approximations involved in Eq. (35), and is necessary for the study of configurations which are not likely to be reached through the ρ resonance.

It is clear that similar results hold for the ω meson, although in this case the allowed strong three-pion and five-pion decay modes are so strongly suppressed by phase space and angular momentum factors that such relatively weak modes as $\omega\to\pi^0+\gamma$ and $\omega\to\pi^++\pi^-$ may become important.[22,26] If the appropriate intermediate states are included in the sum in Eq. (14′), the ω meson contributions to $s_\pi(x)$ clearly becomes

$$s_\pi^\omega(x)\sim(\alpha/12)(4\pi/\gamma_\omega^2)\delta(x-m_\omega^2). \qquad (36)$$

Should the η meson have spin $J=1^-$ and odd G parity, it would contribute a similar term to $s_\pi(x)$.[26a]

IV. RESULTS AND DISCUSSION

Calculation of the pionic corrections to the static magnetic moment of the muon is now straightforward. We will use the expression for $\Delta\kappa$ in terms of $s_\pi(x)$ given in Eq. (19), substituting for $g(4m_\mu^2/x)$ the expansion given in Eq. (21). Because the major contributions to $s_\pi(x)$ arise from rather large values of x, the expansion converges quite rapidly. The pionic

[23] D. D. Carmony, A. H. Rosenfeld, and R. T. Van de Walle, Phys. Rev. Letters **8**, 117 (1962). *Note added in proof*. The 1^{--} assignment necessary if the η meson is to contribute to $s_\pi^{(3)}$ is clearly precluded by more recent data. See, for example, H. Foelsche, E. C. Fowler, H. L. Kraybill, J. R. Sanford, and D. Stonehill, Phys. Rev. Letters **9**, 223 (1962); M. Chrétien *et al.*, Phys. Rev. Letters **9**, 127 (1962).

[24] The coupling constant $f_{\rho\omega\pi}$ in reference 22 corresponds to our $f_{\rho\omega\pi}/m_\omega$.

[25] This possibility is presently being studied [J. Sanford and D. Stonehill (private communications)].

[26] Y. Nambu and J. J. Sakurai, Phys. Rev. Letters **8**, 79 (1962); G. Feldman, T. Fulton, and K. C. Wali (to be published); G. Feinberg, Phys. Rev. Letters **8**, 151 (1962); B. T. Feld, *ibid.* **8**, 181 (1962).

[26a] See however, the note added in proof, footnote 23.

contribution $s_\pi(x)$ to the spectral function for the photon propagator will be approximated in accordance with the results of the preceding section as

$$s_\pi(x) \sim s_\pi{}^\rho(x) + s_\pi{}^\omega(x),$$

with $s_\pi{}^\rho$ and $s_\pi{}^\omega$ given by the expressions in Eqs. (35) and (36). One then obtains

$$\Delta\kappa \sim (\alpha^2/12\pi)[(4\pi/\gamma_\rho{}^2)K(m_\mu/m_\rho) + \tfrac{1}{3}(4\pi/\gamma_\omega{}^2)K(m_\mu/m_\omega)], \quad (37)$$

where

$$K(z) = z^2[1 + z^2(25/4 + 6\ln z) + O(z^4)]. \quad (38)$$

We shall evaluate γ_ρ from Eq. (30), assuming that $\Gamma_\rho \sim \Gamma_{\rho,2\pi}$ and that $\gamma_{\rho\pi\pi} \sim \gamma_\rho$; these conditions are equivalent to the assumption that the structure of the pion form factor $F_\pi(x)$ is determined almost entirely by the resonant two pion intermediate state. The $\omega - \gamma$ coupling constant γ_ω can probably be measured by determining the rates for the rare leptonic decay modes of the ω, $\omega \to e + \bar{e}$ or $\omega \to \mu + \bar{\mu}$,[22,26] but is not yet known. We shall therefore follow Gell-Mann,[6] and assume that γ_ω can be approximated by its limit for the unitary symmetry model of the strong interactions, $\gamma_\omega = \gamma_\rho$. We remark only that this approximation is consistent with the observation that the effective ω-nucleon coupling should be stronger than the ρ-nucleon coupling if the vector mesons are to account for the hard core in the static nucleon-nucleon potential[27]; the derivation of more quantitative conclusions about the coupling constants from potential considerations would appear to be a rather dubious procedure. If, finally, we ignore the $\rho - \omega$ mass difference, and use the expression in Eq. (30), $\Delta\kappa$ may be written in the form

$$\Delta\kappa \sim (\alpha^2/27\pi)(m_\rho{}^2 - 4m_\pi{}^2)^{\frac{3}{2}} m_\mu{}^2 m_\rho{}^{-4}\Gamma_\rho{}^{-1} \times[1 + (m_\mu/m_\rho)^2(25/4 - 6\ln(m_\rho/m_\mu) + \cdots] \sim (0.45\times10^{-6})m_\mu{}^2 m_\rho{}^{-1}\Gamma_\rho{}^{-1}. \quad (39)$$

The width Γ_ρ of the ρ meson resonance is not well known. Using values of the pion-pion scattering cross section derived from the Yale data on the process $\pi^+ + p \to \pi^+ + \pi^0 + p$, and assuming a $J = 1^-$ resonance with an incoherent background, Stonehill[7] obtains a width $\Gamma_\rho = 60 \pm 20$ MeV. However, considerably different widths can be obtained if the background is assumed to be coherent. The widths quoted by other authors are somewhat larger than the above[4,28] but it seems likely in any case that Γ_ρ lies in the range 60 MeV$\lesssim \Gamma_\rho \lesssim$150 MeV. We shall choose $\Gamma_\rho = 60$ MeV in the following calculations; this choice probably gives the largest value of $\Delta\kappa$ consistent with the pion-pion scattering experiments. We then obtain

$$(\Delta\kappa)_\pi \sim 1.1\times10^{-7}, \quad m_\rho = 750 \text{ MeV}, \quad \Gamma_\rho = 60 \text{ MeV}. \quad (40)$$

Should the ζ meson exist and have spin $J = 1^-$ and isotopic spin $T = 1$, it could contribute to the pion form factor as noted previously. If this contribution was of importance comparable to that of the ρ meson, $\Delta\kappa$ would be considerably increased: The reported mass of the ζ is less than that of the ρ, $m_\zeta \sim 570$ MeV, and the width Γ_ζ is possibly 15 MeV or less.[15] Similar remarks may hold for the η meson[14] with respect to the contributions of the three-pion intermediate state.[26a] However, one does not in this case have a sum rule analogous to the relation $F_\pi(0) = 1$ which is basically responsible for the appearance of Γ_ρ in the denominator in Eq. (39), and must consequently estimate the relevant $\gamma - \eta$ coupling constant by other means. Pending further experimental developments, we shall assume that the ζ and η do not contribute to $\Delta\kappa$.

The result in Eq. (40) is to be added to the value of κ calculated to order α^2 assuming a purely electrodynamic theory involving only muons and electrons,[1]

$$\kappa = (\alpha/2\pi) + 0.75(\alpha/\pi)^2 = 11\,655\times10^{-7}, \quad (41)$$

thereby increasing the last figure by one unit. For comparison, we note that the sixth-order electrodynamic corrections are expected to be on the order of $(\alpha/\pi)^3 \sim 1.2\times10^{-8}$, smaller than $\Delta\kappa$ by a factor of 10. The best experimental value of κ is that obtained by the CERN group[2] in a direct measurement of the difference 2κ between the actual gyromagnetic ratio of the muon and its value $g = 2$ for a noninteracting Dirac particle,

$$\kappa_{\text{exp}} = (11\,620\pm50)\times10^{-7}. \quad (42)$$

The pionic contributions to κ are, therefore, smaller than the present limits of experimental error by a factor of 50. The prospects for an immediate improvement in the experimental accuracy sufficient to detect these effects appear to be remote. It would nevertheless be of great interest to detect the pionic effects, since these represent the onset of a region in which the pure electrodynamics of electrons and muons must be supplemented by information from the theory of strongly interacting particles.

[27] The effects of the vector mesons on the nucleon-nucleon potential have been discussed G. Breit, Proc. Nat'l. Acad. Sci. (U.S.) **46**, 746 (1960); and by J. J. Sakurai, Ann. Phys. **11**, 1 (1960). More quantitative results have been obtained by R. S. McKean, Jr., Phys. Rev. **125**, 1399 (1962).

[28] The width of the meson resonance has been quoted as 150–200 MeV (Erwin *et al.*), 90 MeV (Stonehill *et al.*), 130 MeV (Pickup *et al.*), 90 MeV or less (McLeod *et al.*), 150 MeV (Carmony and Van de Walle), where the appropriate references are given in footnote 4. An analysis of the Yale data by Stonehill more detailed than attempted previously (Stonehill *et al.*) yields $\Gamma_\rho = 60\pm20$ MeV. Button *et al.* have reported some evidence that the ρ meson resonance has a double structure, with one peak near the mass of the ω. The second peak occurs at a mass of 720 MeV and has a width of 20–60 MeV. If this structure is real, the high-mass peak can possibly be ascribed to the expected isospin-violating two-pion decay mode of the ω,[22] the low-mass peak, to the ρ meson. Assuming only a single peak to be present, Button *et al.* obtain $m_\rho \sim 765$ MeV, $\Gamma_\rho \sim 100$ MeV. It is perhaps worth noting that the narrower widths are obtained from the study of charged ρ mesons; the two-pion decay mode of the ω cannot contribute to the data in these cases. There remains in every case the problem of removing background events.

It is interesting to compare the resonant pionic contribution to κ, Eq. (40), to that which is obtained by calculating the two-pion contributions to $s_\pi(x)$ in the first Born approximation $[F_\pi(x)=1]$,

$$(\Delta\kappa)_B=(\alpha^2/360\pi^2)(m_\mu/m_\pi)^2[1-0.18+O(m_\mu^4/m_\pi^4)] \sim 0.7\times10^{-8}. \quad (43)$$

We have again used the expansion in Eq. (21) to evaluate the integral in Eq. (19). The ρ and ω resonances lead to a considerable enhancement of the pionic contributions to κ, as would be expected. If we consider only the two-pion system, the enhancement factor is on the order of 12. We note also the contribution from the nucleon pair state, again calculated in first Born approximation,

$$(\Delta\kappa)_{N\bar{N}}=(\alpha^2/45\pi^2)(m_\mu/m_p)^2[1+O(m_\mu^2/m_p^2)] \sim 1.5\times10^{-9}. \quad (44)$$

This is entirely negligible. As we have seen, there is reason to believe that the nonresonant contributions to the photon spectral function, hence, to $\Delta\kappa$, are small for the three-pion and four-pion intermediate states, but in any case, contributions from these and other states can only *increase* the value of $\Delta\kappa$. For that reason, we could obtain a rigorous lower bound for $\Delta\kappa$ by calculating, for example, only the ρ-meson contributions to $s_\pi(x)$; however, this requires more precise knowledge about Γ_ρ and R in Eq. (25), hence, about the electromagnetic form factor of the pion.

We remark finally that the pionic corrections to the photon propagator which we have considered enter also in many other processes, in some of which they are in fact much more important. For example, retaining only the ρ- and ω-meson contributions, one may write the propagator for $q^2>-m_\rho^2$ as

$$D_{\mu\nu}'(q^2)\to\delta_{\mu\nu}[(q^2)^{-1}+(\alpha/4)(4\pi/\gamma_\rho^2)(m_\rho^2+q^2)^{-1} +(\alpha/12)(4\pi/\gamma_\omega^2)(m_\omega^2+q^2)^{-1} +(e,\bar{e}),(\mu,\bar{\mu})+\cdots], \quad (45)$$

where we have denoted by $(e,\bar{e})$ and $(\mu,\bar{\mu})$ the normal second-order electron and muon vacuum polarization contributions.[29] For $\gamma_\omega=\gamma_\rho$, and $\Gamma_\rho=60$ MeV, the pionic corrections to the propagator are about 0.4% of the leading term for $q^2=m_\rho^2$. The lepton pair terms may be approximated by

$$(l,\bar{l})\to(\alpha/3\pi)(q^2)^{-1}[2\ln(q/m_l)-5/3], \quad q^2\gg m_l^2; \quad (46)$$

these yield 1% (electron) and 0.2% (muon) corrections to the photon propagator for $q^2=m_\rho^2$. Thus, as was noted by Brown and Calogero,[30] vacuum polarization effects associated with multipion states can give significant corrections to electron scattering phenomena. However, electron-electron scattering experiments with accuracies of better than 0.4% in the cross sections, and more complete calculation of the purely electrodynamic corrections are required.[31] The exception is the measurement of the partial annihilation cross sections in an electron-positron colliding beam configuration; even rough measurements can in this case give significant information about the $s_\pi(x)$, the pionic contribution to the photon spectral function.[13]

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Note added in proof. A calculation similar to the present one, but based on the results of Brown and Calogero for the photon propagator,[30] has been made by L. Michel and C. Bouchiat [reference 30 (Phys. Rev.), footnote 14]. The results, which used the Frazer-Fulco values of the two pion resonance parameters,[16] were apparently not published. The author would like to thank Professor Laurie Brown for calling this work to his attention.

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[29] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1955), Chap. 9.

[30] L. M. Brown and F. Calogero, Phys. Rev. Letters **4**, 315 (1960); Phys. Rev. **120**, 653 (1960).

[31] Y. S. Tsai, Phys. Rev. **120**, 269 (1960). The expression on the righthand side of Eq. (A13) of this paper is too large by a factor of 2.