Unitarity and High-Energy Inelastic Scattering*

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The effect of approximately enforcing the unitarity requirement on peripheral collision-type approximations is discussed. A matrix formulation of the Fourier-Bessel representation of the scattering amplitude is utilized which automatically satisfies unitarity at high energies. Our results indicate that quantitative agreement of the peripheral collision approximation with experiment can be accidental since the corrections due to unitarity take the same form and magnitude as the phenomenological form factors depending only on the momentum transfer which are introduced to account for "off-mass-shell" effects. Three models are discussed, and one of them has the behavior characteristics of a Regge trajectory.

I. INTRODUCTION AND GENERAL THEORY

RECENTLY, there have been suggestions that high-energy cross sections are dominated by the one-particle exchange contributions for low-momentum transfers. This particle may be of the conventional "elementary" particle type[1] or may be of the "Regge pole" type.[2] Our objective here is to estimate the corrections to such approximations due to the unitarity requirement. We find that in the high-energy range (5–50 BeV) unitarity reduces the inelastic cross section by a factor which is rather insensitive to the details of the original approximation. The example of particle production in proton-proton collisions is used to illustrate our procedure. Let two protons of momenta p_1 and p_2 scatter into two well-defined groups of particles of momenta p_1' and p_2', respectively, which contain a total of n particles as depicted in Fig. 1. Define $s=-(p_1+p_2)^2=-(p_1'+p_2')^2$ as the square of the center-of-mass energy and $t=-(p_2-p_2')^2=-(p_1-p_1')^2$ as the momentum transfer. We are interested in the case of large s and small t and hence ignore the effects of the crossed momentum transfer variables. Let v_n be the remaining variables necessary to specify the amplitude $M_{2n}(s,t;v_n)$ of the process depicted in Fig. 1. Spins are neglected.

Now in order to impose the unitarity condition upon M_{2n} we are also forced to consider the matrix element

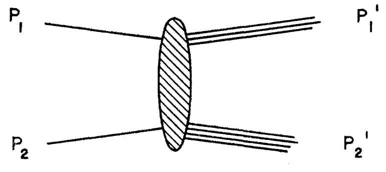

FIG. 1. A production process.

for the process m particles $\rightarrow n$ particles for all m and n, as depicted in Fig. 2, which is described by a matrix element $M_{mn}(s,t;v_{mn})$, where v_{mn} are the remaining variables necessary to specify this multiparticle process. We now assume that the matrix elements of interest have a Fourier-Bessel representation[3] with respect to the variable t:

$$M_{mn}(s,t;v_{mn})=\int_0^\infty bdb\, J_0(b(-t)^{\frac{1}{2}})H_{mn}(s,b;v_{mn}). \quad (1.1)$$

Let h_{mn} denote the absorptive part of the matrix element H_{mn}:

$$2ih_{mn}(s,b;v_{mn})=H_{mn}(s+i\epsilon,b;v_{mn})-H_{mn}(s-i\epsilon,b;v_{mn}), \quad (1.2)$$

where $s>4m^2$, and m is the proton mass.

The advantage of using the Fourier-Bessel transform for expressing the unitarity condition at high energies has been pointed out in reference 3, where it was shown that the two-particle contribution to $h_{22}(s,t)$ in the limit of large s takes on the simple form

$$h_{22}(s,b)=H_{22}(s+i\epsilon;b)\rho_2(s)H_{22}(s-i\epsilon;b), \quad (1.3)$$

where

$$\rho_2(s)=m/ps^{\frac{1}{2}} \quad (1.4)$$

is the approximate two-particle density of states factor using a normalization convention to be given later, and $p^2=(s-4m^2)/4$.

By using analogous arguments one can show that if

$$h_{mn}(s,b;v_{mn})=\sum_{j=2,3,\cdots} H_{mj}(s+i\epsilon,b;v_{mj})\times\rho_j(s;v_j)H_{jn}(s-i\epsilon,b;v_{jn}), \quad (1.5)$$

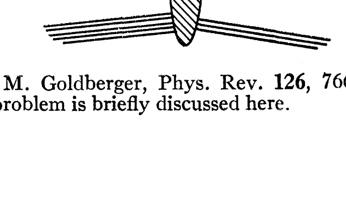

FIG. 2. A multiparticle process.

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[1] S. D. Drell, Revs. Modern Phys. **33**, 458 (1961); G. Salzman, Phys. Rev. Letters **5**, 377 (1960); I. Dremen and D. Chernavskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 229 (1960); D. Amati, S. Fubini, and A. Stanghellini, CERN preprint (1960).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen Phys. Rev. **126**, 2204 (1962).

[3] R. Blankenbecler and M. Goldberger, Phys. Rev. **126**, 766 (1962). The multichannel problem is briefly discussed here.

where $\rho_j(s,v_j)$ is an appropriate density of states factor for a j particle state, and where the summation goes over all numbers of particles and for each j over all the variables v_j necessary to define the intermediate state, then the set of matrix elements M_{mn} of (1.1) approximately satisfies the unitarity equation including *all* intermediate states for $s \gg p_1'^2$ or $p_2'^2$. In deriving (1.5) it was necessary to assume that H_{mj} depends only on v_j and v_m rather than upon the totality of variable v_{jm}. This is not an essential restriction upon the generality of the Eqs. (1.5), as we now show that the solutions of (1.5) are automatically of that form in our approximation.

The Eqs. (1.5) are solved in two stages. Before giving the complete solution we assume that $H_{22}(s,b)$ is known and then proceed to construct H_{2n}, H_{m2}, and H_{mn} $m, n\neq 2$ which satisfy (1.5) in terms of the given H_{22}. We then give explicit expression for H_{22} and the solution of (1.5) is complete.

Let $B_{2n}(s,b;v_m)$ be the Fourier-Bessel transform of the production matrix element in an approximation which neglects initial and final state interactions. The peripheral collision model is a suitable example. $B_{2n}(s,b;v_n)$ is assumed to be given and the object of this work is to show how to correct $B_{2n}(s,b;v_n)$ in order to account for unitarity. In terms of the H_{22}, the solutions are expressible as[4]

$$H_{m2}(s,b;v_m)=B_{m2}(s,b;v_m)[1+I(s)H_{22}(s,b)], \tag{1.6}$$

$$H_{2n}(s,b;v_n)=[1+H_{22}(s,b)I(s)]B_{2n}(s,b;v_n), \tag{1.7}$$

$$H_{mn}(s,b;v_{mn})=B_{m2}(s,b;v_m)I(s)[1+H_{22}(s,b)I(s)] \times B_{2n}(s,b;v_n), \tag{1.8}$$

where

$$I(s)=\int_{4m^2}^{\infty}(ds'/\pi)\rho_2(s')(s'-s)^{-1}, \tag{1.9}$$

and $m, n\neq 2$.

That the functions H_{m2}, H_{2n}, and H_{mn} of the Eqs. (1.6), (1.7), and (1.8) have absorptive parts given by (1.5) is immediately verified by direct substitution. The essence of our approximation is that the multiparticle matrix element, (1.8), is produced only through transitions to a fully interacting two-particle state. Nowhere in Eqs. (1.6), (1.7), and (1.8) does the multiparticle density of states explicitly appear, and it is this fact which allows us to calculate conveniently the corrections due to unitarity. All these complications are contained implicitly in $H_{22}(s,b)$. The second part of the problem is then to construct $H_{22}(s,b)$ from a given approximation $B_{22}(s,b)$ which has none of the physical rescattering singularities due to unitarity. This is most easily accomplished by noting that $H_{22}(s,b)$ is a solution of the following equations[3]:

$$H_{22}D_{22}+\sum_{n\neq 2}H_{2n}D_{n2}=B_{22}(s,b), \tag{1.10}$$

[4] Similar unitarity correction formulas have been derived in a completely different manner by R. D. Amado (to be published).

with

$$D_{22}=1-I(s)B_{22}(s,b) \tag{1.11}$$

and

$$D_{n2}(s,b;v_n)=-\int_{s_n}^{\infty}\frac{ds'}{\pi}\frac{\rho_n(s',v_n)}{s'-s}B_{n2}(s',b;v_n), \tag{1.12}$$

where S_n is the threshold of the n-particle state under discussion. Using Eq. (1.8) for H_{2n} we can solve (1.10) for H_{22} with the result

$$H_{22}=\frac{[B_{22}(s,b)+\sum_{n\neq 2}B_{2n}(s,b;v_n)D_{n2}(s,b;v_n)]}{1-I(s)[B_{22}(s,b)+\sum_{n\neq 2}B_{2n}(s,b;v_n)D_{n2}(s,b;v_n)]}. \tag{1.13}$$

One can now verify directly that the absorptive part of (1.13) satisfies (1.5). Equation (1.13) shows how a strong inelastic force B_{2n} acts twice (or any even number of times) through unitarity with the system ending up in the elastic channel. The inelastic processes are in this way self-damping; in general, the larger the inelastic force B_{2n}, the larger the feedback to the elastic channel. The large elastic scattering which is built up in this manner then acts through unitarity according to (1.7) to damp the inelastic channels at large energies. We estimate the magnitude of this effect when we consider detailed models in Sec. II.

Let us now summarize our formal results. Given any approximations $B_{22}(s,b)$ and $B_{2n}(s,b;v_n)$ to the scattering and production matrix elements which do not have physical singularities in s, we can construct a set of amplitudes from Eqs. (1.7), (1.8), (1.9), and (1.13) which satisfy the approximate unitarity Eqs. (1.5) independent of the specific form of B_{22} and B_{2n}. Because of the complicated form of the solution (1.13) for $H_{22}(s,b)$, we do not apply this complete solution to the particular example considered in this paper. Instead we choose $H_{22}(s,b)$ semiphenomenologically in three different models, and then use our results to give us an estimate of the corrections to inelastic processes due to unitarity.

II. SPECIFIC MODELS

Model A

As a preliminary remark we note that in order that the unitarity equation take the form (1.3), it is necessary that M_{22} be normalized so that[3]

$$2i[M_{22}(s+i\epsilon)-M_{22}(s-i\epsilon)]=2\rho^2\int\frac{d\Omega}{4\pi}M_{22}^*\rho_2M_{22}. \tag{2.1}$$

M_{22} is then related to the differential cross section for the scattering of identical particles by the relation

$$d\sigma/d\Omega=(16m^2/s)|M_{22}|^2. \tag{2.2}$$

As the first model for H_{22}, we choose the empirical fit of p-p scattering given by Cork, Wenzel, and Causey[5]:

$$(d\sigma/d\Omega)^{\frac{1}{2}} = (r^2p/2)e^{tr^2/8}, \qquad (2.3)$$

where $r \gtrsim 10^{-13}$ cm. The recent data obtained at CERN[6] fall considerably below the curve (2.3) for $(-t) \gtrsim 25\mu^2$. This could be accounted for by allowing r to depend upon energy, and we return to this point later. It turns out that we are primarily interested only in the region $t < 25\mu^2$ and (2.3) is an adequate representation of the data in this range. If we assume that M_{22} is purely imaginary, then the choice

$$H_{22}(s,b) = \frac{i}{2\rho_2}\exp(-2b^2/r^2) \qquad (2.4)$$

yields[7]

$$M_{22} = \int bdb\, J_0(b(-t)^{\frac{1}{2}})H_{22}(s,b) = \frac{ir^2}{8m}s^{1/2}p\exp(tr^2/8). \qquad (2.5)$$

Equations (2.5) and (2.2) then give us the desired expression for the differential cross sections.

Then we insert (2.4) into (1.7) to obtain a corrected production amplitude $H_{2n}(s,b,v_n)$ from any given approximation $B_{2n}(s,b;v_n)$:

$$H_{2n}(s,b;v_n) = \left[1 + \frac{iI(s)}{2\rho_2(s)}\exp(-2b^2/r^2)\right]B_{2n}(s,b;v_n). \qquad (2.6)$$

The effect of this correction upon the resulting $M_{2n}(s,t;v_n)$ depends upon the particular form of B_{2n} under consideration. We can make a qualitative estimate of the magnitude of the rescattering effect by assuming $B_{2n}(s,b;v_n)$ to be of the form

$$B_{2n}(s,b;v_n) = f_{2n}(v_n)\exp(-x^2b^2/r^2), \qquad (2.7)$$

where r/x is a measure of the range of the fundamental inelastic scattering. We expect x to be of the order of unity since at high energies the elastic amplitude is driven primarily by the inelastic force acting twice (or any even number of times) as seen explicitly from Eq. (1.13). If the particular state n is the primary driving mechanism for the elastic process, then x must be of order one.

The evaluation of the resulting production matrix element $M_{2n}(s,t;v_n)$ from the formula (1.1) is straightforward:

$$M_{2n}(s,t;v_n) = \int_0^\infty bdb\, J_0(b(-t)^{\frac{1}{2}})\left[1 + \frac{iI}{2\rho_2}e^{-2b^2/r^2}\right] \times f_{2n}(v_n)e^{-x^2b^2/r^2}. \qquad (2.8)$$

If we define

$$M_{2n}^0(s,t;v_n) \equiv \frac{f_{2n}(v_n)}{2x^2}r^2e^{tr^2/4x^2}, \qquad (2.9)$$

where M_{2n}^0 is the uncorrected production amplitude, then the remaining factor in Eq. (2.8) is the correction factor, $K(s,t)$, due to rescattering:

$$M_{2n}(s,t;v_n) = M_{2n}^0(s,t;v_n)K(s,t), \qquad (2.10)$$

where

$$K(s,t) = 1 + \frac{iI(s)}{2\rho_2(s)}\left(\frac{x^2}{2+x^2}\right)\exp[-tr^2/2x^2(2+x^2)]. \qquad (2.11)$$

The deviation of $K(s,t)$ from unity is then a measure of the effect of rescattering on the first approximation $M_{2n}^0(s,t;v_n)$. We might consider $M_{2n}^0(s,t;v_n)$ to be a representation of the peripheral collision approximation. The fact that (2.9) has an exponential dependence on t, while an inverse power dependence is characteristic of the peripheral approximation is not important, since we could not distinguish between these two analytical forms in the range of t of interest.[8]

Now for $s \gg 4m^2$,

$$I(s) \simeq \frac{2m}{s}\left[\frac{1}{\pi}\ln(s/4m^2) + i\right]. \qquad (2.12)$$

As long as $s < 100m^2$, the imaginary part of $I(s)$ gives the dominant contribution in $K(s,t)$. Thus if we set

$$I(s) = i\rho_2(s) \qquad (2.13)$$

and take $x = 1$, our correction factor becomes

$$K(s,t) = 1 - \tfrac{1}{6}\exp(-tr^2/6). \qquad (2.14)$$

We might note several interesting features of this result:

(a) The correction is independent of s in its range of validity $4m^2 \ll s < 100m^2$. At higher energies the factor would develop a logarithmic dependence upon s, but this should not be taken seriously.

(b) In the forward direction $(t=0)$, $K(s,0) = 5/6$. Thus we have a 30% reduction in the forward cross section. The mechanism for this decrease was discussed in Sec. I. Of course, instead of considering the H_{22} given by Eq. (1.13), which is guaranteed to provide the proper feedback of the inelastic channel upon the elastic, we have taken the elastic scattering $H_{22}(s,b)$ from experiment according to Eq. (2.4), assuming that it is purely absorptive. The resulting damping of our given inelastic

[5] B. Cork, W. A. Wenzel, and C. W. Causey, Phys. Rev. **107**, 859 (1957).

[6] G. Cocconi, Reports of CERN Conference on Theoretical Aspects of Very High-Energy Phenomena, June 1961 (unpublished).

[7] Bateman Manuscript Project, California Institute of Technology. *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 2.

[8] In this connection, see the recent work of S. D. Drell and K. Hiida, Phys. Rev. Letters **7**, 199 (1961); and K. Hiida, Phys. Rev. Letters **8**, 149 (1962).

channel according to (1.7) then amounts to about 30%. We try to guess a more realistic form for $H_{22}(b,s)$ and we see that none of the numerical conclusions of this section are affected.

(c) If we look at nonforward directions we find that $K(t)$ decreases from 5/6 at $t=0$ to 1/2 at $(-t)\approx 6/r^2 = 12\mu^2$ to zero at $(-t)\gtrsim 22\mu^2$. As t becomes even larger K becomes large and negative since the exponential term dominates in this region. This happens because the range of the correction term is smaller than the range of the original (peripheral) approximation. However, since our original approximation neglects other short-range corrections, it is clear that our result can only be valid for momentum transfer $(-t)$ not much larger than $1/r^2$. Thus the reduction in the cross section increases from 30% in the forward direction to about 75% at $(-t)\approx 12\mu^2$, and for momentum transfers larger than this we cannot make any reliable statements.

(d) In the peripheral approximation, form factors are introduced in order to obtain quantitative agreement with experiment. The inclusion of the unitarity correction factor $K(t,s)$ allows a fit to experiment with form factors having a weaker momentum transfer dependence. The magnitude of this effect depends strongly upon the details of the experiment.

(e) A further point concerns the test of the peripheral collision approximation suggested by Treiman and Yang.[9] They point out that by studying experimentally the dependence of the matrix element upon all its variables, one should be able to show that it depends only upon v_n and t as demanded by Eq. (2.9). However, the type of unitarity corrections considered here do not change this functional dependence in the range of s under consideration. Thus their type of test would not demonstrate the absence of these corrections to the peripheral collision approximation in this energy range. However, their test is important in examining the dependence on the other subsidiary variables.

(f) If we went to ultrahigh energies, say $E_1\approx 10^6$ BeV, the logarithm term would become the dominant part of $I(s)$ according to Eq. (2.12). At such energies our correction factor would become an enhancement rather than a reduction. We know that this is absurd. The difficulty is that we have not chosen an H_{22} which is unitary, and so we now turn to a second model where H_{22} is chosen to approximately satisfy unitarity. With this choice we see that the observations (a)–(e) remain essentially unaltered.

Model B

Our object in this section is to make a selection for H_{22} which more closely resembles the exact solution (1.13) of our unitarity equations. We choose $H_{22}(s,b)$ to be of the form

$$H_{22}(s,b)=N(s,b)[1-I(s)N(s,b)]^{-1}. \quad (2.15)$$

[9] S. B. Treiman and C. N. Yang, Phys. Rev. Letters 8, 140 (1962).

By giving $N(s,b)$ a suitable imaginary part, we can simulate the effects of the $[B_{2n}B_{n2}]$ terms in the numerator and denominator of (1.13 which account for the contributions of inelastic process to the unitarity condition. This is because the absorptive part $h_{22}(s,b)$ of $H_{22}(s,b)$ satisfies

$$h_{22}=\rho_2(s)|H_{22}|^2+\mathrm{Im}N(s,b)|1-I(s)N(s,b)|^{-2}. \quad (2.16)$$

Now, of course, a choice of $\mathrm{Im}N(s,b)$ which would make the second term on the right-hand side of the above equation precisely equal to the contributions of the inelastic states to the right-hand side of Eq. (1.5) would be tantamount to using the complete solution (1.15). However, we simply choose $N(s,b)$ so that we fit the experimental forward cross section given in Eq. (2.4) and we give it the same Gaussian dependence upon impact parameter e^{-2b^2/r^2} as the $H_{22}(s,b)$ directly determined in (2.4). Then, however, our resulting expression for $M_{22}(s,t)$ does not precisely agree with empirical fit (2.3) for nonforward directions. This difference is quite small for reasonable values of t and simply forces us to choose a slightly different value of the range parameter. The resulting $H_{22}(s,b)$ better approximates the unitary $H_{22}(s,b)$ of (1.13) than does the $H_{22}(s,b)$ resulting from the direct simple fit (2.5). We, therefore, set

$$N(s,b)=-n(s)\exp(-2b^2/r^2), \quad (2.17)$$

and find

$$M_{22}(s,t)=\int_0^\infty bdb\, J_0(b(-t)^{\frac{1}{2}})[-n(s)e^{-2b^2/r^2}] \times[1+Ine^{-2b^2/r^2}]^{-1}. \quad (2.18)$$

In the forward direction, the scattering amplitude becomes

$$M_{22}(s,0)=-\frac{r^2}{4I(s)}\ln[1+n(s)I(s)]. \quad (2.19)$$

Comparison with (2.5) then yields the following equation for $n(s)$:

$$\ln[1+n(s)I(s)]=is^{1/2}pI(s)/2m. \quad (2.20)$$

If we go to the high-energy limit $s\gg 4m^2$, (2.20) becomes, using (2.12),

$$\ln[1+n(s)I(s)]\simeq\tfrac{1}{2}-\frac{i}{2\pi}\ln(s/4m^2). \quad (2.21)$$

Now for $s<100m^2$, the imaginary part of (2.21) is not very important and

$$nI\simeq 0.65. \quad (2.22)$$

In this energy range we have

$$H_{22}(s,b)\simeq\frac{is}{m}\left(\frac{0.65}{2}\right)e^{-2b^2/r^2}[1+0.65e^{-2b^2/r^2}]^{-1}. \quad (2.23)$$

When (2.23) is compared to model A, e.g., (2.4), we see that (2.23) represents a 30% increase at large impact parameters and a 30% decrease at low impact parameters. Thus it does not differ substantially from model A.

Now our corrected production amplitude $H_{2n}(s,b;v_n)$, formed from Eq. (1.7) using the unitary $H_{22}(s,b)$, is given by

$$H_{2n}(s,b;v_n)=B_{2n}(s,b;v_n)[1+I(s)n(s)e^{-2b^2/r^2}]^{-1}. \quad (2.24)$$

If we now assume the same form for B_{2n} as in model A, and take $x=1$ for the same reason as before, we can carry out the resulting integral for $M_{2n}(s,0;v_n)$ with the result

$$M_{2n}(s,0;v_n)=M_{2n}{}^0(s,0;v_n)K'(s,0), \quad (2.25)$$

where $M_{2n}(s,0;v_n)$ is the uncorrected production amplitude given by Eq. (2.9) with $x=1$, $t=0$, and

$$K'(s,0)=\tan^{-1}[n(s)I(s)]^{\frac{1}{2}}/[n(s)I(s)]^{\frac{1}{2}} \quad (2.26)$$

is the correction factor in the forward direction. If we are in the energy range $4m^2\ll s<100m^2$ where (2.22) applies, we obtain

$$K'(s,0)\simeq 0.84, \quad (2.27)$$

which does not differ significantly from the result $K(s,0)=5/6$ of model A, as expected.

For nonforward directions, we can evaluate the Bessel transform of (2.24) by expanding the denominator in a power series. The resulting correction factor $K'(s,t)$ is then a sum of terms, each corresponding to a successively shorter range which ultimately approaches zero. We can approximate the series by the expression

$$K'(s,t)=1-e^{-tr^2/5}[1-K'(s,0)]. \quad (2.28)$$

This was obtained by giving all terms in the series after the first the t dependence $\exp(-tr^2/5)$, which is an average dependence of these terms. This approximation produces no significant errors in the region of interest, $(-tr^2)\leq 6$. If we insert our evaluation of nI, the correction factor becomes

$$K'(s,0)\simeq 1-0.16e^{-tr^2/5}. \quad (2.29)$$

This result does not differ significantly from our previous estimate for $K(s,t)$, and the conclusions (a)–(e) remain true in model B also. The essential difference between K and K' is that the deviation of K' from unity comes from iterated terms of shorter range than the term which is the sole contribution to $K(s,t)$. However, since we are only interested in the low momentum transfer region, our result is not sensitive to this distinction.

Model C

We conclude by applying the unitarity corrections to the currently popular hypothesis that high-energy scattering is dominated by Regge poles. This hypothesis gives a high-energy behavior of the matrix element M_{22} of the form

$$M_{22}(s,t)\simeq i(s/4m^2)^{\alpha(t)}F(t), \quad (2.30)$$

with $\alpha(0)=1$ in order that total cross section be constant at high energies.

One could proceed as in model B by modifying (2.30) in a manner similar to the way in which model A was made unitary. However, we saw in that case that such a modification did not change our conclusions regarding the effect of unitarity upon the inelastic scattering. Any similar modification of (2.30) should not affect our conclusions regarding its effect upon the inelastic scattering either. Thus, we first look for an $H_{22}(s,b)$ which yields an $M_{22}(s,t)$ of the form (2.30), and then we proceed as before to use this $H_{22}(s,b)$ to obtain a corrected production amplitude. We choose

$$H_{22}(s,b)=n(s)[b^2+a^2(s)]^{-\frac{1}{2}} \times\exp\{-\beta[b^2+a^2(s)]^{\frac{1}{2}}\}. \quad (2.31)$$

The resulting scattering amplitude $M_{22}(s,t)$ is then given by[7]

$$M_{22}(s,t)=n(s)(\beta^2-t)^{-\frac{1}{2}}\exp[-a(s)(\beta^2-t)^{\frac{1}{2}}]. \quad (2.32)$$

Now if we set

$$a(s)=\delta+\epsilon\ln(s/4m^2) \quad (2.33)$$

$$n(s)=iC(s/4m^2)^{1+\beta\epsilon}, \quad (2.34)$$

then $M_{22}(s,t)$ is in the Regge form with

$$\alpha(t)=1+\beta\epsilon-\epsilon(\beta^2-t)^{\frac{1}{2}} \quad (2.35)$$

and

$$F(t)=C(\beta^2-t)^{-\frac{1}{2}}\exp[-\delta(\beta^2-t)^{\frac{1}{2}}]. \quad (2.36)$$

ϵ, C, δ, and β are constants which must be chosen to fit the elastic scattering data. Now $\alpha(t)$ has a cut at $t=4\mu^2$ corresponding to the two-pion intermediate state in the crossed reaction, and hence we might expect that $\beta\gtrapprox 2\mu$. We do not expect our ansatz to be a reasonable form for $\alpha(t)$ when t gets large and negative, since then $\alpha(t)$ also becomes very negative. The current viewpoint frowns on this behavior although it does not object to $\alpha(t)$ passing through zero for negative t as some experiments seem to allow. In the small t region of interest to us, such problems are unimportant.

Let us assume C, δ, ϵ, and β have been determined, and proceed to use $H_{22}(b,s)$ to correct the inelastic scattering amplitude. We choose

$$B_{2n}=f_{2n}(v_n)\exp[-\tfrac{1}{2}\beta(b^2+a^2)^{\frac{1}{2}}]. \quad (2.37)$$

The form (2.37) gives the inelastic process half the range of the elastic process and, most important, allows all the integrals to be performed analytically. The corrected production amplitude is then given in the form

$$M_{2n}(s,t;v_n)=M_{2n}{}^0(s,t;v_n)K''(s,t), \quad (2.38)$$

where

$$M_{2n}{}^0=\frac{f_{2n}\beta}{2[\frac{1}{4}\beta^2-t]^{\frac{3}{2}}}[1+a(s)(\tfrac{1}{4}\beta^2-t)^{\frac{1}{2}}] \times\exp[-a(\tfrac{1}{4}\beta^2-t)^{\frac{1}{2}}] \quad (2.39)$$

TABLE I. The unitarity correction.

δ	β^2/μ^2	$\epsilon\beta$	t/μ^2 value when $\alpha=2$	t/μ^2 value when $\alpha=0$	$K''(s,0)$
0	10	0.75	···	−44	0.90
0	20	1.50	18	−36	0.80
ϵ	10	0.65	···	−53	0.90
ϵ	20	1.17	20	−49	0.80
2ϵ	10	0.55	···	−69	0.90
2ϵ	20	0.96	···	−66	0.80

is the uncorrected production amplitude, and

$$K''=1+\frac{2I(s)n(s)(\frac{1}{4}\beta^2-t)^{\frac{3}{2}}}{\beta[(9/4)\beta^2-t]^{\frac{1}{2}}[1+a(\frac{1}{4}\beta^2-t)^{\frac{1}{2}}]}\times\exp[-a\{[(9/4)\beta^2-t]^{\frac{1}{2}}-(\frac{1}{4}\beta^2-t)^{\frac{1}{2}}\}] \quad (2.40)$$

is the correction factor due to unitarity. In the forward direction this becomes

$$K''(s,0)=1+iI(s)C\beta se^{-\delta\beta}/24m^2. \quad (2.41)$$

In the high-energy region $4m^2 \ll s < 100m^2$, we can again approximate I by $2im/s$ and get

$$K''(s,0)=1-c\beta e^{-\delta\beta}/12m. \quad (2.42)$$

In order to fit the forward elastic cross section with (2.30), we must have

$$C\beta e^{-\delta\beta}=M\beta^2/8\mu^2,$$

which yields

$$K''(s,0)=1-\beta^2/96\mu^2. \quad (2.43)$$

In order to estimate the unitarity correction, we must choose the parameters δ, β, and ϵ to fit the elastic scattering data. There is quite a wide latitude in the possible parameter values and typical reasonable fits to the results of reference 6 are presented in Table I (for lab energies of 10–20 BeV). We see that the correction is essentially the same as found for the previous models. Thus, our conclusion that there are rather large unitarity corrections to the peripheral collision model remains unchanged.